CP2 ELECTROMAGNETISM

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LECTURE 16:

INDUCTION EXAMPLES & SELF INDUCTION

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 16. INDUCTION EXAMPLES & SELF INDUCTION

16.1 Example : the Homopolar Generator (Faraday's disk)

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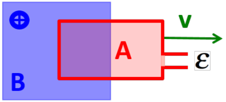
Faraday's and Lenz's Laws summary

Faraday's Law of electromagnetic induction:

The induced electromotance $\mathcal{E}$ in any closed circuit is equal to the negative of the time rate of change of the magnetic flux Φ through the circuit.



$$\varepsilon = \frac{d\Phi}{dt} = -\frac{d}{dt}\oint_S \boldsymbol{B}\,.\,\boldsymbol{da}$$

In terms of E- and B-fields:

Integral form: $$\oint \boldsymbol{E}\,.\,\boldsymbol{d\ell} = -\frac{d}{dt}\oint_S \boldsymbol{B}\,.\,\boldsymbol{da}$$

Differential form: $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Lenz's Law:

An induced electromotance always gives rise to a current whose magnetic field opposes the original change in magnetic flux.

Unit of magnetic flux Weber $[Wb] = [Tm^2] = [kg\,m^2 s^{-2} A^{-1}]$

16.1 Example : the Homopolar Generator (Faraday's disk)

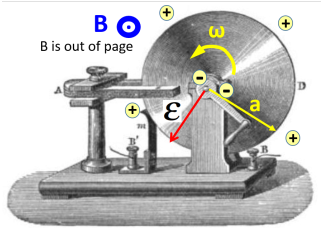
1. Determine voltage using Lorenz force

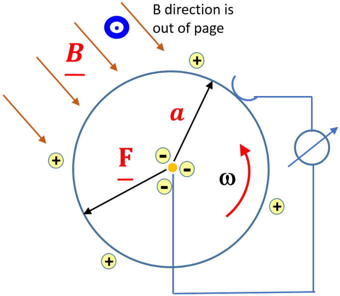
- Metal disk mechanically rotated (performing work)
- A B-field is present with $\underline{\mathbf{B}}$ perpendicular to the disk area.
- Voltage pick-up between the centre and rim of disk.
- EMF is *radial*, with identical potential along each circumference element, radius r

$$\mathcal{E} = \int_{r=0}^{r=a} (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{dr}}$$

where $\underline{\mathbf{v}} \perp \underline{\mathbf{B}} \perp \underline{\mathbf{dr}}$ and $v = r\,\omega$

- $\mathcal{E} = \int_0^a \omega B\, r\, dr = \frac{1}{2}\omega a^2 B$





The Homopolar Generator continued

2. Determine using Faraday's Law

$$\mathcal{E} = -\frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$$

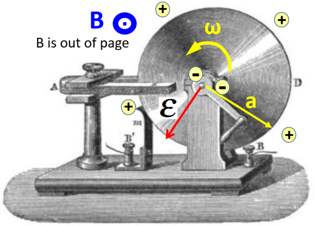
- Consider area element $\Delta A = r\,\Delta\theta\,\Delta r$

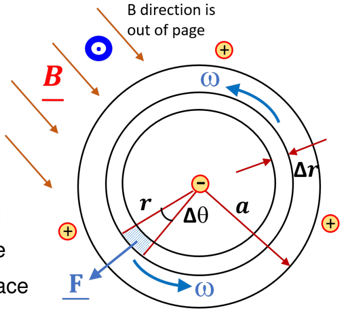
  $\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(r\,\Delta\theta\,\Delta r) = \omega r\,\Delta r$
- Add up all contributions $\Delta r \to dr$

  (There is a +/- sign ambiguity depending on direction of $\underline{\mathbf{da}}$. Take direction such that $\mathcal{E}$ is positive.)

- $\mathcal{E} = \int_0^a \omega B\, r\, dr = \frac{1}{2}\omega a^2 B$

  same result as before *.

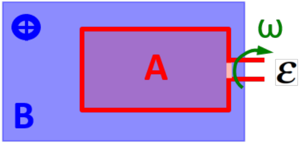




* Strictly speaking, this method from Faraday's Law is not entirely sensible since the current is continuous across the disk and $\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$ is in principle only applicable for a surface bounding a closed current path (see for example Griffiths).

16.2 Example : coil rotating in a B-field

Coil, N turns, rotating at angular frequency ω in a uniform B-field



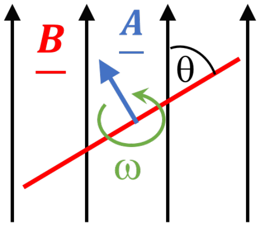
- Magnetic flux $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = N\,A\,B\,\sin\theta$

  where $\theta = \omega\,t$

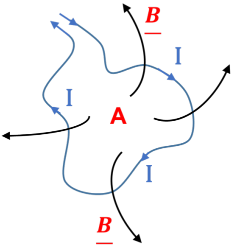
  [$\times N$ since each turn of the coil links flux]

$$\mathcal{E} = -\frac{d\Phi}{dt} = -N\,A\,B\,\omega\,\cos\omega\,t$$

- This is a generator/dynamo (incorporated into most aspects of electrical power generation).



16.3 Self inductance



- Take a closed-loop circuit through which current flows
- The current I has an associated magnetic field which penetrates the circuit, $B \propto I$
- If the current changes, there will be a changing B-field through the loop.
  - Faraday : The changing magnetic flux Φ induces an EMF (voltage) in the loop *itself* : $\mathcal{E} = -\frac{d\Phi}{dt}$, where $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$
  - Lenz : This EMF will act in a direction so as to oppose the change in flux which caused it
  - EMF induced $\mathcal{E} = -\frac{d\Phi}{dt}$; Note that $\Phi \propto B \propto I$
  - Define *self inductance* $L = \frac{\Phi}{I}$

    Since $\Phi \propto I$, can also be written $L = \frac{d\Phi}{dI} = \frac{d\Phi}{dt} / \frac{dI}{dt} = -\mathcal{E} / \frac{dI}{dt}$
  - L depends solely on the geometry of the circuit. (Compare with circuit theory : $V = L\frac{dI}{dt}$)

16.4 Example : self induction of a long coil

Calculate the self inductance of a long coil, area A, length ℓ, with N turns

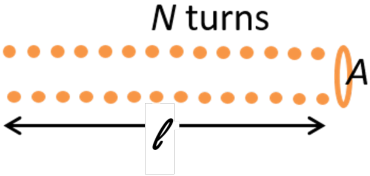
- From Ampere's law

  $B = \mu_0 \frac{N}{\ell} I$

- Magnetic flux $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$

  $\Phi = N A B = \mu_0 \frac{N^2}{\ell} A I$

  (since each of the N coils links its own flux)



- Hence $L = \frac{\Phi}{I} = \mu_0 \frac{N^2}{\ell} A$

- EMF induced in coil :

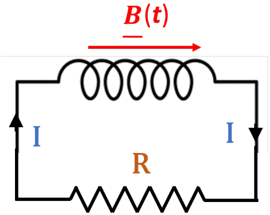
  $\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 \frac{N^2}{\ell} A \frac{dI}{dt} = -L \frac{dI}{dt}$

16.5 Example : long coil in varying B with resistive load

- Consider a long coil, area A, length ℓ, with N turns.
- Coil is immersed in axial time-varying magnetic field : $B(t) = B_0 \cos \omega t$
- EMF is induced in coil, coil is connected across a resistor $\rightarrow$ current will flow
- EMF induced :

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(N\,A\,B_0 \cos \omega t\right)$$

$$= N\,A\,\omega\,B_0 \sin \omega t$$



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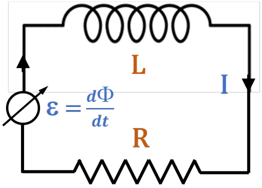


Long coil in varying B with resistive load, continued

- Self inductance of coil : $L = \mu_0 \frac{N^2}{\ell} A$
- Back EMF induced due to L opposes the changing current (Lenz)
- Ohm's Law for current flowing in the coil

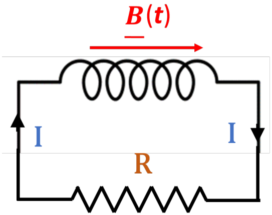
$$\underbrace{\mathcal{E}}_{\text{induced emf}} = I R + \underbrace{L \frac{dI}{dt}}_{\text{back emf}}$$

Alternatively can write $\mathcal{E} = I Z$
where $Z = R + j\omega L \;\rightarrow\; Z = |Z|\, e^{j\phi}$

- $\mathcal{E} = Im\left[\mathcal{E}_0\, e^{j\omega t}\right]$ where $\mathcal{E}_0 = (N A \omega B_0)$
- Current $I = I_0\; Im\left[e^{j(\omega t - \phi)}\right]$

where $I_0 = \mathcal{E}_0/|Z| = \mathcal{E}_0/\sqrt{R^2 + (\omega L)^2}$

and phase angle : $\tan\phi = (\omega L)/R$



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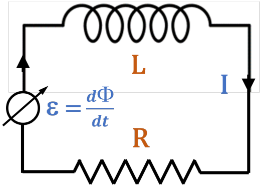
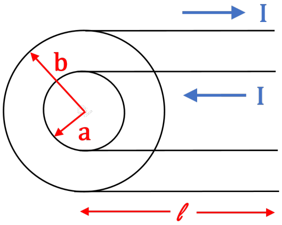


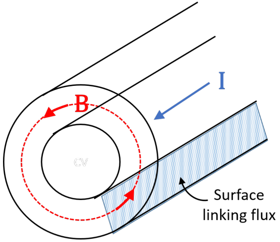
16.6 Example : self induction of a coaxial cable

Calculate the self inductance of a coaxial cable, inner/outer radii a & b, length ℓ



- From Ampere's law, for $a \geq r \geq b$:
  $B = \frac{\mu_0 I}{2\pi r}$
- Note that the area linking flux is *radial* :
  $da = \ell\, dr$
- Magnetic flux :

$$\Phi = \int_a^b \frac{\mu_0 I}{2\pi r} \ell\, dr$$
$$= \frac{\mu_0 I}{2\pi} log_e\left(\frac{b}{a}\right) \ell$$



$$L = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} log_e\left(\frac{b}{a}\right) \ell$$