CP2 ELECTROMAGNETISM

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LECTURE 3:

ELECTRIC MULTIPOLES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 3. ELECTRIC MULTIPOLES

3.1 The potential due to an electric dipole

3.2 The field of an electric dipole

3.3 The torque on a dipole in an external E-field

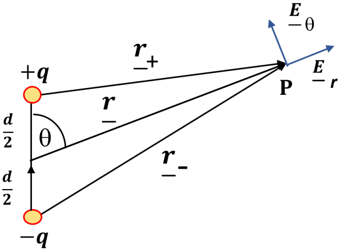
3.4 The energy of a dipole in an external E-field

3.5 The quadrupole potential

3.6 The general multipole expansion

3.1 The potential due to an electric dipole

- Two charges $+q$ and $-q$ separated by (small) distance d
- Define *dipole moment* : $\underline{\mathbf{p}} = q\,\underline{\mathbf{d}}$
- Potential at P : $V = \frac{q}{4\pi\epsilon_0 r_+} - \frac{q}{4\pi\epsilon_0 r_-}$
- Cosine rule : $r_{+/-} = \sqrt{r^2 + (d/2)^2 \mp dr\cos\theta}$



  - $V = \frac{q}{4\pi\epsilon_0 r}\left(\frac{1}{\sqrt{1+(d/2r)^2-(d/r)\cos\theta}} - \frac{1}{\sqrt{1+(d/2r)^2+(d/r)\cos\theta}}\right)$
  - Look at the field $d << r$:

    Expand : $\frac{1}{\sqrt{1+x}} = (1 - \frac{1}{2}x + \frac{1}{2!}(-\frac{1}{2})(-\frac{3}{2})x^2 + \cdots)$,
    retain terms only up to first order of d/r
  - $V = \frac{q}{4\pi\epsilon_0 r}((1 + (d/2r)\cos\theta) - (1 - (d/2r)\cos\theta))$

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\underline{\mathbf{p}}\cdot\underline{\mathbf{r}}}{4\pi\epsilon_0 r^3}$$

3.2 The field of an electric dipole

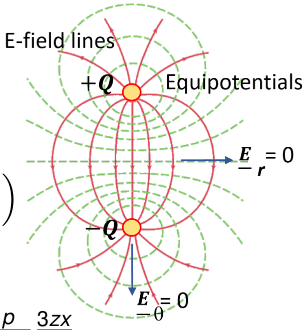
Use $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = -\underline{\nabla} V$; $\quad V = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}$

1. Spherical polar coordinates $\quad \underline{\nabla} \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2qd\cos\theta}{4\pi\epsilon_0 r^3} = \frac{2\underline{\mathbf{p}}\cdot\underline{\mathbf{r}}}{4\pi\epsilon_0 r^4}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{qd\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$



2. Cartesian coordinates $\quad \underline{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$$\underline{\mathbf{p}}\cdot\underline{\mathbf{r}} = pz \; ; \; r = (x^2+y^2+z^2)^{\frac{1}{2}} = \frac{z}{\cos\theta}$$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{pz}{4\pi\epsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}} \right) = \frac{p}{4\pi\epsilon_0}\frac{3zx}{r^5}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{p}{4\pi\epsilon_0}\frac{3zy}{r^5}$$

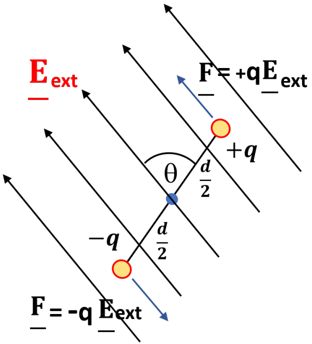
$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right) = \frac{p}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)$$

3.3 The torque on a dipole in an external **E**-field

- Torque (couple) on the dipole : $\underline{\mathbf{T}} = \sum_i \underline{\mathbf{r_i}} \times \underline{\mathbf{F_i}}$
- Taking moments about the centre point between the charges :

  $\underline{\mathbf{T}} = 2\left((\underline{\mathbf{d}}/2) \times q\underline{\mathbf{E}}_{\mathbf{ext}}\right) = q\underline{\mathbf{d}} \times \underline{\mathbf{E}}_{\mathbf{ext}}$
- $\underline{\mathbf{T}} = \underline{\mathbf{p}} \times \underline{\mathbf{E}}_{\mathbf{ext}}$
- Magnitude of torque from cross product :

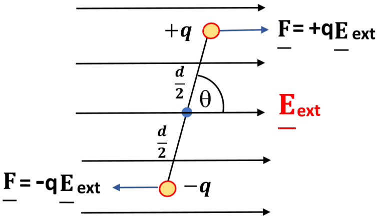
  $|\underline{\mathbf{T}}| = |\underline{\mathbf{p}}|\,|\underline{\mathbf{E}}_{\mathbf{ext}}|\,\sin\theta$
- There is *only* a couple : no translational force.



3.4 The energy of a dipole in an external E-field

- Calculate the work done by an applied force to rotate the dipole from angle $\pi/2$ to θ

  (take $\theta = \pi/2$ as the zero of potential energy)

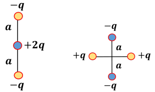


- $W = \int_{\frac{\pi}{2}}^{\theta} T d\theta' = \int_{\frac{\pi}{2}}^{\theta} p\, E_{ext} \sin\theta' d\theta'$
- $W = [-p\, E_{ext} \cos\theta']_{\pi/2}^{\theta}$

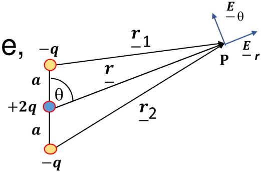
  $= -p\, E_{ext} \cos\theta$
- Hence potential energy of $\underline{\mathbf{p}}$ in $\underline{\mathbf{E}}_{\mathbf{ext}}$:

$$U = -p\, E_{ext} \cos\theta = -\underline{\mathbf{p}} \cdot \underline{\mathbf{E}}_{\mathbf{ext}}$$

3.5 The quadrupole potential



- Two charge configurations of the quadrupole, which both look identical at long distance
- Cosine rule : $r_{1/2} = \sqrt{r^2 + a^2 \mp 2ar\cos\theta}$



- Potential at P :

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} - \frac{q}{r\sqrt{1+(a/r)^2-2(a/r)\cos\theta}} - \frac{q}{r\sqrt{1+(a/r)^2+2(a/r)\cos\theta}}\right]$$

- Expand : $\frac{1}{r\sqrt{1+x}} = \frac{1}{r}\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right)$

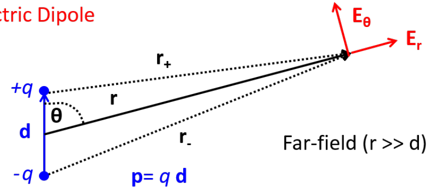
  Retain only up to powers of $(a/r)^2$

- $$V = \frac{1}{4\pi\epsilon_0 r}\left[2q - q\left\{1 - \frac{1}{2}(\frac{a}{r})^2 + \frac{a}{r}\cos\theta + \frac{3}{8}(\frac{a}{r})^2 4\cos^2\theta\right\}\right]$$
  $$- \left[q\left\{1 - \frac{1}{2}(\frac{a}{r})^2 - \frac{a}{r}\cos\theta + \frac{3}{8}(\frac{a}{r})^2 4\cos^2\theta\right\}\right]$$
  $$= \frac{1}{4\pi\epsilon_0 r}\left[+(\frac{a}{r})^2 - \frac{3}{4}(\frac{a}{r})^2 4\cos^2\theta\right]$$

- Quadrupole potential : $V = \frac{q\,a^2}{4\pi\epsilon_0 r^3}(1 - 3\cos^2\theta)$

Summary: Electric dipole and quadrupole

Electric Dipole

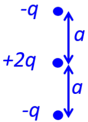


Far-field (r >> d) potential: $$V = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}$$

Far-field (r >> d) electric field: $$\mathbf{E}_r = \frac{2\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^4} \quad \mathbf{E}_\theta = \frac{qd\sin\theta}{4\pi\varepsilon_0 r^3} \quad \mathbf{E}_\phi = 0$$

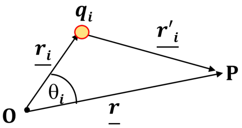
Energy of dipole in external electric field $\mathbf{E}_{ext}$: $$W = -\mathbf{E}_{\mathrm{ext}}\cdot\mathbf{p}$$

Electric Quadrupole:



$$V = \frac{qa^2}{4\pi\varepsilon_0 r^3}(1 - 3\cos^2\theta)$$

3.6 The general multipole expansion



- Potential at P : $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\underline{\mathbf{r}}'_\mathbf{i}|}$

  where $\underline{\mathbf{r}}'_\mathbf{i} = \underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}$

- Cosine rule : $r'_i = \sqrt{r^2 + r_i^2 - 2\, r\, r_i \cos\theta_i}$

  $= r\sqrt{1 - 2\frac{r_i\cos\theta_i}{r} + \frac{r_i^2}{r^2}} \equiv r\sqrt{1+x}$

- For points P far from the charge assembly $r_i << r \;\to x << 1$

- Expand : $\frac{1}{r\sqrt{1+x}} = \frac{1}{r}\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right)$

  Retain only up to powers of $(r_i/r)^2$

- $\frac{1}{|\underline{\mathbf{r}}'_\mathbf{i}|} \approx \frac{1}{r}\left[1 + \frac{r_i\cos\theta_i}{r} - \frac{r_i^2}{2r^2} + \frac{3}{2}\frac{r_i^2}{r^2}\cos^2\theta_i + \cdots\right]$

  $$= \frac{1}{r} + \frac{r_i\cos\theta_i}{r^2} + \frac{r_i^2}{r^3}\frac{1}{2}\left(3\cos^2\theta_i - 1\right) + \cdots$$

The general multipole expansion

- $$V(\underline{\mathbf{r}}) = \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r}}_{\text{monopole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i r_i \cos\theta_i}{r^2}}_{\text{dipole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{\frac{1}{2} q_i r_i^2 \left(3\cos^2\theta_i - 1\right)}{r^3}}_{\text{quadrupole term}} + \cdots$$
- So any assembly of charges can be described in terms of the sum over contributions from multipoles
- The n-th multipole potential falls off with $1/r^n$