CP2 ELECTROMAGNETISM

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LECTURE 21:

ELECTROMAGNETIC WAVES & ENERGY FLOW

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 21. ELECTROMAGNETIC WAVES & ENERGY FLOW

21.1 Divergence, time derivative, and curl of $\underline{\mathbf{E}}$ *and* $\underline{\mathbf{B}}$

21.2 Electromagnetic waves : speed of propagation

21.3 Relationship between $\underline{\mathbf{E}}$ *and* $\underline{\mathbf{B}}$

21.4 Characteristic impedance of free space

21.5 Polarisation

21.6 Energy flow and the Poynting Vector

21.7 Example : Poynting Vector for a long resistive cylinder

21.1 Divergence, time derivative, and curl of $\underline{\mathbf{E}}$ *and* $\underline{\mathbf{B}}$

21.2 Electromagnetic waves : speed of propagation

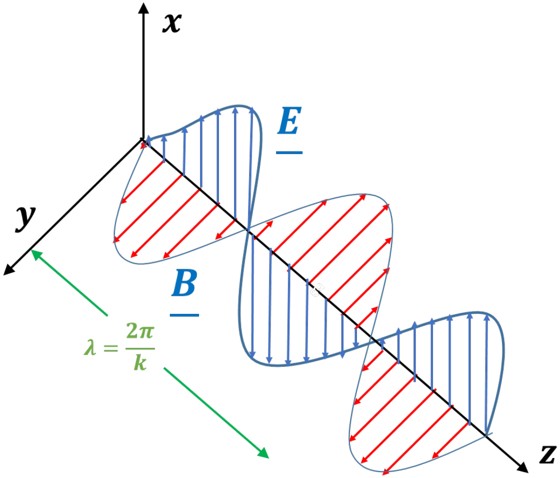
- To get speed of propagation, substitute $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ into the wave equation $\underline{\nabla}^2 \underline{\mathbf{E}} = \epsilon_0 \, \mu_0 \frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}$

21.3 Relationship between $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$

- Electric and magnetic fields in vacuum are *perpendicular* to direction of propogation $\rightarrow$ *EM waves are transverse*

- $\underline{\mathbf{E}}$, $\underline{\mathbf{B}}$ & $\underline{\mathbf{k}}$ are mutually orthogonal (NB. $\underline{\mathbf{k}} \times \underline{\mathbf{B}} = kB \sin\frac{\pi}{2}\,\hat{\underline{\mathbf{E}}}$)
- $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are in phase and lie in the plane of the wavefront
- Field magnitude ratio : $$|\underline{\mathbf{E}}|/|\underline{\mathbf{B}}| = \frac{c^2}{\omega}k = c = \frac{1}{\sqrt{\mu_0\,\epsilon_0}}$$

Electromagnetic wave travelling along the z direction



$$\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \sin\left(\omega(t - z/c)\right) \, \underline{\hat{\mathbf{x}}}$$

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}_0 \sin\left(\omega(t - z/c)\right) \, \underline{\hat{\mathbf{y}}}$$

21.4 Characteristic impedance of free space

$$Z = \mu_0 \frac{|\mathbf{E}|}{|\mathbf{B}|} = \mu_0\, c = \frac{\mu_0}{\sqrt{\mu_0\, \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\,\Omega$$

Electromagnetic waves : summary

In vacuum, free of charge or currents (ρ, **J** = 0):

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

→

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \ddot{\mathbf{E}}$$

$$\nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \ddot{\mathbf{B}}$$

Wave equations in **E, B**!

Electromagnetic waves propagate in free space:

Plane EM wave fronts: $\mathbf{E} = \mathbf{E_0} \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}$ with wavelength $\lambda = \frac{2\pi}{k}$

Propagation velocity of wave fronts: $c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8\,\mathrm{m\,s^{-1}}$

Relationship between E and B:
(in phase and mutually orthogonal with wave vector **k**) $\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E}$ $\quad \mathbf{E} = -c^2 \frac{\mathbf{k}}{\omega} \times \mathbf{B}$ $\quad \frac{|\mathbf{E}|}{|\mathbf{B}|} = c$

Impedance of free space: $Z = \frac{|\mathbf{E}|}{|\mathbf{B}|/\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\,\Omega$

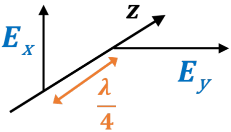
21.5 Polarisation

- Linearly (or plane) polarised wave :

  E has one specific orientation

- Circularly polarised wave :

  Two linear components of *E* superimposed at a right angle and phase shifted by $\pi/2$

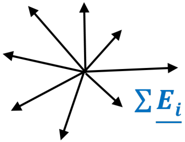


- Elliptically polarised wave :

  As above but with unequal amplitudes

- Unpolarised :

  E superimposed with all orientations (with no fixed phase relationships between components)

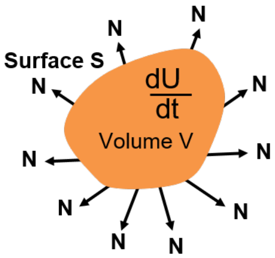


21.6 Energy flow and the Poynting Vector

- Recall : Energy of the electric field $U_e = \int_{\mathcal{V}} \frac{1}{2}\,\epsilon_0\,\underline{\mathbf{E}}^2\,d\mathcal{V}$

  Energy of the magnetic field $U_m = \int_{\mathcal{V}} \frac{1}{2\,\mu_0}\,\underline{\mathbf{B}}^2\,d\mathcal{V}$

Energy flow and the Poynting Vector continued



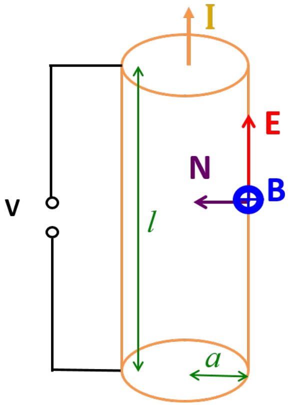
Poynting vector $\underline{\mathbf{N}}$ is the power per unit area flowing through the surface bounded by volume $\mathcal{V}$. (It also gives the direction of flow). Units of $\underline{\mathbf{N}}$: $[W\, m^{-2}]$

- For EM waves, the intensity is the time-average of $|\underline{\mathbf{N}}|$

$$\Im = <|\underline{\mathbf{N}}|> = \frac{1}{\mu_0} E_0\, B_0 \underbrace{< \cos^2(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}}) >}_{1/2} = \frac{1}{2\,\mu_0\, c}\, E_0^2$$

21.7 Example : Poynting Vector for a long resistive cylinder

- Calculate Poynting Vector at the surface of the wire with applied potential difference V and current I



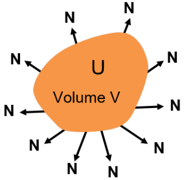
- Total power dissipated in wire : $P = \int_S \underline{\mathbf{N}} \cdot d\underline{\mathbf{a}} = V\, I$ as expected from circuit theory.

Poynting Vector : summary

Total electromagnetic energy *U* contained in volume *V*:

$$U = \int_V \frac{1}{2}\left(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \mathrm{d}V$$

energy density $U_V = \frac{dU}{dV}$



$$\frac{dU}{dt} = -\oint_S \mathbf{N}\,.\,\mathbf{da} \quad \text{with} \quad \mathbf{N} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

Poynting vector

Energy flow rate out of volume V

Power per unit area through area bounding V

$[N]=W/m^2$

The intensity *I* of an EM wave is given by the time-average over the magnitude of the Poynting vector:

$$I = \langle|\mathbf{N}|\rangle = \frac{1}{2c\mu_0}E_0^{\,2}$$