CP2 ELECTROMAGNETISM

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LECTURE 5:

GAUSS LAW

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 5. GAUSS LAW

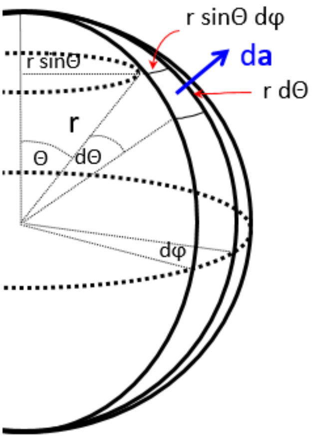
5.1 Introduction to solid angles

- Consider an element of area on a sphere. Define a vector of surface element $\underline{\mathbf{da}}$ normal to the surface :
- $\underline{\mathbf{da}} = (r \sin\theta \, d\phi) \times (r d\theta) \, \underline{\hat{\mathbf{r}}}$

  $\underline{\mathbf{da}} = r^2 \underbrace{\sin\theta \, d\theta \, d\phi} \, \underline{\hat{\mathbf{r}}}$
- Define $d\Omega = \sin\theta \, d\theta \, d\phi$ as a *solid angle* element.

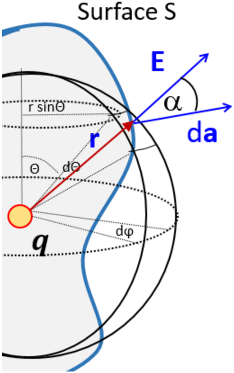
  (note that $d\Omega$ is *independent* of r)
- Hence :

  $\int_{surface} d\Omega = \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi$



5.2 Gauss' Law

Calculate the *flux* $d\Phi_E = \underline{\mathbf{E}} \cdot \underline{\mathbf{da}}$ through an infinitesimal area $\underline{\mathbf{da}}$ of surface S at a distance r away from a point charge q



- $d\Phi_E = \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot da \cos\alpha$
- Note that $(da \cos\alpha)$ is the surface element $\underline{\mathbf{da}}$ of S resolved onto the sphere centred on charge q
- Hence $d\Phi_E = \left(\frac{q}{4\pi\epsilon_0}\frac{\hat{\underline{\mathbf{r}}}}{r^2}\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{\mathbf{r}}}\right)$

$$= \frac{q}{4\pi\epsilon_0} \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega} \text{ independent of } r$$

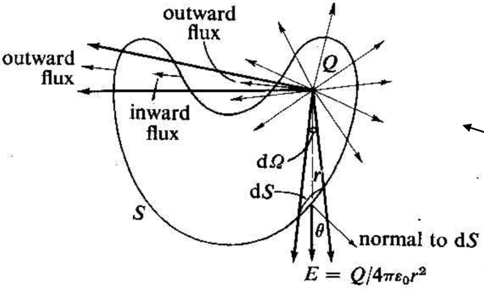
  - Therefore for any *closed* surface

    $\oint_{\substack{closed\\surface}} \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{q}{\epsilon_0} \oint \frac{d\Omega}{4\pi}$. Hence $\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{q}{\epsilon_0}$
  - It does not matter WHERE q is inside the surface for this to hold (because flux $d\Phi_E$ is independent of r) !

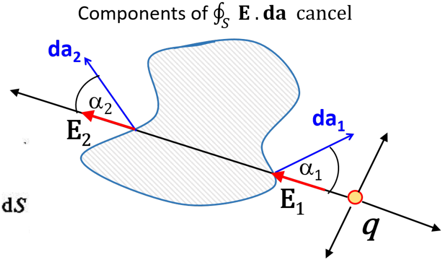
Gauss' Law

$$\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \begin{cases} \frac{q}{\epsilon_0} & \textit{if q is INSIDE closed surface} \\ 0 & \textit{if q is OUTSIDE closed surface} \end{cases}$$

INSIDE



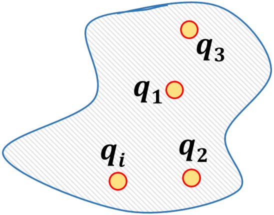
OUTSIDE



5.3 Gauss' Law for a collection of charges

- $\oint \underline{\mathbf{E_i}} \cdot \underline{\mathbf{da}} = \frac{q_i}{\epsilon_0}$ for any charge enclosed
- Apply the principle of superposition

  $\oint \sum_i \underline{\mathbf{E_i}} \cdot \underline{\mathbf{da}} = \frac{\sum_i q_i}{\epsilon_0}$ ($\sum_i \underline{\mathbf{E_i}}$ is the sum of field components on the surface)

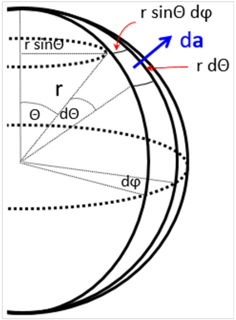


- **Gauss Law :** $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{Q_\mathcal{V}}{\epsilon_0}$

  $\underline{\mathbf{E}}$ is the field at surface S

  $Q_\mathcal{V} = \sum_i q_i$ is the total charge in the volume $\mathcal{V}$ enclosed by surface S
- For a continuous charge distribution, density ρ : $Q_\mathcal{V} = \int_\mathcal{V} \rho(\underline{\mathbf{r}}) d\mathcal{V}$
  - Gauss Law allows finding the total charge enclosed inside a closed surface if the field is known on the surface, and vice versa
  - Allows a straightforward calculation of field for symmetrical charge configurations

Gauss Law : summary



Area and Solid angle elements:

$$da = r^2 \sin\Theta \, d\Theta \, d\varphi = r^2 \, d\Omega$$

Calculate electric field flux dΦ through area **da** for a point charge q_i a distance r away from **da**:

$$d\Phi = \mathbf{E}_i \,.\, \mathbf{da} = \frac{q_i}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \cdot \hat{\mathbf{r}}\, r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{q_i}{4\pi\varepsilon_0} \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega}$$

Independent of r !

Integrate over a closed surface:

$$\oint_S \mathbf{E_i} \,.\, \mathbf{da} = \frac{q_i}{\varepsilon_0} \frac{\oint d\Omega}{4\pi} = \begin{cases} \dfrac{q_i}{\varepsilon_0} & \text{if } q_i \text{ is enclosed} \\ 0 & \text{if } q_i \text{ is not enclosed} \end{cases}$$

Principle of superposition →

Gauss' Law;

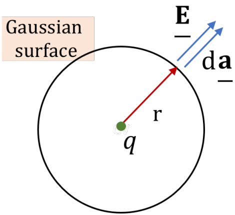
$$\oint_S \mathbf{E} \,.\, \mathbf{da} = \frac{Q}{\varepsilon_0}$$

E-field on closed surface

total charge enclosed

5.4 Example : Spherically symmetric charge distributions

1. Point charge
   - By symmetry : $\underline{\mathbf{E}} = E\,\hat{\underline{\mathbf{r}}}$
     $d\underline{\mathbf{a}} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\underline{\mathbf{r}}} = r^2\, d\Omega\, \hat{\underline{\mathbf{r}}}$
   - $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \oint_S r^2 d\Omega = E 4\pi r^2 = \frac{q}{\epsilon_0}$
   - $E = \frac{q}{4\pi\epsilon_0 r^2}$ as expected



2. Hollow sphere, radius a, with q evenly distributed on surface.
   - Inside sphere ($r < a$):
     $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 4\pi r^2 = \frac{q_{enclosed}}{\epsilon_0} = 0$
   - No charge enclosed $\rightarrow$ $E = 0$
   - Outside sphere ($r > a$):
     $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 4\pi r^2 = \frac{q_{enclosed}}{\epsilon_0} = \frac{q}{\epsilon_0}$
   - $E = \frac{q}{4\pi\epsilon_0 r^2}$ as for point charge