CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 18:

TRANSFORMER, MAGNETIC ENERGY

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 18. TRANSFORMER, MAGNETIC ENERGY

18.1 Coaxial solenoids sharing the same area

18.2 Inductors in series and parallel

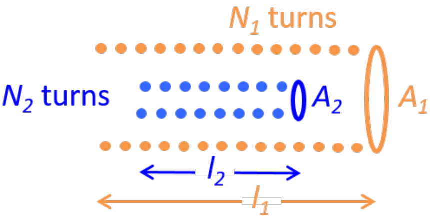
18.3 The transformer

18.4 Energy of the magnetic field

18.1 Coaxial solenoids sharing the same area

From before : mutual inductance between coils :

$$M_{21} = M_{12} = \mu_0 \frac{N_1 N_2}{\ell_1} A_2 \; (= M)$$



- Self inductance of coils 1 & 2

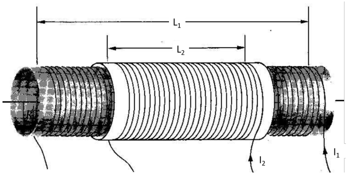
$$L_1 = \mu_0 \frac{N_1^2}{\ell_1} A_1 \quad \text{and}$$

$$L_2 = \mu_0 \frac{N_2^2}{\ell_2} A_2$$

- If $A_1 = A_2$

$$M = \left(\sqrt{\frac{\ell_2}{\ell_1}}\right) \sqrt{(L_1 L_2)}$$

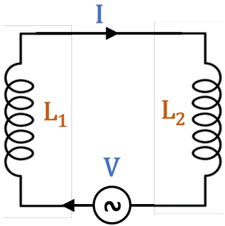
If $\ell_1 = \ell_2$ then $M = \sqrt{(L_1 L_2)}$



- Hence the mutual inductance is proportional to the geometrical mean of the self inductances.

In general circuits may not be tightly coupled, hence $M = k\sqrt{(L_1 L_2)}$ where $k < 1$. k is the *coefficient of coupling*.

18.2 Inductors in series and parallel



- 1. In series with no mutual inductance between coils :

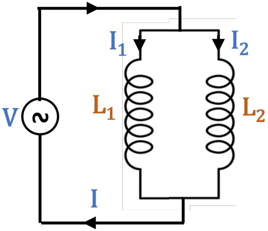
$$V = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} = (L_1 + L_2) \frac{dI}{dt}$$

$$L = L_1 + L_2$$

- 2. In series with mutual inductance between coils :

$$V = (L_1 + M) \frac{dI}{dt} + (L_2 + M) \frac{dI}{dt}$$
$$= (L_1 + L_2 + 2M) \frac{dI}{dt}$$

$$L = L_1 + L_2 + 2M$$



- 3. In parallel, no mutual inductance :

$$V = L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} \quad \text{where } I = I_1 + I_2$$

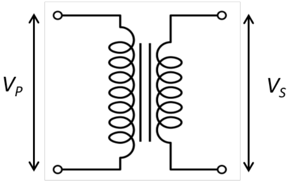
Write $V = L \frac{dI}{dt} \rightarrow V = L\left(\frac{dI_1}{dt} + \frac{dI_2}{dt}\right) = L\left(\frac{V}{L_1} + \frac{V}{L_2}\right)$

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

(with mutual inductance $L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$)

18.3 The transformer

- Primary coil creates flux $\Phi_p = A_p B_p$ per winding $\rightarrow$ secondary coil gives EMF per winding $\mathcal{E}_S = -\frac{d\Phi_S}{dt}$
- The coils are coupled : $\Phi_S = k\Phi_P$ where $k = 1$ for an ideal transformer (k depends on geometry, coupling etc.)



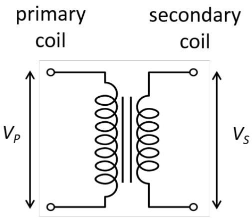
- Ratio of EMFs :

$$\mathcal{E}_P = -N_P \frac{d\Phi_P}{dt} \qquad \mathcal{E}_S = -N_S \frac{d\Phi_S}{dt} \rightarrow \quad \frac{V_S}{V_P} = \underbrace{\frac{d\Phi_S}{d\Phi_P}}_{\text{k}} \times \underbrace{\frac{N_S}{N_P}}_{\text{winding ratio}}$$

  - Transformer will step up or step down applied voltage V_P by the winding ratio
  - Ideally there is no power dissipated in the transformer if coils have zero resistance

$$\rightarrow \quad V_S I_S = V_P I_P \rightarrow \frac{I_S}{I_P} = \frac{V_P}{V_S} = \frac{1}{k}\frac{N_P}{N_S}$$

Transformer summary



Primary coil creates flux which permeates secondary coil, coupling their voltages:

Voltage Ratio:

$$\frac{V_S}{V_P} = \frac{d\Phi_S}{d\Phi_P}\frac{N_S}{N_P}$$

Current Ratio:

$$\frac{I_S}{I_P} = \frac{d\Phi_P}{d\Phi_S}\frac{N_P}{N_S}$$

18.4 Energy of the magnetic field

Consider the energy stored in an inductor L :

- Change in current results in a back EMF $\mathcal{E}$
- We need to do work to change the current : $dW = VdQ$

  Power = work per unit time = $V\frac{dQ}{dt} = VI$

  Energy expended $U = \int \underbrace{V\,I}_{\text{power}}\,dt = \int \underbrace{L\,\frac{dI}{dt}}_{\text{Back EMF}}\,I\,dt$

- $U = \frac{1}{2}\,L\,I^2 = \frac{1}{2}\,\Phi\,I$ $\quad (L = \frac{\Phi}{I})$

  regardless of circuit / current geometry
- For a coil : $L = \mu_0\,\frac{N^2}{\ell}\,A$ and $B = \mu_0\,\frac{N}{\ell}\,I$ (Ampere Law)

  $\rightarrow\ U = \frac{1}{2}\left(\mu_0\,\frac{N^2}{\ell}\,A\right)\left(\frac{B^2}{\mu_0^2\,\frac{N^2}{\ell^2}}\right) = \frac{1}{2}\,\frac{B^2}{\mu_0}\,A\,\ell = \frac{1}{2}\,\frac{B^2}{\mu_0}\mathcal{V}\ \leftarrow$ volume
- In the general case : $U = \frac{1}{2\,\mu_0}\int B^2\,d\mathcal{V}$ over all space

Summary of energy in E and B fields

Electric field energy

- In terms of circuits :

$$U_e = \frac{1}{2}\, C\, V^2$$
$$= \frac{1}{2}\, Q\, V$$

- In terms of fields :

$$U_e = \frac{\epsilon_0}{2} \int_{all\ \ space} E^2\, d\mathcal{V}$$

Magnetic field energy

- In terms of circuits :

$$U_m = \frac{1}{2}\, L\, I^2$$
$$= \frac{1}{2}\, \Phi\, I$$

- In terms of fields :

$$U_m = \frac{1}{2\,\mu_0} \int_{all\ \ space} B^2\, d\mathcal{V}$$