CP2 ELECTROMAGNETISM

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LECTURE 20:

MAXWELL'S EQUATIONS & ELECTROMAGNETIC WAVES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 20. MAXWELL'S EQUATIONS & ELECTROMAGNETIC WAVES

20.1 Example : Ampere's Law and a charging capacitor

20.2 Example : B-field of a short current-carrying wire

20.3 Electromagnetic waves in vacuum

20.4 Electromagnetic waves : 3D plane wave solutions

Summary : Ampere's Law

Ampere's law does not comply with the Equation of Continuity:

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply div: $\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{=0 \text{ always}} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{= -\frac{\partial \rho}{\partial t}}$

$= 0$ only for statics!

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\underbrace{\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{\text{displacement current } \mathbf{J_D}})$$

Obtain **Ampere's law with "displacement current"**:

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

Using Stokes theorem: $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot d\underline{a}$

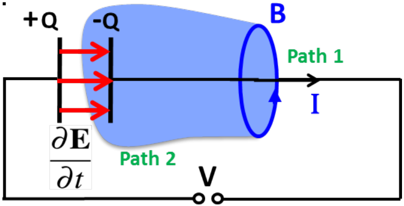
Gives integral form : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \underbrace{\mu_0 \int_S \underline{\mathbf{J}} \cdot d\underline{a}}_{\mu_0 I_{encl.}} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{a}$

20.1 Example : Ampere's Law and a charging capacitor

- This is the first example showing why Ampere's Law fails without adding the displacement current : a straight wire, and add a capacitor into the circuit
- Before we used Ampere's Law to calculate magnetic field along Amperian loop $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 \, I_{encl.}$
- But there is not one unique path :

  (i) Path 1: the smallest area (plane surface) $\rightarrow \; I_{encl.} = I$

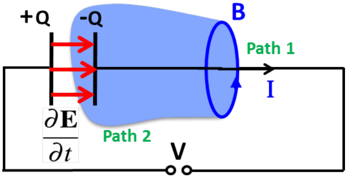
  (ii) Path 2: via a "bulged" surface that passes between the capacitor plates $\rightarrow \; I_{encl.} = 0$



- The $\underline{\mathbf{B}}$ field has to be the same no matter which path we choose
- The issue is that the $\underline{\mathbf{E}}$ field is changing in the capacitor !

A charging capacitor and Ampere's Law, continued

- Gauss Law for a parallel plate capacitor : $E = \frac{Q}{\epsilon_0 A}$



- $\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A} \frac{\partial Q}{\partial t} = \frac{1}{\epsilon_0 A} I$
- Add $I_D = \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\underline{\mathbf{a}}$ to Ampere's Law

  - $$\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \underbrace{\mu_0 I_{encl.}}_{\text{Term 1}} + \underbrace{\mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}}_{\text{Term 2}}$$
  - For the surface around the wire :

    Term 1 = $\mu_0 I$, Term 2 = 0
  - For the surface around the capacitor

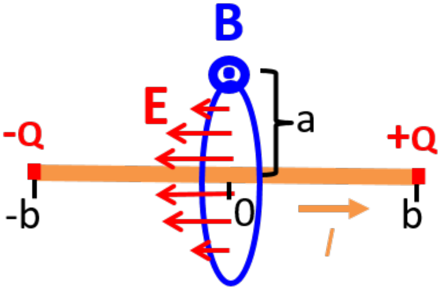
    Term 1 = 0 , Term 2 = $\mu_0 \epsilon_0 \times \frac{1}{\epsilon_0 A} I \times A = \mu_0 I$

    $\rightarrow$ RHS = $\mu_0 I$, regardless of choice of path ✓✓

  In differential form : $$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t} \right)$$

20.2 Example : B-field of a short current-carrying wire

- Recall B-field from Biot-Savart Law $\rightarrow$ $B = \frac{\mu_0 I}{2\pi a} \frac{b}{\sqrt{b^2+a^2}}$
- Again, Ampere's law fails depending on what path we use. Need to use displacement current.
- $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 I_{encl.} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$
- Wire is short, so charge builds up at the ends giving time-varying $\underline{\mathbf{E}}$-field

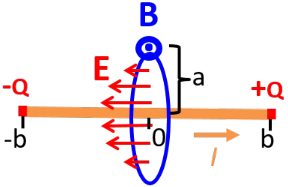


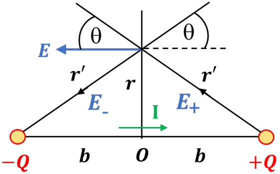
B-field of a short current-carrying wire, continued

- Integrate $\frac{\partial E}{\partial t}$ over area, radius a
- Calculate $\mathbf{E}$-field due to two point charges at wire ends, $\pm b$

$$E(r) = -\underbrace{\frac{2Q/(4\pi\epsilon_0)}{(r^2+b^2)}}_{r'^2}\ \underbrace{\frac{b}{\sqrt{r^2+b^2}}}_{\cos\theta}$$

(2 field components E_+ and E_-, and note I_D and I have opposite signs)





- $I_D = \epsilon_0 \int_0^a \frac{\partial E(r)}{\partial t}\, 2\pi\, r\, dr = \epsilon_0 \frac{\partial Q}{\partial t} \int_0^a -\frac{b/(2\pi\epsilon_0)}{(r^2+b^2)^{\frac{3}{2}}}\, 2\pi\, r\, dr$
- $I_D = \frac{\partial Q}{\partial t} \left[\frac{b}{\sqrt{(r^2+b^2)}} \right]_{r=0}^{r=a} = I \left[\frac{b}{\sqrt{(a^2+b^2)}} - 1 \right]$
- $\oint_C \mathbf{B} \cdot d\underline{\ell} = B \cdot 2\pi\, a = \mu_0\, I + \mu_0\, I \left[\frac{b}{\sqrt{(a^2+b^2)}} - 1 \right]$
- So : $B = \frac{\mu_0\, I}{2\pi\, a} \frac{b}{\sqrt{b^2+a^2}}$ as from Biot-Savart Law ✓✓

Summary of Maxwell's Equations

$$\oint_S \mathbf{E} \, . \, \mathbf{da} = \frac{Q}{\varepsilon_0} \quad \leftrightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss' law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint \mathbf{E} \, . \, \mathbf{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{B} \, . \, \mathbf{da}$$

$$\leftrightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law: time-varying magnetic fields create electric fields (induction).

$$\oint_S \boldsymbol{B} \, . \, \boldsymbol{da} = 0 \quad \leftrightarrow \quad \nabla \cdot \mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint \mathbf{B} \, . \, \mathbf{d}\boldsymbol{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{E} \, . \, \mathbf{da}$$

$$\leftrightarrow \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Ampere's law including displacement current: electric currents and time-varying electric fields generate magnetic fields.

20.3 Electromagnetic waves in vacuum

- In the absence of electric charge or current
  $\rightarrow \quad \rho = 0$ and $\underline{\mathbf{J}} = 0$:
- Maxwell's Equations become :

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \epsilon_0\, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

(note the symmetry between the $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields)

- Apply curl to Faraday's law :

$$\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial}{\partial t} \underline{\nabla} \times \underline{\mathbf{B}} = -\mu_0\, \epsilon_0\, \frac{\partial^2}{\partial t^2}\, \underline{\mathbf{E}}$$

- Use the vector identity : $\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{E}} = \underline{\nabla}\, \underbrace{(\underline{\nabla} \cdot \underline{\mathbf{E}})}_{= 0} - \underline{\nabla}^2\, \underline{\mathbf{E}}$

- This gives us a *wave equation* in $\underline{\mathbf{E}}$:

$$\boxed{\underline{\nabla}^2\, \underline{\mathbf{E}} - \epsilon_0\, \mu_0\, \ddot{\underline{\mathbf{E}}} = 0}$$

Electromagnetic waves in vacuum, continued

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \epsilon_0 \, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

- Apply curl to Ampere's law :

  $$\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{B}} = \epsilon_0 \, \mu_0 \, \frac{\partial}{\partial t} \underline{\nabla} \times \underline{\mathbf{E}} = -\mu_0 \, \epsilon_0 \, \frac{\partial^2}{\partial t^2} \underline{\mathbf{B}}$$

- Use the vector identity : $\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\nabla} \underbrace{(\underline{\nabla} \cdot \underline{\mathbf{B}})}_{=0} - \underline{\nabla}^2 \, \underline{\mathbf{B}}$
- This gives us a *wave equation* in $\underline{\mathbf{B}}$:

  $$\underline{\nabla}^2 \, \underline{\mathbf{B}} - \epsilon_0 \, \mu_0 \, \ddot{\underline{\mathbf{B}}} = 0$$

  together with : $$\underline{\nabla}^2 \, \underline{\mathbf{E}} - \epsilon_0 \, \mu_0 \, \ddot{\underline{\mathbf{E}}} = 0$$

- These equations have general solutions (in 1D) of the form:
- $E(x, t) = F(x - ct) + G(x + ct)$ and
  $B(x, t) = F'(x - ct) + G'(x + ct)$

  where F, G, F', G' are *any* functions of $(x - ct), (x + ct)$

20.4 Electromagnetic waves : 3D plane wave solutions

- Consider the simplest form of solution : 3D plane waves of the form

  $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ and

  $\underline{\mathbf{B}} = \underline{\mathbf{B}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$

  Real part : $Re[\underline{\mathbf{E}}] = \underline{\mathbf{E}}_0 \cos\underbrace{(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})}_{\text{phase}}$

- $\underline{\mathbf{k}}$ is in the direction normal to the wave-fronts
- All points P form a wave-front with the same phase
- Maxima are separated by the wavelength λ where $\lambda = 2\pi/k$
- Phase velocity (or propagation velocity) of wave-fronts given by $c = \omega/k$

Plane waves