CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 3:

ELECTRIC MULTIPOLES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 3. ELECTRIC MULTIPOLES

3.1 The potential due to an electric dipole

3.2 The field of an electric dipole

3.3 The torque on a dipole in an external E-field

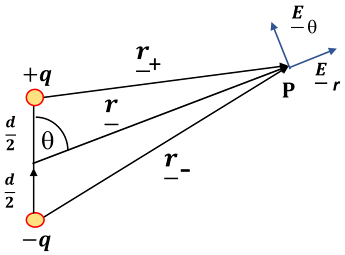
3.4 The energy of a dipole in an external E-field

3.5 The quadrupole potential

3.6 The general multipole expansion

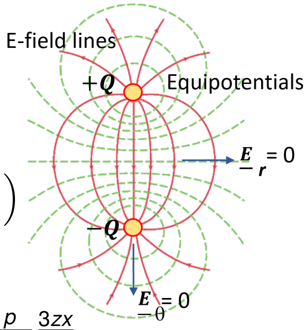
3.1 The potential due to an electric dipole

- Two charges $+q$ and $-q$ separated by (small) distance d



$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\underline{\mathbf{p}}\cdot\underline{\mathbf{r}}}{4\pi\epsilon_0 r^3}$$

3.2 The field of an electric dipole



2. Cartesian coordinates $\underline{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\underline{\mathbf{p}} \cdot \underline{\mathbf{r}} = pz \; ; \; r = (x^2 + y^2 + z^2)^{\frac{1}{2}} = \frac{z}{\cos\theta}$$

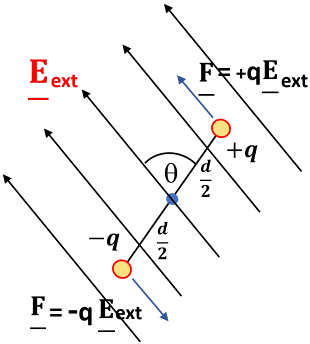
$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{pz}{4\pi\epsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}}\right) = \frac{p}{4\pi\epsilon_0}\frac{3zx}{r^5}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{p}{4\pi\epsilon_0}\frac{3zy}{r^5}$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) = \frac{p}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)$$

3.3 The torque on a dipole in an external E-field

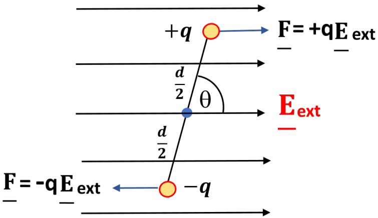
$$\underline{\mathbf{T}} = \underline{\mathbf{p}} \times \underline{\mathbf{E}}_{\mathrm{ext}}$$



3.4 The energy of a dipole in an external E-field

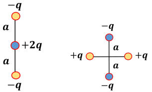
- Calculate the work done by an applied force to rotate the dipole from angle $\pi/2$ to θ

  (take $\theta = \pi/2$ as the zero of potential energy)

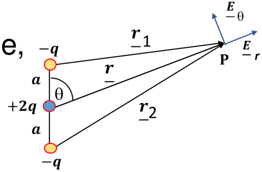


$$U = -p\,E_{ext}\cos\theta = -\underline{\mathbf{p}}\cdot\underline{\mathbf{E}}_{\mathrm{ext}}$$

3.5 The quadrupole potential



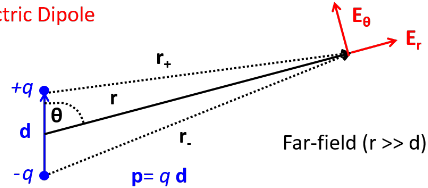
- Two charge configurations of the quadrupole, which both look identical at long distance



- Quadrupole potential : $V = \frac{q\, a^2}{4\pi\epsilon_0 r^3}(1 - 3\cos^2\theta)$

Summary: Electric dipole and quadrupole

Electric Dipole

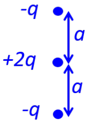


Far-field (r >> d) potential: $$V = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}$$

Far-field (r >> d) electric field: $$\mathbf{E}_r = \frac{2\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^4} \quad \mathbf{E}_\theta = \frac{qd\sin\theta}{4\pi\varepsilon_0 r^3} \quad \mathbf{E}_\phi = 0$$

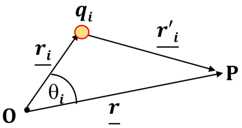
Energy of dipole in external electric field $\mathbf{E}_{ext}$: $$W = -\mathbf{E}_{\text{ext}}\cdot\mathbf{p}$$

Electric Quadrupole:



$$V = \frac{qa^2}{4\pi\varepsilon_0 r^3}(1 - 3\cos^2\theta)$$

3.6 The general multipole expansion



- Potential at P : $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\underline{\mathbf{r}}'_\mathbf{i}|}$

  where $\underline{\mathbf{r}}'_\mathbf{i} = \underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}$
- Cosine rule : $r'_i = \sqrt{r^2 + r_i^2 - 2\, r\, r_i \cos\theta_i}$

  $= r\sqrt{1 - 2\frac{r_i\cos\theta_i}{r} + \frac{r_i^2}{r^2}} \equiv r\sqrt{1+x}$
- For points P far from the charge assembly

  $r_i << r \;\; \rightarrow x << 1$
- Expand : $\frac{1}{r\sqrt{1+x}} = \frac{1}{r}\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right)$

  Retain only up to powers of $(r_i/r)^2$
- $\frac{1}{|\underline{\mathbf{r}}'_\mathbf{i}|} \approx \frac{1}{r}\left[1 + \frac{r_i\cos\theta_i}{r} - \frac{r_i^2}{2r^2} + \frac{3}{2}\frac{r_i^2}{r^2}\cos^2\theta_i + \cdots\right]$

  $$= \frac{1}{r} + \frac{r_i\cos\theta_i}{r^2} + \frac{r_i^2}{r^3}\frac{1}{2}\left(3\cos^2\theta_i - 1\right) + \cdots$$

The general multipole expansion

- $$V(\underline{\mathbf{r}}) = \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r}}_{\text{monopole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i r_i \cos\theta_i}{r^2}}_{\text{dipole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{\frac{1}{2} q_i r_i^2 \left(3\cos^2\theta_i - 1\right)}{r^3}}_{\text{quadrupole term}} + \cdots$$
- So any assembly of charges can be described in terms of the sum over contributions from multipoles
- The *n*-th multipole potential falls off with $1/r^n$