CP2 ELECTROMAGNETISM

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LECTURE 15:

ELECTROMAGNETIC INDUCTION

Neville Harnew[1]

University of Oxford

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

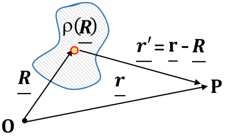
[1] With thanks to Prof Laura Herz

OUTLINE : 15. ELECTROMAGNETIC INDUCTION

15.1.1 Summarizing where we are : electrostatics

1. Coulomb's Law :

$$\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})}{(\underline{\mathbf{r}}-\underline{\mathbf{R}})^3}(\underline{\mathbf{r}}-\underline{\mathbf{R}})\,d\mathcal{V}$$



- An electric charge generates an electric field. Electric field lines begin and end on charge or at ∞.

2. Gauss Law :

$$\underbrace{\oint_S \underline{\mathbf{E}}\cdot\underline{\mathbf{da}} = Q_{encl.}/\epsilon_0}_{\text{integral form}} \;\rightarrow\; \underbrace{\underline{\nabla}\cdot\underline{\mathbf{E}} = \rho/\epsilon_0}_{\text{differential form}}$$

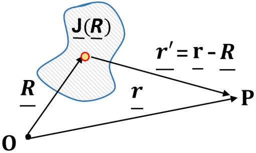
3. The electric field is conservative :

- A well-defined potential V such that $\underline{\mathbf{E}} = -\underline{\nabla}\,V$
  $\rightarrow\ \oint \underline{\mathbf{E}}\cdot\underline{\mathbf{d\ell}} = 0$ (work done is independent of path)
- Using the vector identity : $\underline{\nabla}\times\underline{\mathbf{E}} = -\underline{\nabla}\times\underline{\nabla}\,V = 0$
- Hence $\underline{\nabla}\times\underline{\mathbf{E}} = 0$

15.1.2 Summarizing where we are : magnetostatics

1. Biot-Savart Law :

$$\underline{\mathbf{B}}(\underline{\mathbf{r}}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\underline{\mathbf{J}}(\underline{\mathbf{R}})}{(\underline{\mathbf{r}}-\underline{\mathbf{R}})^3} \times (\underline{\mathbf{r}} - \underline{\mathbf{R}})\, d\mathcal{V}$$



- There are no magnetic monopoles. Magnetic field lines form closed loops.

2. Gauss Law of magnetostatics :

$$\underbrace{\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0}_{\text{integral form}} \rightarrow \underbrace{\underline{\nabla} \cdot \underline{\mathbf{B}} = 0}_{\text{differential form}}$$

3. Ampere's Law :
   - Magnetic fields are generated by electric currents.

$$\rightarrow \oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0\, I_{encl.} \rightarrow \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \underline{\mathbf{J}}$$

4. Continuity equation :
   - $\int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = -\frac{d}{dt} \int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V} \rightarrow \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{d}{dt}(\rho)$

     (charge conserved)

Vector and scalar potential

Off syllabus, but worth a mention

Magnetic vector potential A defined through: $\mathbf{B} = \nabla \times \mathbf{A}$

Such **A** always exists because: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

Inserting into Ampere's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A})$$
$$= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

There is a certain degree of freedom in which **A** to choose – set: $\nabla \cdot \mathbf{A} = 0$

Poisson equations for magnetostatics:
(one for each **J** & **A** coordinate)

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Magnetic scalar potential V_m**:** $\mathbf{B} = -\mu_0 \nabla V_m$ ⟷ $V_m = -\frac{1}{\mu_0} \int_A^B \mathbf{B} \cdot d\mathbf{l}$

Caution: V_m is *pathway-dependent and not single-valued* because $\nabla \times \mathbf{B} \neq 0$.

But V_m can be used with care in simply-connected, current-free regions.

Being a scalar, V_m is mathematically easier to use than the vector potential.

15.2 Electromagnetic induction - outline

Up to now we have considered stationary charges and steady currents. We now focus on what happens when either the *E*-field or *B*-field varies with time.

1. Introduction: Electromagnetic Induction
2. Faraday's and Lenz's Laws of Induction
3. Self-Inductance and Mutual Inductance
4. The Transformer
5. Energy of the Magnetic Field
6. Charged Particles in E- and B-Fields

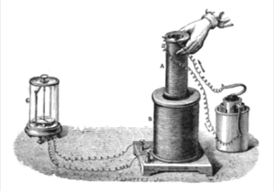
Items 1–5: Problem Set 4

Item 6: Problem Set 5

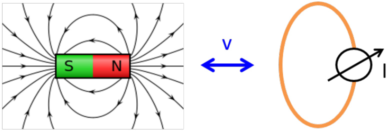
Origins of electromagnetic induction

1831: Michael Faraday carries out a series of experiments and observes:

Two coils are arranged in a way so that the magnetic flux density of coil A penetrates through coil B. He found that if the B-field in coil A is changing, this induces an electrical current in coil B.



Moving a bar magnet through a circuit element (wire loop) generates a current in the circuit. Moving instead the circuit w.r.t. the bar magnet, gives the same result.



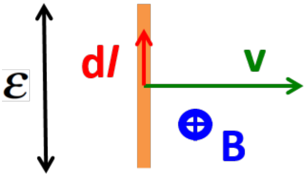
→ A change with time in the magnetic flux density through a circuit causes an "electromotive force" that moves charges along the circuit.

15.3 Faraday and Lenz's Laws of Induction

15.3.1 Electromotive force (EMF)

- Consider a wire moving with velocity $\underline{\mathbf{v}}$ through a B-field.
- Free charges in the wire experience a Lorenz force, perpendicular to $\underline{\mathbf{v}}$ & $\underline{\mathbf{B}}$:

$$\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$$

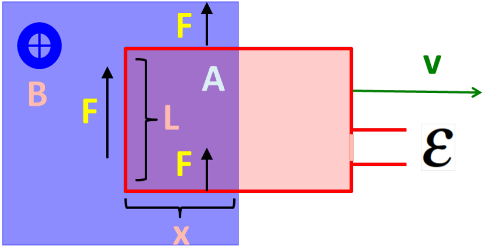


  - This moves charge to one side/end of the wire, which will create an electric potential drop along the wire :
    $\mathcal{E} = \int_\ell \frac{dW}{q} = \int_\ell \frac{\underline{\mathbf{F}} \cdot \underline{\mathbf{d\ell}}}{q}$ (by definition, $V =$ work/unit charge)
  - Hence $\mathcal{E} = \int_\ell (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d\ell}}$

    $\mathcal{E}$ is the *electromotive force* (or *electromotance*) (EMF)
  - Note that $\mathcal{E}$ is *not* a force but a line integral over a force (i.e. a potential) !

15.3.2 Magnetic flux

- Now consider a wire circuit loop being pulled with velocity $\underline{\mathbf{v}}$ out of a region containing a B-field.



- EMF on vertical side :

$$\mathcal{E} = \int_\ell (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d}\ell}$$
$$= v\,B\,L$$

  - No contribution to EMF from horizontal sides
  - Define *magnetic flux* $\Phi = \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$
  - Rate of change of flux $\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = \frac{d}{dt}\int_S B\,da$ (since $\underline{\mathbf{B}}$ is $\|$ to $\underline{\mathbf{da}}$)
  - $\frac{d\Phi}{dt} = \frac{d}{dt}(BA) = \frac{d}{dt}(B\,Lx) = B\frac{dx}{dt}L = -v\,B\,L = -\mathcal{E}$ (negative since x *decreases* with positive v)
  - In general, $\mathcal{E}$ from magnetic flux $\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = -\mathcal{E}$

15.4 Faraday's and Lenz's Laws

- Faraday's Law

  The induced electromotance (EMF) $\mathcal{E}$ in any closed circuit is equal to (the negative of) the time rate of change of the magnetic flux Φ through the circuit.

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = -\mathcal{E}$$

- Lenz's Law

  The induced electromotance always gives rise to a current whose magnetic field opposes the original change in magnetic flux.

15.5 Faraday's Law in differential form

- Net potential around a *closed* circuit loop = 0

  $\mathcal{E} = \oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell}$, hence $V = -\mathcal{E} = -\oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell}$

- Faraday's Law in integral form

  $\mathcal{E} = \oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\frac{d}{dt} \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$

  Apply Stokes' theorem to LHS :

  $\int_S (\underline{\nabla} \times \underline{\mathbf{E}}) \cdot \underline{\mathbf{da}} = -\frac{d}{dt} \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$

- Gives Faraday's Law in differential form

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

- Any time-varying magnetic field (or change in magnetic flux) generates an electric field which results in an electric potential $\mathcal{E}$.

  (In contrast $\underline{\nabla} \times \underline{\mathbf{E}} = 0$ for electro/magnito-statics)