CP2 ELECTROMAGNETISM

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LECTURE 2:

ELECTRIC FIELD AND POTENTIAL

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

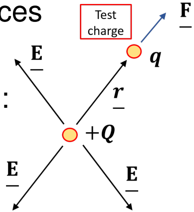
[1] With thanks to Prof Laura Herz

OUTLINE : 2. THE ELECTRIC FIELD AND POTENTIAL

2.1 The Electric Field

- The electric field at point $\underline{\mathbf{r}}$, generated by a distribution of charges q_i is defined as the force per unit charge that a test charge would experience if placed at $\underline{\mathbf{r}}$.

  $\rightarrow$ a point test charge q due to a field $\underline{\mathbf{E}}$ experiences a force $\underline{\mathbf{F}} = q \cdot \underline{\mathbf{E}} = \frac{q \cdot Q}{4\pi\epsilon_0 r^2}\,\underline{\hat{\mathbf{r}}}$



- Electric field due to a point charge Q at the origin: always points away from + charge (radial)

  $\underline{\mathbf{E}} = \underline{\mathbf{F}}/q = \frac{Q}{4\pi\epsilon_0 r^2}\,\underline{\hat{\mathbf{r}}}$

- The principle of superposition holds for the electric field : the electric field generated by a distribution of charges is equal to the sum of the electric fields generated by the individual charges.

2.2 The Electrostatic Potential

- Work done to move a point test charge q from A to B

  $W_{AB} = -\int_{\underline{\mathbf{r}}_\mathbf{A}}^{\underline{\mathbf{r}}_\mathbf{B}} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}}\ell = -q \int_{\underline{\mathbf{r}}_\mathbf{A}}^{\underline{\mathbf{r}}_\mathbf{B}} \underline{\mathbf{E}} \cdot \underline{\mathbf{d}}\ell$
- The electrostatic potential difference between A and B is defined as the the work done to move a unit charge between A and B
- Since the work done is related linearly to the electric field, the principle of superposition also holds for the work done.
- Note that *any* field configuration can be made up of the sum of infinitesimal point charges. So we can now check that the work done by moving a test charge through the electric field of a *point* charge does not depend on the path taken.

The Electrostatic Potential

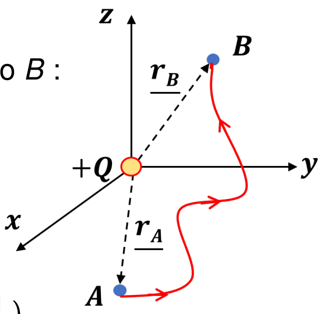
For a point charge charge Q :

- $\underline{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\underline{\hat{\mathbf{r}}}$
- Work done to move charge q from A to B :

  $W_{AB} = -q\int_A^B \underline{\mathbf{E}}\cdot\underline{\mathbf{d}}\ell$
- In spherical coordinates :

  $\underline{\mathbf{d}}\ell = dr\,\underline{\hat{\mathbf{r}}} + r\,d\theta\underline{\hat{\theta}} + r\sin\theta d\phi\underline{\hat{\phi}}$
- Hence $\underline{\mathbf{E}}\cdot\underline{\mathbf{d}}\ell = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}dr$

  $W_{AB} = -q\int_A^B \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = q\frac{Q}{4\pi\epsilon_0}(\frac{1}{r_B} - \frac{1}{r_A})$



- Hence energy required to move test charge from A to B depends only on initial and final radial separation, and independent of path.
- Electric field is conservative

2.3 The Potential Difference

Define electrostatic potential difference

$$V_{AB} = \frac{W_{AB}}{q} = -\int_A^B \underline{\mathbf{E}} \cdot \underline{\mathbf{d}}\ell = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

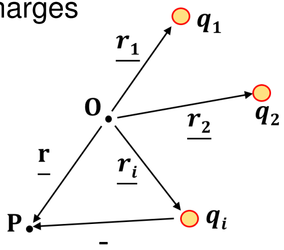
- The potential of a point charge Q at a *general* point $\underline{\mathbf{r}}$ is given by : $V(\underline{\mathbf{r}}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r_A}\right)$
  here the second term is a constant (which is often set to zero by taking $V(r \to \infty) = 0$)
- Again, since $\underline{\mathbf{E}}$ and V are linearly related, the Principle of Superposition also holds for V.

Potential at point P due to an assembly of charges

- $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}|} +$ constant

The field due to the assembly :

- $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{(\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i})^2}\widehat{\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}}$
  where $\widehat{\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}} = \frac{\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}}{|\underline{\mathbf{r}} - \underline{\mathbf{r}}_\mathbf{i}|}$



Summary of Relationship between Electric Field and Potential

The electric field **E** at a point **r**, generated by a distribution of charges q_i, is equal to the force **F** per unit charge q that a small test charge q would experience if it was placed at **r**:

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q}$$

The electric potential V at a point **r** is the energy W required per unit charge q to move a small test charge q from a reference point to **r**. For a system of charges:

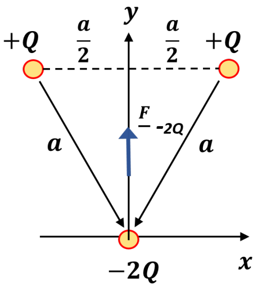
$$V(\mathbf{r}) = \frac{W(\mathbf{r})}{q}$$

The electric field and potential are related through:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \quad \longleftrightarrow \quad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

2.4 Calculating the field from the potential

Revisit example 1 : Calculate the electric field generated by charges $+Q$ and $+Q$, located on an equilateral triangle, and felt by "test charge" $-2Q$ at the origin.



- Reminder from before, force on $-2Q$

$$\underline{\mathbf{F}}_{-2\mathbf{Q}} = \frac{\sqrt{3}Q^2}{2\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- Field $\underline{\mathbf{E}}$ at $\underline{\mathbf{r}} = (0,0,0)$ = force/unit charge

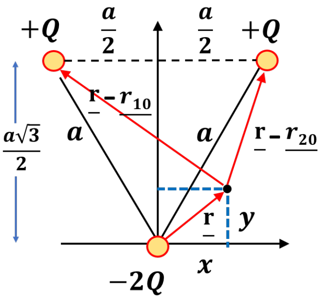
$$\underline{\mathbf{E}} = \frac{1}{-2Q}\frac{\sqrt{3}Q^2}{2\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\frac{\sqrt{3}Q}{4\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- But now adopt a different approach : derive the electric field from $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = -\underline{\nabla} V$ and evaluate $\underline{\mathbf{E}}$ at the origin.

1. Calculate the potential due to Q and Q at a position $\underline{\mathbf{r}}$

- $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_{10}|} + \frac{Q}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_{20}|}\right)$

$$= \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{\left[(a/2+x)^2+(a\sqrt{3}/2-y)^2+z^2\right]^{1/2}} + \frac{1}{\left[(a/2-x)^2+(a\sqrt{3}/2-y)^2+z^2\right]^{1/2}}\right\}$$



2. Derive $\underline{\mathbf{E}}(\underline{\mathbf{r}})$ from $V(\underline{\mathbf{r}})$:

$$\underline{\mathbf{E}} = -\underline{\nabla} V = -\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} V$$

- $\underline{\mathbf{E}} = -\frac{Q}{4\pi\epsilon_0}\times$

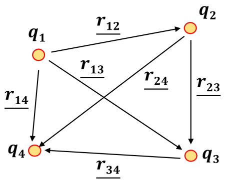
$$\left\{\frac{-1/2}{\left((a/2+x)^2+(a\sqrt{3}/2-y)^2+z^2\right)^{3/2}}\begin{pmatrix} 2(a/2+x) \\ -2(a\sqrt{3}/2-y) \\ 2z \end{pmatrix} + \frac{-1/2}{\left((a/2-x)^2+(a\sqrt{3}/2-y)^2+z^2\right)^{3/2}}\begin{pmatrix} -2(a/2-x) \\ -2(a\sqrt{3}/2-y) \\ 2z \end{pmatrix}\right\}$$

3. Calculate the field at position $\underline{\mathbf{r}} = (0, 0, 0)$

- $$\underline{\mathbf{E}} = +\frac{Q}{4\pi\epsilon_0} \times \left\{ \frac{1}{(a^2/4+3a^2/4)^{3/2}} \begin{pmatrix} a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} + \frac{1}{(a^2/4+3a^2/4)^{3/2}} \begin{pmatrix} -a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} \right\}$$
  $$= \frac{Q}{4\pi\epsilon_0 a^3} \left\{ \begin{pmatrix} a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} \right\}$$
- Hence $\underline{\mathbf{E}} = -\frac{Q\sqrt{3}}{4\pi\epsilon_0 a^2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
  Which is identical to the previous calculation using vector sum over fields.

2.5 Energy of a system of charges

- Calculate the energy to bring i charges up from infinity whilst keeping all the other charges fixed in space



$U =$ the first charge q_1 : none

$+$ the second charge q_2 : $q_2 \left(\frac{q_1}{4\pi\epsilon_0 r_{12}} \right)$

$+$ the third charge q_3 : $q_3 \left(\frac{q_1}{4\pi\epsilon_0 r_{13}} + \frac{q_2}{4\pi\epsilon_0 r_{23}} \right)$

$+$ the fourth charge q_4 : $q_4 \left(\frac{q_1}{4\pi\epsilon_0 r_{14}} + \frac{q_2}{4\pi\epsilon_0 r_{24}} + \frac{q_3}{4\pi\epsilon_0 r_{34}} \right)$

- $+$ etc, up to the i^{th} charge
- Compare to W, the sum over potential energies experienced by *each* charge from *all other* charges:

$$W = \sum_i q_i \sum_{j(\neq i)} \frac{q_j}{4\pi\epsilon_0 r_{ij}}$$

In matrix form :

- $U =$

$$(q_1\ q_2\ q_3 \cdots q_{i-1} q_i) \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1/r_{12} & 0 & \cdots & 0 & 0 \\ 1/r_{13} & 1/r_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/r_{1,i-1} & 1/r_{2,i-1} & \cdots & 0 & 0 \\ 1/r_{1i} & 1/r_{2i} & \cdots & 1/r_{i-1,i} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \cdots \\ q_{i-1} \\ q_i \end{pmatrix}$$

and where

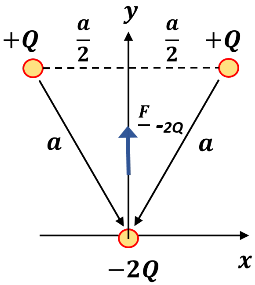
$W =$

$$(q_1\ q_2\ q_3 \cdots q_{i-1} q_i) \begin{pmatrix} 0 & 1/r_{21} & \cdots & 1/r_{i-1,1} & 1/r_{i1} \\ 1/r_{12} & 0 & \cdots & 1/r_{i-1,2} & 1/r_{i2} \\ 1/r_{13} & 1/r_{23} & \cdots & 1/r_{i-1,3} & 1/r_{i3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/r_{1,i-1} & 1/r_{2,i-1} & \cdots & 0 & 1/r_{i,i-1} \\ 1/r_{1i} & 1/r_{2i} & \cdots & 1/r_{i-1,i} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \cdots \\ q_{i-1} \\ q_i \end{pmatrix}$$

- Hence $U = \frac{1}{2} W = \sum_i \frac{1}{2} q_i V_i$ where $V_i = \sum_{j(\neq i)} \frac{q_j}{4\pi\epsilon_0 r_{ij}}$
- The energy U required to assemble a system of charges from infinity (whilst keeping all charges fixed in space) is half the energy of the sum W over potential energies experienced by each charge from all other charges.

Energy to assemble the system in Example 1

1. Charge $-2Q$ in potential of Q & Q
   $q_1 V_1 = -2Q\left(\frac{Q}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0 a}\right) = -\frac{Q^2}{\pi\epsilon_0 a}$
2. Charge Q in potential of Q & $-2Q$
   $q_2 V_2 = Q\left(\frac{Q}{4\pi\epsilon_0 a} + \frac{-2Q}{4\pi\epsilon_0 a}\right) = -\frac{Q^2}{4\pi\epsilon_0 a}$
3. Charge Q in potential of $-2Q$ & Q
   $q_2 V_2 = q_3 V_3 = -\frac{Q^2}{4\pi\epsilon_0 a}$ (symmetry)



- $U = \frac{1}{2}\sum q_i V_i = \frac{1}{2} \times \frac{-Q^2}{\pi\epsilon_0 a}(1 + 2 \times \frac{1}{4})$
  $= -\frac{3Q^2}{4\pi\epsilon_0 a}$
- Negative, since predominantly attractive forces.

2.6 Summary: assembly of discrete charge systems

The Electric field **E** and Potential V of a distribution of point charges q_i placed at positions $\mathbf{r}_i$ are:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{(\mathbf{r}-\mathbf{r}_i)^2}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}$$

The energy U required to assemble a system of point charges q_i by bringing them to positions $\mathbf{r}_i$ from infinity is given by:

$$U = \frac{1}{8\pi\varepsilon_0}\sum_i q_i \sum_{i\neq j}\frac{q_j}{r_{ji}} = \frac{1}{2}\sum_i q_i V_i$$

where V_i is the potential experienced by q_i at $\mathbf{r}_i$ from all *other* charges q_j.