CP2 ELECTROMAGNETISM

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LECTURE 19:

MOTION IN E & B FIELDS, DISPLACEMENT CURRENT

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 19. MOTION IN E & B FIELDS, DISPLACEMENT CURRENT

19.1 Motion of charged particles in E and B fields

- Force on a charged particle in an $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ field :

$$\underline{\mathbf{F}} = q\left(\underbrace{\underline{\mathbf{E}}}_{\text{along } \underline{\mathbf{E}}} + \underbrace{\underline{\mathbf{v}} \times \underline{\mathbf{B}}}_{\perp \text{ to both } \underline{\mathbf{v}} \text{ and } \underline{\mathbf{B}}}\right)$$

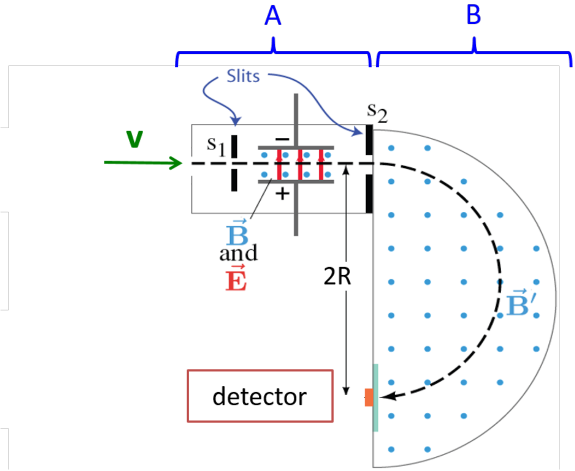
- Newton second law provides equation of motion :

$$\underline{\mathbf{F}} = m\,\underline{\mathbf{a}} = m\,\underline{\ddot{\mathbf{r}}} = q\,(\,\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}})$$

- Will demonstrate with 2 examples :
  1. Mass spectrometer
  2. Magnetic lens

19.2 Example : the mass spectrometer

Used for detecting small charged particles (molecules, ions) by their mass *m* .



Stage A : The velocity filter

- The particle will pass through both slits if it experiences no net force inside the filter

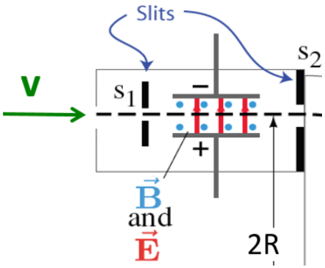
- The region has both $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields

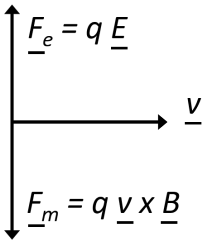
$$\underline{\mathbf{F}} = q\,(\,\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = 0$$

$$\rightarrow \quad \text{need } \underline{\mathbf{E}} = -\underline{\mathbf{v}} \times \underline{\mathbf{B}} \rightarrow \; v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

$$(\underline{\mathbf{E}} \perp \underline{\mathbf{v}} \,\&\, \underline{\mathbf{B}})$$

- Will filter particles with $v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$ and the spread $\pm\Delta v$ is given by the slit width





Stage B : The mass filter

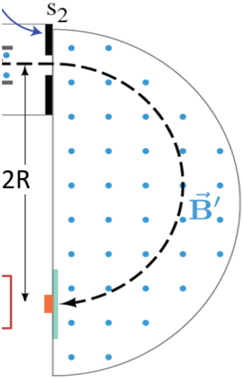
- This region has only a $\underline{\mathbf{B}}$ field

$$m\,\ddot{\underline{\mathbf{r}}} = q\,\dot{\underline{\mathbf{r}}} \times \underline{\mathbf{B}}$$

$$\text{with } \underline{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \text{ and } \dot{\underline{\mathbf{r}}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{q}{m} \begin{pmatrix} \dot{y}\,B \\ -\dot{x}\,B \\ 0 \end{pmatrix}$$

$$\rightarrow \ \ddot{z} = 0 \ \rightarrow \ v_z = \text{constant} \ (= 0)$$



- $\ddot{r}^2 = \ddot{x}^2 + \ddot{y}^2 = \frac{q^2}{m^2} \underbrace{\left(\dot{x}^2 + \dot{y}^2\right)}_{v^2} B^2$

- Circular motion in $x - y$ plane with : $\ddot{r} = \frac{q}{m} v\,B$

  For circular motion $\ddot{r} = \frac{v^2}{R} \ \rightarrow \ R = \frac{mv}{qB}$

- Since q and v are constant, then $R \propto m$

Mass spectrometer summary

In the presence of both E- and B-fields, a charge experiences the force:

$$\mathbf{F}_{\mathrm{EM}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Mass Spectrometer.

A. velocity filter:

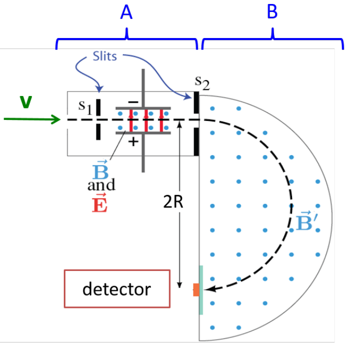
E&B-fields present. Charged particles pass through Stage A if their velocity equals the amplitude ratio:

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

B. Filter stage:

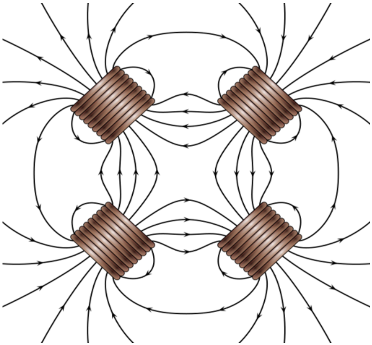
Only B-field present. Charged particles are forced on circular path with radius:

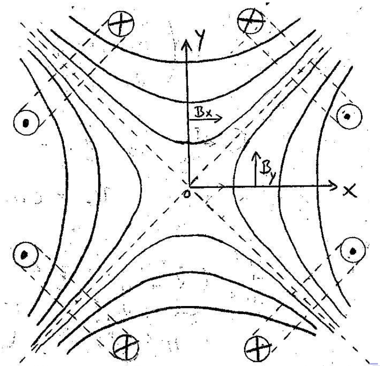
$$R = \frac{mv}{qB}$$



19.3 Example : magnetic lenses

- Magnetic lenses are used for focusing and collimating charged particle beams. Used in electron microscopes, particle accelerators etc.
- Quadrupole lens : four identical coils aligned in z-direction.
- Sum of 4 dipole fields : for small values of x, y close to the axis of symmetry, $B_x \propto y$, $B_y \propto x$

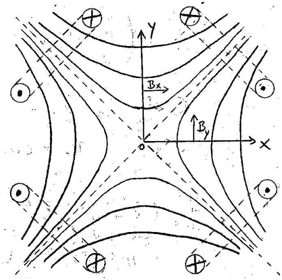




Quadrupole lens

- Along x-axis : only B_y component
- Along y-axis : only B_x component
- No z-component (symmetry)
- Inside the lens, close to the z-axis

$$\underline{\mathbf{B}} = \begin{pmatrix} k\,y \\ k\,x \\ 0 \end{pmatrix}$$ where k is a constant



- Equation of motion $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ k\,y & k\,x & 0 \end{vmatrix} = q\,k \begin{pmatrix} -x\,\dot{z} \\ y\,\dot{z} \\ x\,\dot{x} - y\,\dot{y} \end{pmatrix}$$

- Assume particle travels at a small angle wrt the z-axis :

$$\rightarrow\ \dot{x}, \dot{y} \approx 0\ \rightarrow\ \ddot{z} = 0\ \rightarrow\ \dot{z} = v = \text{constant}\ \rightarrow\ z = v\,t$$

- Equations of motion in the $x - y$ plane :

$$\ddot{x} = -\frac{q}{m} k\,v\,x \quad \text{and} \quad \ddot{y} = \frac{q}{m} k\,v\,y$$

Quadrupole lens continued

- Equ. of motion : $\ddot{x} = -\alpha^2 x$ & $\ddot{y} = \alpha^2 y$, where $\alpha = \sqrt{\frac{q\,k\,v}{m}}$
- Solutions : $x(t) = A \sin\alpha\, t + B \cos\alpha\, t$

  $$y(t) = C \sinh\alpha\, t + D \cosh\alpha\, t$$

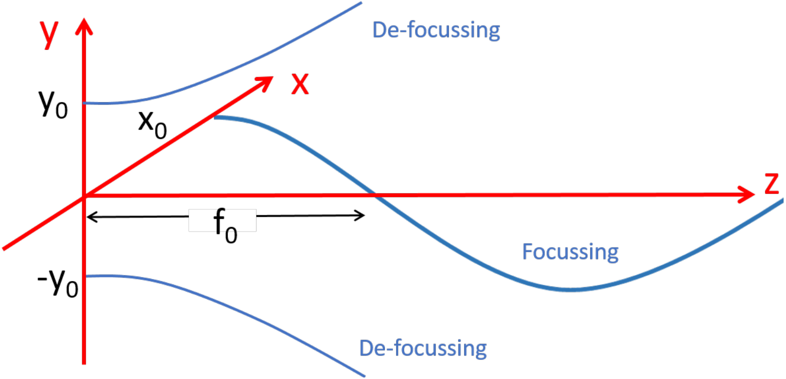
  where $\cosh y, \sinh y = (e^y \pm e^{-y})/2$
- Boundary conditions :
  At $t = 0 \rightarrow z = 0,\ x = x_0$ and $\dot{x} = 0, y = y_0$ and $\dot{y} = 0$
- Solutions : $x(t) = x_0 \cos\alpha t = x_0 \cos\frac{\alpha}{v} z$: focusing

  $$y(t) = y_0 \cosh\alpha t = y_0 \cosh\frac{\alpha}{v} z \quad \text{: de-focusing}$$

  (where $t = z/v$) $\rightarrow x = 0$ for $\frac{\alpha}{v} z = \frac{\pi}{2} + n\pi$
- Focal points in z direction (x=0) at $f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$
- Use lens pair with 90° angle for collimating a charged beam

Quadrupole lens continued

The lens pulls the beam on-axis in x and removes particles deviating in y

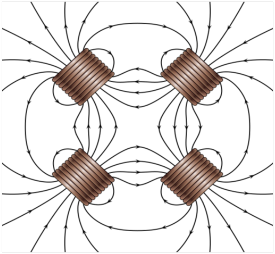


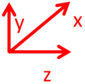
$$f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$$

Magnetic lens summary

Magnetic Lens.

$$\mathbf{B} = (\mathrm{k}\, y, \mathrm{k}\, x, 0)$$





Equation of Motion: $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$

Solutions:

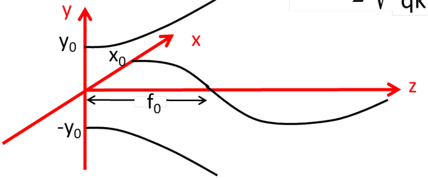
$$y(z) = y_0 \cosh \sqrt{\frac{q\,\mathrm{k}}{vm}}z$$

de-focusing

$$x(z) = x_0 \cos \sqrt{\frac{q\,\mathrm{k}}{vm}}z$$

focusing with

$$f_0 = \frac{\pi}{2}\sqrt{\frac{vm}{q\mathrm{k}}}$$



19.4 Electrodynamics "before Maxwell"

1. Gauss Law :

$$\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = Q_{encl.}/\epsilon_0 \quad \rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{E}} = \rho/\epsilon_0$$

2. No magnetic monopoles :

$$\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0 \quad \rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

3. Faraday's Law :

$$\oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\frac{\partial}{\partial t} \oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} \quad \rightarrow \quad \underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

4. Ampere's Law :

$$\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 \, I_{encl.} \quad \rightarrow \quad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \underline{\mathbf{J}}$$

Time-varying *B*-fields generate *E*-fields. *However*, time-varying *E*-fields do not seem to create *B*-fields in this version.
Is there something wrong ?

19.5 Revisit Ampere's Law

- Ampere's Law : $\rightarrow \quad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \underline{\mathbf{J}}$

  Apply Div : $\rightarrow \quad \underbrace{\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{B}})}_{\text{always zero}} \quad = \quad \underbrace{\mu_0 \, \underline{\nabla} \cdot \underline{\mathbf{J}}}_{\text{not always zero !!}}$

- Recall the continuity equation :

  $$\int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = -\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho(\mathcal{V}) \, d\mathcal{V} \quad \rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$$

  [Current leaving volume] = [Rate of change of charge]
  through surface inside volume

- Therefore Ampere's Law in its current form violates the continuity equation (and hence charge conservation) !

- But this is not surprising since we derived Ampere's Law assuming that $\frac{\partial}{\partial t}(\rho) = 0$

  $\rightarrow$ We have to "fix" Ampere's Law !

19.6 Fixing Ampere's Law : displacement current

- Add a term to Ampere's Law to make it compatible with the continuity equation :
- $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$

  Apply Gauss Law $\underline{\nabla} \cdot \underline{\mathbf{E}} = \rho/\epsilon_0$

  $\rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\epsilon_0 \underline{\nabla} \cdot \underline{\mathbf{E}}) = -\underline{\nabla} \cdot (\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t})$

  $\rightarrow \quad \underline{\nabla} \cdot (\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}) = 0$
- Implies we need to add $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ to $\underline{\mathbf{J}}$ in Ampere's law.

  $$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$$

  $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ is called the *displacement current* $\underline{\mathbf{J}}_D$ (but is actually a time-varying electric field)
- Time-varying $\underline{\mathbf{E}}$ fields now generate $\underline{\mathbf{B}}$ fields and vice versa. Also satisfies charge conservation.