CP2 ELECTROMAGNETISM

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LECTURE 7:

LAPLACE & POISSON EQUATIONS

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE :7. LAPLACE & POISSON EQUATIONS

7.1 Poisson and Laplace Equations

7.2 Uniqueness Theorem

7.3 Laplace equation in cartesian coordinates

7.4 Laplace Equation in spherical coordinates

7.1 Poisson and Laplace Equations

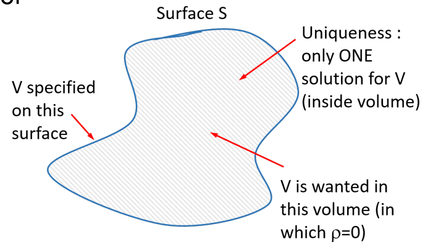
- The expression derived previously is the “integral form" of Gauss’ Law

  $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$ over volume $\mathcal{V}$
- We can express Gauss’ Law in differential form using the Divergence Theorem :

7.2 Uniqueness Theorem

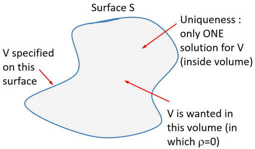
This states : *The solution to Laplace's equation in some volume is uniquely determined if the potential V is specified on the boundary surface S.* Why is this so?

- Suppose there are TWO solutions V_1 and V_2 to Laplace's equation for potential inside the volume



Uniqueness Theorem continued

From the previous page :



- $\underline{\nabla}^2 V_1 = 0$ & $\underline{\nabla}^2 V_2 = 0$ with $V_1 = V_2$ on the surface
- $V_3 = V_1 - V_2$ (which $= 0$ on the surface)
- $\underline{\nabla}^2 V_3 = 0$ everywhere.
    - The $\underline{\nabla}^2$ operator is a three-dimensional second derivative of a function - when a function has an extrema, the second derivative will be negative for a maximum and positive for a minimum.
    - The fact that the second derivative is always zero therefore indicates that there are no such minima or maxima in the region of interest
    - Hence solutions to Laplace's equation do not have minima or maxima.
    - Since $V_3 = 0$ on the surface, the maximum and minimum values of V_3 must also be zero everywhere inside it.

Hence $V_3 = 0$ everywhere, and **V must be unique**

- Note the same applies to Poisson's equation.
- If $\underline{\nabla}^2 V_1 = -\rho/\epsilon_0$ and $\underline{\nabla}^2 V_2 = -\rho/\epsilon_0$, then $\underline{\nabla}^2 V_3 = 0$ as before.

Poisson and Laplace Equations : summary

Gauss' law: $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ Definition of Potential: $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson equation

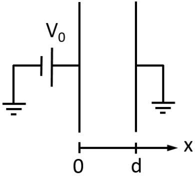
In regions where ρ=0: $\nabla^2 V = 0$ Laplace equation

Uniqueness Theorem:
The potential V inside a volume is *uniquely* determined, if the following are specified:

(i) The charge density throughout the region
(ii) The value of V on all boundaries

7.3 Laplace equation in cartesian coordinates

Example : Solutions to Laplace's equation for a parallel plate capacitor. Symmetry suggests use of cartesian coordinates.



7.4 Laplace Equation in spherical coordinates

... assuming azimuthal symmetry.

General solutions to Laplace's equation for charge distributions with azimuthal symmetry (mainly for information here : see second year).

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}}_{=0} = 0$$

Separation of variables yields the general solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

where A_l, B_l are constants determined by boundary conditions and P_l are Legendre Polynomials in cos θ, i.e.:

$P_0(\cos\theta) = 1$
$P_1(\cos\theta) = \cos\theta$
$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta-1)$
Etc ...

$$V(r,\theta) = A_0 + \frac{B_0}{r} + A_1 r\cos\theta + \frac{B_1}{r^2}\cos\theta$$

$$+ A_2 r^2 \frac{1}{2}(3\cos^2\theta - 1) + \frac{B_2}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \cdots$$

Laplace equation examples in spherical coordinates

1. Take a defined small spherical volume which contains some azimuthally symmetric charge distribution :
   - Outside the volume $\rho = 0$
   - Boundary condition on potential : $V \to 0$ as $r \to \infty$
   - Hence $A_\ell = 0$ for all ℓ
   - Retain just multipole expansion terms (monopole + dipole+ quadrupole + $\cdots$ terms)
2. Special case of spherically symmetric charge distribution inside the volume :
   - Outside the volume $\rho = 0$, $\nabla^2 V = 0$ with no θ dependence
   - $A_\ell = B_\ell = 0$ for $\ell \neq 0$
   - $V(r) = A_0 + B_0/r$ as expected from Gauss' Law