CP2 ELECTROMAGNETISM

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LECTURE 20:

MAXWELL'S EQUATIONS & ELECTROMAGNETIC WAVES

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 20. MAXWELL'S EQUATIONS & ELECTROMAGNETIC WAVES

20.1 Example : Ampere's Law and a charging capacitor

20.2 Example : B-field of a short current-carrying wire

20.3 Electromagnetic waves in vacuum

20.4 Electromagnetic waves : 3D plane wave solutions

Summary : Ampere's Law

Ampere's law does not comply with the Equation of Continuity:

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply div: $\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{=0 \text{ always}} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{= -\frac{\partial \rho}{\partial t}}$

$= 0$ only for statics!

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\underbrace{\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{\text{displacement current } \mathbf{J}_D})$$

Obtain **Ampere's law with "displacement current"**:

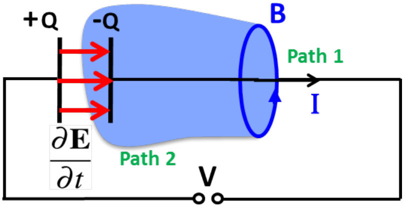
$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

Using Stokes theorem: $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot d\underline{a}$

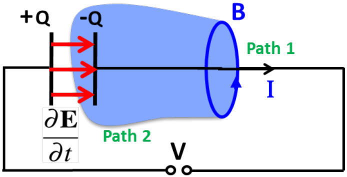
Gives integral form : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 \underbrace{\int_S \underline{\mathbf{J}} \cdot d\underline{a}}_{\mu_0 I_{encl.}} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{a}$

20.1 Example : Ampere's Law and a charging capacitor

- This is the first example showing why Ampere's Law fails without adding the displacement current : a straight wire, and add a capacitor into the circuit
- Before we used Ampere's Law to calculate magnetic field along Amperian loop $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I_{encl}$.



A charging capacitor and Ampere's Law, continued

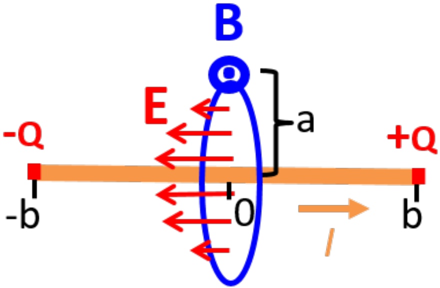


In differential form :

$$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t} \right)$$

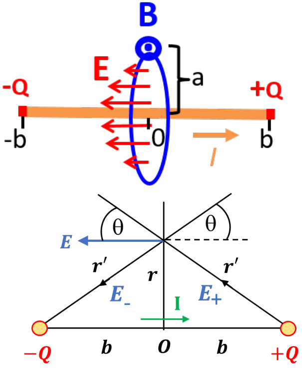
20.2 Example : B-field of a short current-carrying wire

- Recall B-field from Biot-Savart Law $\rightarrow$ $B = \frac{\mu_0 I}{2\pi a} \frac{b}{\sqrt{b^2+a^2}}$
- Again, Ampere's law fails depending on what path we use. Need to use displacement current.
- $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 I_{encl.} + \mu_0\epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$
- Wire is short, so charge builds up at the ends giving time-varying $\underline{\mathbf{E}}$-field



B-field of a short current-carrying wire, continued

- Integrate $\frac{\partial E}{\partial t}$ over area, radius a
- Calculate $\underline{\mathbf{E}}$-field due to two point charges at wire ends, $\pm b$



Summary of Maxwell's Equations

$$\oint_S \mathbf{E} \,.\, \mathbf{da} = \frac{Q}{\varepsilon_0} \quad \leftrightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss' law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint \mathbf{E} \,.\, \boldsymbol{dl} = -\frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{B} \,.\, \mathbf{da} \quad \leftrightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law: time-varying magnetic fields create electric fields (induction).

$$\oint_S \boldsymbol{B} \,.\, \boldsymbol{da} = 0 \quad \leftrightarrow \quad \nabla \cdot \mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint \mathbf{B} \,.\, \mathbf{d}\boldsymbol{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{E} \,.\, \mathbf{da} \quad \leftrightarrow \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Ampere's law including displacement current: electric currents and time-varying electric fields generate magnetic fields.

20.3 Electromagnetic waves in vacuum

- In the absence of electric charge or current $\rightarrow \quad \rho = 0$ and $\underline{\mathbf{J}} = 0$:
- Maxwell's Equations become :

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \epsilon_0 \, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

(note the symmetry between the $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields)

Electromagnetic waves in vacuum, continued

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \epsilon_0 \, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

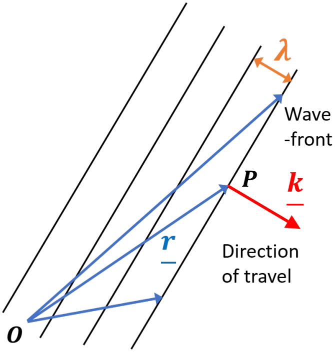
These equations have general solutions (in 1D) of the form:

$E\,(x,t) = F\,(x - ct) + G\,(x + ct)$ and
$B\,(x,t) = F'\,(x - ct) + G'\,(x + ct)$

where F, G, F', G' are *any* functions of $(x - ct), (x + ct)$

20.4 Electromagnetic waves : 3D plane wave solutions

Plane waves



- $\underline{\mathbf{k}}$ is in the direction normal to the wave-fronts
- All points P form a wave-front with the same phase
- Maxima are separated by the wavelength λ where $\lambda = 2\pi/k$
- Phase velocity (or propagation velocity) of wave-fronts given by $c = \omega/k$