CP2 ELECTROMAGNETISM

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LECTURE 6:

GAUSS LAW EXAMPLES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 6. GAUSS LAW EXAMPLES

6.1 Gauss theorem : uniform volume charge

6.2 Gauss Theorem : Long, uniformly charged rod

6.3 Uniformly charged infinite plate

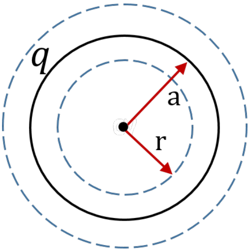
6.4 Electric field inside a conductor

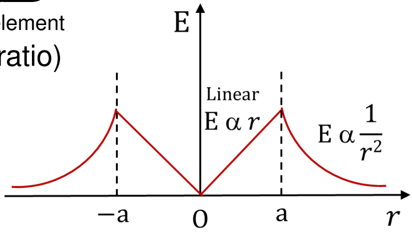
6.5 Revisit the electric field inside a hollow sphere

6.1 Gauss theorem : uniform volume charge

Sphere with uniform volume charge density

- $\rho = \begin{cases} \frac{q}{(4/3)\,\pi\, a^3} & \textit{for } 0 \le r \le a \;(\textit{inside}) \\ 0 & \textit{for } a \le r \;(\textit{outside}) \end{cases}$
- $\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$
  (volume $\mathcal{V}$ bounded by surface)
- Inside sphere :
  $E \cdot 4\,\pi\, r^2 = \frac{1}{\epsilon_0} \int_0^r \frac{q}{(4/3)\,\pi\, a^3} \underbrace{4\,\pi\, r'^2 dr'}_{\text{volume element}}$
  $= \frac{q}{\epsilon_0} \int_0^r \frac{3\, r'^2}{a^3} dr' = \frac{q}{\epsilon_0} \frac{r^3}{a^3}$ (volume ratio)
- $E = \frac{q}{4\pi\epsilon_0 a^3} r$ (radial)
- Outside sphere :
  $\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{q}{\epsilon_0}$
- $E = \frac{q}{4\pi\epsilon_0 r^2}$ (point charge again)



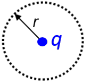


Summary Gauss Law : spherical symmetry

Spherically symmetric charge distributions.

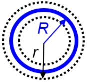
$$\oint_S \mathbf{E} \, . \, \mathbf{da} = E_r \times 4\pi r^2 = \frac{1}{\varepsilon_0} \int_V \rho \, dV \longrightarrow E_r = \frac{1}{4\pi\varepsilon_0 r^2} \int_V \rho \, dV$$

(i) point charge q:



$$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$ for any r

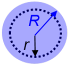
(ii) hollow sphere with q spread evenly across surface:



For 0 < r < R (inside sphere): $E_r = 0$

For R < r (outside sphere): $$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$

(iii) Sphere carrying uniform volume charge ρ:



For 0 < r < R (inside sphere): $$E_r = \frac{q}{4\pi\varepsilon_0 R^2} \frac{r}{R}$$

For R < r (outside sphere): $$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$

6.2 Gauss Theorem : Long, uniformly charged rod

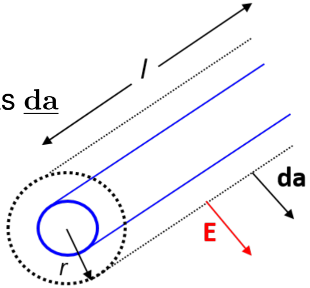
- Long, uniformly charged cylindrical rod with surface charge q
- Choose cylindrical Gaussian surface

  Symmetry : $\underline{\mathbf{E}}$ is in the same direction as $\underline{\mathbf{da}}$

  $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 2\pi\, r \cdot \ell = \frac{q}{\epsilon_0}$

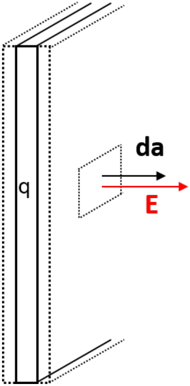
- $$E = \frac{q}{\ell}\frac{1}{2\,\pi\,\epsilon_0 r} = \frac{\lambda}{2\,\pi\,\epsilon_0 r}$$ (radial)

  λ is the charge per unit length



6.3 Uniformly charged infinite plate

1. Uniformly charged "infinite" plate of area A



- By symmetry : $\underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot da \;\; (\underline{\mathbf{E}} \parallel \underline{\mathbf{da}})$

  $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 2\,A = \frac{q}{\epsilon_0}$

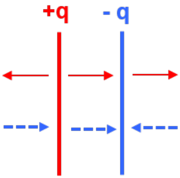
  (factor 2 due to both sides)

  $$E = \frac{1}{2\,\epsilon_0}\frac{q}{A} = \frac{\sigma}{2\,\epsilon_0}$$

  Field is uniform. σ is the charge per unit area

- As the plates become very large, the contribution from the edges become negligible

2. The capacitor



- Principle of superposition between the plates

  $E = \frac{\sigma}{2\,\epsilon_0} - \frac{-\sigma}{2\,\epsilon_0} = \frac{\sigma}{\epsilon_0}$

- Outside the plates $E = \frac{\sigma}{2\,\epsilon_0} + \frac{-\sigma}{2\,\epsilon_0} = 0$

6.4 Electric field inside a conductor

Inside a conductor, one or more electrons per atom are free to move throughout the material (copper, gold, and other metals). We are considering electroSTATICS (static charge). As a result:

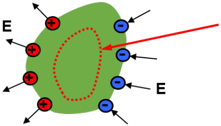
(i) **E**=0 inside a conductor (free charge moves to surface until the internal electric field is cancelled).

(ii) ρ =0 inside a conductor (from Gauss' law: **E**=0 hence ρ=0).

(iii) Therefore any net charge resides on the surface.

(iv) A conductor is an equipotential (since **E**=0, V($\mathbf{r}_1$)=V($\mathbf{r}_2$)).

(v) At the surface of a conductor, **E** is perpendicular to the surface (otherwise charges will flow until the tangential component becomes zero when equilibrium is reached).



$$\oint_S \mathbf{E} \, . \, \mathbf{da} = 0$$

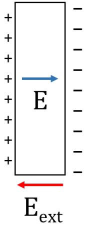
↓

ρ=0

inside *V*

Properties of conductors

1. $\mathbf{E} = 0$ inside a conductor
   - We are dealing with electroSTATICS - charges can move in an $\mathbf{E}$-field !
   - They will move to the surface, creating surface charge which opposes applied field.
   - Equilibrium reached with $\mathbf{E} = 0$ inside conductor.

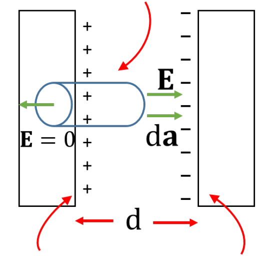


2. $\rho = 0$ inside a conductor :
   - $\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$
   - $\mathbf{E} = 0 \, , \; \rho = 0$

- Alternative treatment for the capacitor :

$$EA + 0 = \frac{q}{\epsilon_0} \;\rightarrow\; E = \frac{q}{A\,\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

Where the "0" term is the field *inside* the plate

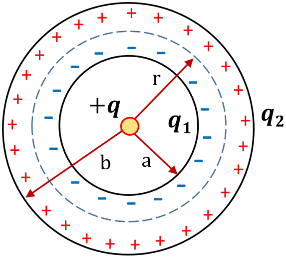


- Potential difference between plates

$V = -\int_0^d \mathbf{E} \cdot \mathbf{d\ell} = -Ed$

6.5 Revisit the electric field inside a hollow sphere

Consider an uncharged hollow metal sphere of finite thickness, with point charge $+q$ at its centre.



- Inside hollow :

$$\oint_S \mathbf{E} \cdot \mathbf{da} = E_r \cdot 4\pi\, r^2 = \frac{q}{\epsilon_0}$$

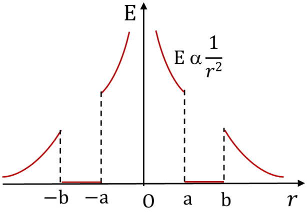
- Inside metal $\quad \mathbf{E} = 0$:

$$\oint_S \mathbf{E} \cdot \mathbf{da} = (q + q_1)/\epsilon_0 = 0$$

$\rightarrow$ Inner surface charge $q_1 = -q$ must be induced on inner surface

- Outside sphere :

$$\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{q+q_1+q_2}{\epsilon_0}$$



- Because there is no net charge on the sphere

$\rightarrow$ Outer surface charge given by $q_1 + q_2 = 0$

- $\rightarrow$ $q_2 = +q$ is induced on the outer surface

$$\rightarrow \quad E_r = \frac{q}{4\pi\epsilon_0\, r^2}$$