CP2 ELECTROMAGNETISM

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LECTURE 21:

ELECTROMAGNETIC WAVES & ENERGY FLOW

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 21. ELECTROMAGNETIC WAVES & ENERGY FLOW

21.1 Divergence, time derivative, and curl of $\underline{\mathbf{E}}$ *and* $\underline{\mathbf{B}}$

- The divergence of $\underline{\mathbf{E}}$: $\underline{\nabla} \cdot \underline{\mathbf{E}} = \underline{\nabla} \cdot \underline{\mathbf{E}}_0 \exp\left[i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right]$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \underline{\mathbf{E}}_0 \exp\left(i(\omega t - k_x x - k_y y - k_z z)\right)$$

$$= \left[(-i)\, k_x E_x + (-i)\, k_y E_y + (-i)\, k_z E_z\right] \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$$

$$= (-i)\, \underline{\mathbf{k}} \cdot \underline{\mathbf{E}} \quad : \quad \text{hence} \quad \boxed{\underline{\nabla} \equiv -i\, \underline{\mathbf{k}}}$$

- The time derivative of $\underline{\mathbf{E}}$: $\frac{\partial}{\partial t} \underline{\mathbf{E}} = \frac{\partial}{\partial t} \underline{\mathbf{E}}_0 \exp\left[i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right]$

$$= i\,\omega\, \underline{\mathbf{E}} \quad : \quad \text{hence} \quad \boxed{\frac{\partial}{\partial t} \equiv i\,\omega}$$

- The curl of $\underline{\mathbf{E}}$:

$$\underline{\nabla} \times \underline{\mathbf{E}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} =$$

$$(-i) \begin{pmatrix} k_y E_z - k_z E_y \\ k_z E_x - k_x E_z \\ k_x E_y - k_y E_x \end{pmatrix} = (-i)\, \underline{\mathbf{k}} \times \underline{\mathbf{E}} \quad \text{\& again} \quad \boxed{\underline{\nabla} \equiv -i\, \underline{\mathbf{k}}}$$

21.2 Electromagnetic waves : speed of propagation

- To get speed of propagation, substitute $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ into the wave equation $\underline{\nabla}^2 \underline{\mathbf{E}} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}$
- Use $\underline{\nabla} \equiv -i\, \underline{\mathbf{k}} \;\rightarrow\; \underline{\nabla}^2 \equiv (-i\, \underline{\mathbf{k}})^2 = -k^2$

  $$\frac{\partial}{\partial t} \equiv i\,\omega \;\rightarrow\; \frac{\partial^2}{\partial t^2} \equiv (i\,\omega)^2 = -\omega^2$$
- $-k^2\, \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right) = -\omega^2 \epsilon_0 \mu_0\, \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$

  $$\rightarrow \quad k^2 = \omega^2 \epsilon_0 \mu_0$$
- Fields of this form are solutions to the wave equation with velocity of propagation :

  $$c = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 \;\; \mathrm{m\,s^{-1}}$$

  i.e. the speed of light $\rightarrow$ speed of an EM wave in vacuum

21.3 Relationship between $\underline{\mathbf{E}}$ *and* $\underline{\mathbf{B}}$

- Substitute $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ into Maxwell eqn's :

  $\underline{\nabla} \cdot \underline{\mathbf{E}} = -i\,\underline{\mathbf{k}} \cdot \underline{\mathbf{E}} = 0$

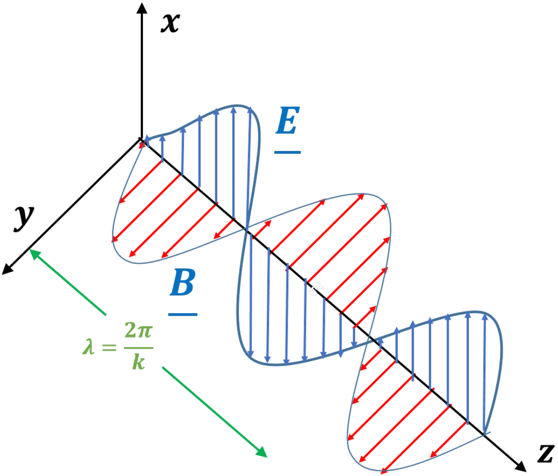
  $\underline{\nabla} \cdot \underline{\mathbf{B}} = -i\,\underline{\mathbf{k}} \cdot \underline{\mathbf{B}} = 0$

  Hence $\underline{\mathbf{k}} \cdot \underline{\mathbf{E}} = 0$ and $\underline{\mathbf{k}} \cdot \underline{\mathbf{B}} = 0$
- Electric and magnetic fields in vacuum are *perpendicular* to direction of propogation $\rightarrow$ *EM waves are transverse*
- Substitute into Faraday's Law : $\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$

  $-i\,\underline{\mathbf{k}} \times \underline{\mathbf{E}} = -i\,\omega\,\underline{\mathbf{B}} \quad \rightarrow \quad \boxed{\underline{\mathbf{B}} = \frac{1}{\omega}\underline{\mathbf{k}} \times \underline{\mathbf{E}}}$
- Substitute into Ampere's Law : $\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\,\epsilon_0\,\frac{\partial \underline{\mathbf{E}}}{\partial t}$

  $-i\,\underline{\mathbf{k}} \times \underline{\mathbf{B}} = i\,\omega\,\mu_0\,\epsilon_0\,\underline{\mathbf{E}} \quad \rightarrow \quad \boxed{\underline{\mathbf{E}} = -\frac{c^2}{\omega}\underline{\mathbf{k}} \times \underline{\mathbf{B}}}$
- $\underline{\mathbf{E}}$, $\underline{\mathbf{B}}$ & $\underline{\mathbf{k}}$ are mutually orthogonal (NB. $\underline{\mathbf{k}} \times \underline{\mathbf{B}} = kB\sin\frac{\pi}{2}\,\hat{\underline{\mathbf{E}}}$)
- $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are in phase and lie in the plane of the wavefront
- Field magnitude ratio : $\boxed{|\underline{\mathbf{E}}|/|\underline{\mathbf{B}}| = \frac{c^2}{\omega}k = c = \frac{1}{\sqrt{\mu_0\,\epsilon_0}}}$

21.4 Electromagnetic wave travelling along the z direction



$$\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \sin\left(\omega(t - z/c)\right)\ \underline{\hat{\mathbf{x}}}$$

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}_0 \sin\left(\omega(t - z/c)\right)\ \underline{\hat{\mathbf{y}}}$$

21.5 Characteristic impedance of free space

- Take the ratio $Z = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{H}}|}$ where $|\underline{\mathbf{H}}| = \frac{1}{\mu_0} |\underline{\mathbf{B}}|$

- Z has units $[V\, m^{-1}] / [A\, m^{-1}] =$ Ohms.

- Z is called the *characteristic impedance of free space*

$$Z = \mu_0 \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|} = \mu_0\, c = \frac{\mu_0}{\sqrt{\mu_0\, \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\, \Omega$$

Electromagnetic waves : summary

In vacuum, free of charge or currents (ρ, **J** = 0):

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \ddot{\mathbf{E}}$$

$$\nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \ddot{\mathbf{B}}$$

Wave equations in **E, B**!

Electromagnetic waves propagate in free space:

Plane EM wave fronts: $\mathbf{E} = \mathbf{E_0} \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}$ with wavelength $\lambda = \frac{2\pi}{k}$

Propagation velocity of wave fronts: $c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \,\mathrm{m\,s^{-1}}$

Relationship between E and B:
(in phase and mutually orthogonal
with wave vector **k**)

$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E} \qquad \mathbf{E} = -c^2 \frac{\mathbf{k}}{\omega} \times \mathbf{B} \qquad \frac{|\mathbf{E}|}{|\mathbf{B}|} = c$$

Impedance of free space: $Z = \frac{|\mathbf{E}|}{|\mathbf{B}|/\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\,\Omega$

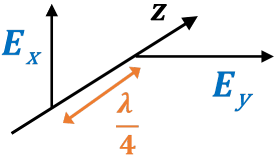
21.6 Polarisation

- Linearly (or plane) polarised wave :

  E has one specific orientation

- Circularly polarised wave :

  Two linear components of E superimposed at a right angle and phase shifted by $\pi/2$



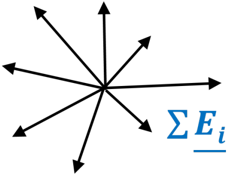
$$\begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} \sin(\omega t) + \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \sin(\omega t + \pi/2) = E \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}$$

- Elliptically polarised wave :

  As above but with unequal amplitudes

- Unpolarised :

  E superimposed with all orientations (with no fixed phase relationships between components)



21.7 Energy flow and the Poynting Vector

- Recall : Energy of the electric field $U_e = \int_{\mathcal{V}} \frac{1}{2}\epsilon_0 \underline{\mathbf{E}}^2 \, d\mathcal{V}$

  Energy of the magnetic field $U_m = \int_{\mathcal{V}} \frac{1}{2\mu_0} \underline{\mathbf{B}}^2 \, d\mathcal{V}$
- Total EM energy in volume $\mathcal{V}$:

$$U = \int_{\mathcal{V}} \underbrace{\frac{1}{2}\left(\epsilon_0 \underline{\mathbf{E}} \cdot \underline{\mathbf{E}} + \frac{1}{\mu_0} \underline{\mathbf{B}} \cdot \underline{\mathbf{B}}\right)}_{\text{Energy density}} d\mathcal{V}$$

- In free space ($\underline{\mathbf{J}} = 0, \rho = 0$)

$$\underbrace{\frac{\partial \underline{\mathbf{E}}}{\partial t} = \frac{1}{\mu_0 \epsilon_0} \underline{\nabla} \times \underline{\mathbf{B}}}_{\text{Ampere's Law}} \quad ; \quad \underbrace{\frac{\partial \underline{\mathbf{B}}}{\partial t} = -\underline{\nabla} \times \underline{\mathbf{E}}}_{\text{Faraday's Law}}$$

- Calculate the rate of change of energy in $\mathcal{V}$:

$$\begin{aligned} \frac{dU}{dt} &= \int_{\mathcal{V}} \left(\epsilon_0 \underline{\mathbf{E}} \cdot \underline{\dot{\mathbf{E}}} + \frac{1}{\mu_0} \underline{\mathbf{B}} \cdot \underline{\dot{\mathbf{B}}}\right) d\mathcal{V} \\ &= \int_{\mathcal{V}} \left(\frac{\epsilon_0}{\mu_0 \epsilon_0} (\underline{\mathbf{E}} \cdot \underline{\nabla} \times \underline{\mathbf{B}}) - \frac{1}{\mu_0} \underline{\mathbf{B}} \cdot (\underline{\nabla} \times \underline{\mathbf{E}})\right) d\mathcal{V} \\ &= -\frac{1}{\mu_0} \int_{\mathcal{V}} \underline{\nabla} \cdot (\underline{\mathbf{E}} \times \underline{\mathbf{B}}) \, d\mathcal{V} \end{aligned}$$

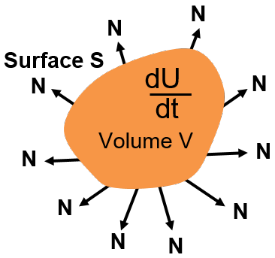
Energy flow and the Poynting Vector continued

- Energy flow out of volume $\mathcal{V}$ per unit time :

$$\frac{dU}{dt} = -\frac{1}{\mu_0}\int_{\mathcal{V}} \underline{\nabla}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}})\,d\mathcal{V}$$

- Apply the divergence theorem :

$$\frac{dU}{dt} = -\oint_S \underbrace{\left(\frac{1}{\mu_0}\underline{\mathbf{E}}\times\underline{\mathbf{B}}\right)}_{\text{Poynting Vector, }\underline{\mathbf{N}}}\cdot d\underline{\mathbf{a}}$$



$$\frac{dU}{dt} = -\oint_S \underline{\mathbf{N}}\cdot\,d\underline{\mathbf{a}}$$ where $$\underline{\mathbf{N}} = \frac{1}{\mu_0}\underline{\mathbf{E}}\times\underline{\mathbf{B}}$$

Poynting vector $\underline{\mathbf{N}}$ is the power per unit area flowing through the surface bounded by volume $\mathcal{V}$. (It also gives the direction of flow). Units of $\underline{\mathbf{N}}$: $[W\,m^{-2}]$

- For EM waves, the intensity is the time-average of $|\underline{\mathbf{N}}|$

$$\Im = <|\underline{\mathbf{N}}|> = \frac{1}{\mu_0}E_0\,B_0\underbrace{<\cos^2(\omega t-\underline{\mathbf{k}}\cdot\underline{\mathbf{r}})>}_{1/2} = \frac{1}{2\,\mu_0\,c}\,E_0^2$$

21.8 Example : Poynting Vector for a long resistive cylinder

- Calculate Poynting Vector at the surface of the wire with applied potential difference V and current I

$\underline{\mathbf{N}} = \frac{1}{\mu_0}\underline{\mathbf{E}} \times \underline{\mathbf{B}}$

- Electric field along wire axis : $E = V/\ell$

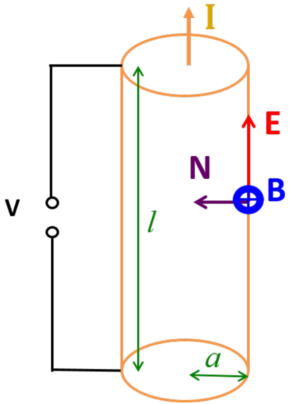
  Magnetic flux density at wire surface :
  $\oint \underline{\mathbf{B}} \cdot d\underline{\ell} = B \cdot 2\pi a = \mu_0 I$

  (note that this is *tangential* - along circumference)

- $N = \frac{1}{\mu_0}\frac{V}{\ell}\frac{\mu_0 I}{2\pi a}$

  (in radial direction pointing *inwards* - i.e. wire heats up !)

- Hence $N = (V\,I)/\underbrace{2\pi\ell a}_{\text{surface area}}$



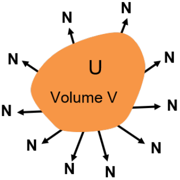
- Total power dissipated in wire : $P = \int_S \underline{\mathbf{N}} \cdot d\underline{\mathbf{a}} = V\,I$ as expected from circuit theory.

Poynting Vector : summary

Total electromagnetic energy U contained in volume V:

$$U = \int_V \frac{1}{2}\left(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \mathrm{d}V$$

energy density $U_V = \dfrac{dU}{dV}$



$$\frac{dU}{dt} = -\oint_S \mathbf{N}\,.\,\mathbf{da} \quad \text{with} \quad \mathbf{N} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

Poynting vector

Energy flow rate out of volume V

Power per unit area through area bounding V

$[N]=W/m^2$

The intensity I of an EM wave is given by the time-average over the magnitude of the Poynting vector:

$$I = \langle|\mathbf{N}|\rangle = \frac{1}{2c\mu_0}E_0{}^2$$