CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 17:

SELF & MUTUAL INDUCTANCE

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 17. SELF & MUTUAL INDUCTANCE

17.1 Example : self inductance of two parallel wires

17.2 Mutual inductance

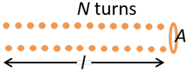
17.3 Mutual induction of two coaxial solenoids

Self inductance summary

Self-inductance L is the ratio of the voltage (emf) produced in a circuit by self-induction, to the rate of change in current causing the induction.

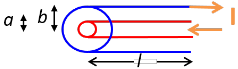
$$L = \frac{\frac{d\Phi}{dt}}{\frac{dI}{dt}} = \frac{d\Phi}{dI} = \frac{-\varepsilon}{\dot{I}}$$

Self-inductance of a long coil.



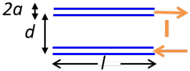
$$L = \frac{d\Phi}{dI} = \mu_0 \frac{N^2}{l} A$$

Self-inductance of a coaxial cable.



$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) l$$

Self-inductance of two parallel wires.

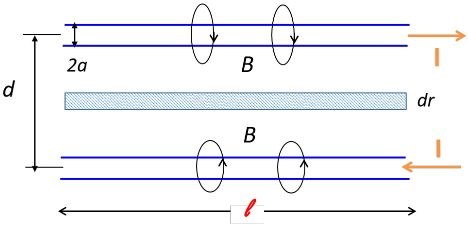


$$L = \frac{\mu_0}{\pi} \ln\left(\frac{d-a}{a}\right) l$$

Units of self inductance : the Henry $[H] \equiv [kg\, m^2 s^{-2} A^{-2}]$.
When the current changes at one ampere per second ($A\, s^{-1}$), an inductance of 1 H results in the generation of one volt (1 V) of potential difference.

17.1 Example : self inductance of two parallel wires

Calculate the self inductance of two parallel wires,radius a, separation to the centres d, and of length ℓ



- From Ampere's law, outside each wire :

  $B = \frac{\mu_0 I}{2\,\pi\, r}$

  Radial area element $\ell\, dr$

- Magnetic flux :

  $$\Phi = 2 \times \int_a^{d-a} B\, \ell\, dr$$

  (factor 2 because same contribution from 2 wires):

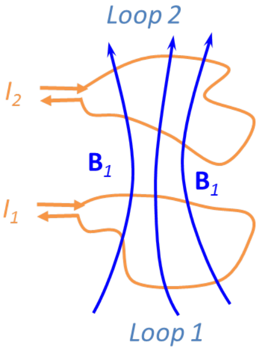
  $$= 2 \times \int_a^{d-a} \frac{\mu_0 I\, \ell}{2\,\pi} \frac{1}{r}\, dr \;=\; \frac{\mu_0\, \ell}{\pi} log_e \left(\frac{d-a}{a}\right)\; I$$

- $$L = \frac{\Phi}{I} = \frac{\mu_0}{\pi}\, \ell\; log_e \left(\frac{d-a}{a}\right)$$

  $$\approx \frac{\mu_0}{\pi}\, \ell\, log_e \left(\frac{d}{a}\right) \quad \text{for } a << d$$

17.2 Mutual inductance

- Current I_1 through circuit loop 1 generates magnetic field density B_1 which penetrates circuit loop 2
- A change in current I_1 will induce an EMF in circuit loop 2



- Define *mutual inductance M*

$$M_{21} = \frac{\Phi_2}{I_1} \;;\; M_{12} = \frac{\Phi_1}{I_2} \;;\; M_{12} = M_{21}$$

- Since $\Phi \propto I$, can also be written $M_{21} = \frac{d\Phi_2}{dI_1}$; $M_{12} = \frac{d\Phi_1}{dI_2}$

17.3 Mutual induction of two coaxial solenoids

1. Current through coil 1 creates magnetic field through coil 2.

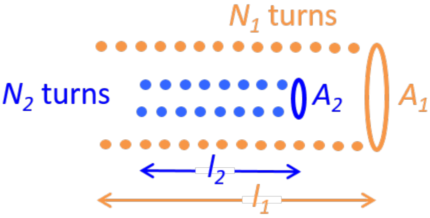
$$B_1 = \mu_0 \frac{N_1}{\ell_1} I_1$$

- A_2 : area of pick-up coil 2
- Flux experienced by coil 2

$$\Phi_2 = N_2 A_2 B_1 = \mu_0 \frac{N_1}{\ell_1} I_1 N_2 A_2$$

- Mutual inductance :

$$M_{21} = \frac{\Phi_2}{I_1} = \mu_0 \frac{N_1 N_2}{\ell_1} A_2$$

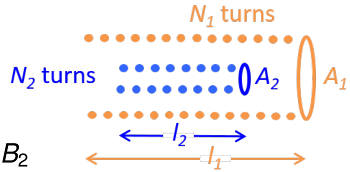


- EMF induced in coil 2 :
  - $\mathcal{E} = -\frac{d\Phi_2}{dt} = -\mu_0 \frac{N_1}{\ell_1} A_2 N_2 \frac{dI_1}{dt}$

    $\mathcal{E} = -M_{21} \frac{dI_1}{dt}$ (compare to $\mathcal{E} = -L\frac{dI}{dt}$ for self inductance)

Mutual induction of two coaxial solenoids continued

2. Current through coil 2 creates magnetic field through coil 1.



- $\Phi_1 = \int B_2 \, da_1'$ (da_1' is "effective" area)

  Now it's more complicated as B_2 is not uniform through coil 1 !
- Flux experienced by coil 1 $\Phi_1 = N_1' \, A_1' \, B_2$

  Overlap with volume over which B is "strongest"
  - Approximate : neglect stray fields of B_2 outside coil 2

    then $A_1' = A_2$ and $N_1' = N_1 \frac{\ell_2}{\ell_1}$ and $B_2 = \mu_0 \frac{N_2}{\ell_2} I_2$
  - Mutual inductance :

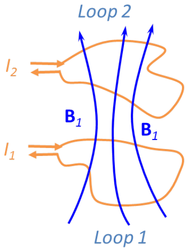
    $$M_{12} = \frac{\Phi_1}{I_2} = \frac{N_1 \, (\ell_2/\ell_1) \, A_2 \, \mu_0 \, (N_2/\ell_2) \, I_2}{I_2} = \mu_0 \frac{N_1 \, N_2}{\ell_1} A_2 = M_{21}$$
  - $M_{12} = M_{21}$ This is Neumann's theorem. (It turns out even if we had done the exact calculation the result would have been the same)

Mutual inductance summary

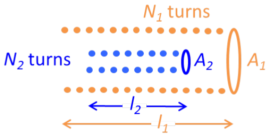
Mutual Inductance M: is the ratio of the voltage (emf) produced in a circuit by self-induction, to the rate of change in current causing the induction.

$$M_{12} = \frac{d\phi_1}{dI_2} \quad \xleftrightarrow{\text{Neumann formula}} \quad M_{21} = \frac{d\phi_2}{dI_1}$$

$$M_{12} = M_{21}$$



Mutual inductance of two coaxial solenoids.



$$M_{12} = \mu_0 \frac{N_1 N_2}{l_1} A_2$$

Units of mutual inductance :
again the Henry $[H] \equiv [kg\, m^2 s^{-2} A^{-2}]$.