CP2 ELECTROMAGNETISM

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LECTURE 4:

CONTINUOUS CHARGE DISTRIBUTIONS

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 4. CONTINUOUS CHARGE DISTRIBUTIONS

4.1 Continuous Charge Distributions

4.2 Example 1 : uniformly charged annulus

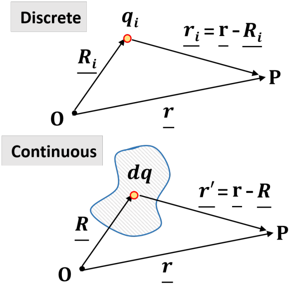
4.3 Example 2 : uniformly charged rod

4.1 Continuous Charge Distributions

- Reminder : the potential at P , position vector $\underline{\mathbf{r}}$, due to assembly of charges :

  $V(\underline{\mathbf{r}}) = \sum_i V_i(q_i) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\underline{\mathbf{r_i}}|}$

  (where $\underline{\mathbf{r_i}} = \underline{\mathbf{r}} - \underline{\mathbf{R_i}}$).



- And the field : $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{(\underline{\mathbf{r_i}})^2} \frac{\underline{\mathbf{r_i}}}{|\underline{\mathbf{r_i}}|}$
- For a continuous charge distribution $\sum_i V_i \rightarrow \int dV$

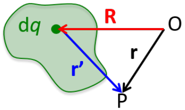
  Hence $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}|}$

- Integrate over all infinitesimal dq over the charge distribution, noting that $q \equiv q(\underline{\mathbf{R}})$
- Alternatively $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})\, d\mathcal{V}}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}|}$ over volume $\mathcal{V}$, where $\rho(\underline{\mathbf{R}})$ is the charge density.
- Similarly for the electric field $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})(\underline{\mathbf{r}} - \underline{\mathbf{R}})}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}|^3} d\mathcal{V}$

Continuous Charge Distributions

Continuous charge distribution:

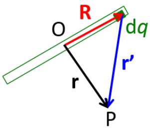
$$\sum V_i \rightarrow \int dV$$



$$V = \int \frac{dq}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{R}|}$$

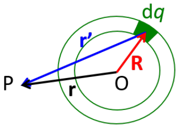
Line charge:

$dq = \lambda \, dl$



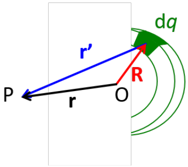
Surface charge:

$dq = \sigma \, dA$



Volume charge:

$dq = \rho \, dV$

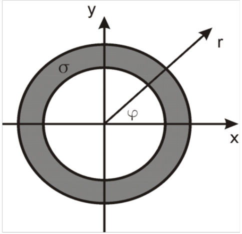


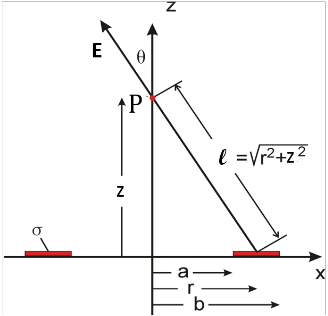
Choose a convenient origin O suiting the geometry of the charge distribution!

Adopt the notation: λ = charge density for 1D distribution, σ = charge density for 2D, ρ = charge density for 3D

4.2 Example 1 : uniformly charged annulus

Uniformly charged ring.





$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right]$$

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right]\hat{\mathbf{z}}$$

Uniformly charged annulus

Annulus contains charge q. Calculate the potential V on the annulus axis at a distance z above its centre. Note the radial symmetry.

- Charge density σ . Charge dq contained in infinitesimally thin ring of radius r :

  $\rightarrow \quad dq = \text{area} \times \text{charge density} = 2\pi r\, dr\, \sigma$

- Potential at P : $\quad V = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{dq}{\ell(r)}$

$$V = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{2\pi r\, dr\, \sigma}{\sqrt{r^2+z^2}}$$

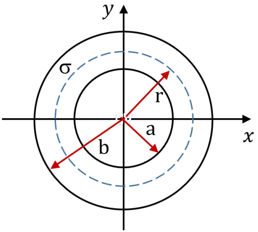
$$= \frac{\sigma}{2\epsilon_0} \int_a^b \frac{r\, dr}{\sqrt{r^2+z^2}}$$

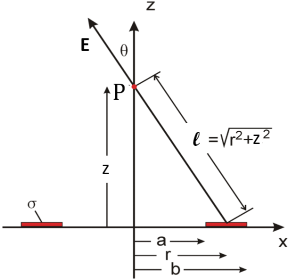
$$V = \frac{\sigma}{2\epsilon_0} \sqrt{r^2 + z^2}\Big|_a^b$$

$$= \frac{\sigma}{2\epsilon_0} \left[\sqrt{b^2 + z^2} - \sqrt{a^2 + z^2}\right]$$

- From symmetry, E field points along z-axis: $\underline{\mathbf{E}} = -\underline{\hat{\mathbf{z}}}\, \frac{\partial}{\partial z} V(z)$

$$\underline{\mathbf{E}} = \frac{\sigma}{2\epsilon_0} \left\{ \frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}} \right\} \underline{\hat{\mathbf{z}}}$$



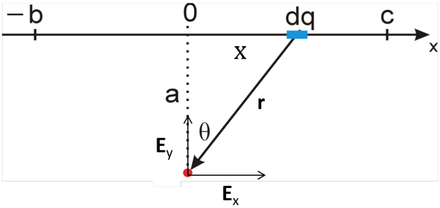


Special cases

1. $a = 0$ (disk)
   - $V = \frac{\sigma}{2\epsilon_0}\left[\sqrt{b^2+z^2} - z\right]$
   - $\underline{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{b^2+z^2}}\right]\underline{\hat{\mathbf{z}}}$
2. Disk ($a = 0$) for $z >> b$ (far away)
   - $V = \frac{\sigma}{2\epsilon_0}\left[z\sqrt{(b/z)^2+1} - z\right]$
     Use $\sqrt{1+(b/z)^2} \approx 1 + \frac{1}{2}(b/z)^2 + \cdots$
   - $V = \frac{\sigma}{2\epsilon_0}(z + \frac{b^2}{2z} - z) = \frac{\sigma b^2}{4\epsilon_0 z}$. But $\sigma = \frac{q}{\pi b^2}$:
   - Hence $V = \frac{q}{4\pi\epsilon_0 z}$ (point charge)
   - Using same method : $\underline{\mathbf{E}} = \frac{q}{4\pi\epsilon_0 z^2}\underline{\hat{\mathbf{z}}}$
3. Disk ($a = 0$) for $z << b$ (close to plate)
   - $V = \frac{\sigma}{2\epsilon_0}b$ & $\underline{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\underline{\hat{\mathbf{z}}}$ ("infinite" charged plane)

4.3 Example 2 : uniformly charged rod

Uniformly charged rod.

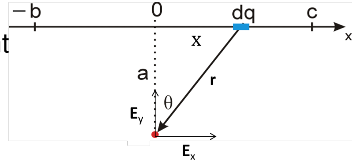


$$\mathbf{E}_x = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+c^2}} - \frac{1}{\sqrt{a^2+b^2}}\right]$$

$$\mathbf{E}_y = \frac{-\lambda}{4\pi\varepsilon_0 a}\left[\frac{c}{\sqrt{a^2+c^2}} + \frac{b}{\sqrt{a^2+b^2}}\right]$$

Uniformly charged rod

- Calculate the field $\underline{\mathbf{E}}$ at a distance a from a uniformly charged rod, with length between coordinates $-b$ and c.
- Charge dq contained in a small element dx, where $dq = \lambda dx$:



- $r = \sqrt{a^2 + x^2}$, $\underline{\hat{\mathbf{r}}} = \frac{1}{r}\begin{pmatrix} -x \\ -a \end{pmatrix}$
  - $\underline{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{-b}^{c} \frac{\underline{\hat{\mathbf{r}}}}{r^2} dq$

    Integrating components:
  - $E_x = \frac{1}{4\pi\epsilon_0} \int_{-b}^{c} \frac{-x\lambda}{(a^2+x^2)^{\frac{3}{2}}} dx = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{a^2+c^2}} - \frac{1}{\sqrt{a^2+b^2}}\right]$

    $E_y = \frac{1}{4\pi\epsilon_0} \int_{-b}^{c} \frac{-a\lambda}{(a^2+x^2)^{\frac{3}{2}}} dx \quad \left[\text{use } \frac{d}{dx}\frac{x}{a^2(a^2+x^2)^{\frac{1}{2}}} = \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\right]$
  - $E_y = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-x/a}{(a^2+x^2)^{\frac{1}{2}}}\right]_{-b}^{c} = \frac{-\lambda}{4\pi\epsilon_0 a}\left[\frac{c}{\sqrt{a^2+c^2}} + \frac{b}{\sqrt{a^2+b^2}}\right]$
  - $E_z = 0 \quad$ (symmetry)

Special cases

1. $b = c = \ell/2$
   - $E_x = 0$ (cancellation by symmetry)
   - $E_y = -\frac{\lambda}{4\pi\epsilon_0 a}\frac{\ell}{\sqrt{a^2+(\ell/2)^2}}$
2. $b = c \to \infty$
   - $E_x = 0$ (symmetry)
   - $E_y = -\frac{\lambda}{2\pi\epsilon_0 a}$ (radial field)

   - Note that this configuration is most easily solved via Gauss Law (see next lecture)