CP2 ELECTROMAGNETISM

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LECTURE 14:

AMPERE'S CIRCUITAL LAW, CHARGE CONSERVATION

Neville Harnew[1]

University of Oxford

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

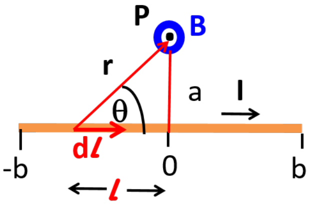
[1] With thanks to Prof Laura Herz

OUTLINE : 14. AMPERE'S CIRCUITAL LAW & CHARGE CONSERVATION

14.1 Ampere's Circuital Law

- Ampere's Circuital Law can be derived formally from the Biot-Savart Law and vector calculus but is beyond the scope of this course.
- But for a special case, we return to the *B*-field due to an infinite straight wire with current I, previously derived.

$\boxed{B = \frac{\mu_0\, I}{2\pi\, a}}$ $|\underline{\mathbf{B}}|$ const. at radius *a*



- We can form the closed-loop integral :

$$\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \frac{\mu_0\, I}{2\pi\, a} \times 2\,\pi\, a = \mu_0 I$$

- This gives us *Ampere's Circuital Law* which is also applicable for the *general case* :

$$\boxed{\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I = \mu_0 \int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}}$$ for current density $\underline{\mathbf{J}}$

Note Ampere's Law needs to be amended in the presence of any time-varying electric field (see later).

Ampere's Circuital Law continued

- Ampere's Circuital Law in integral form

  $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I = \mu_0 \int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$ for current density $\underline{\mathbf{J}}$

- Stokes Theorem : $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{da}}$

  $\rightarrow \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{da}} = \mu_0 \oint_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$

- Ampere's Law in differential form :

  $\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \underline{\mathbf{J}}$

- Ampere's Law : an integral of magnetic flux density $\underline{\mathbf{B}}$ over a closed loop bounding a surface equals the current flowing through the surface.
- Allows straightforward calculations of B-fields along loops where B is constant.

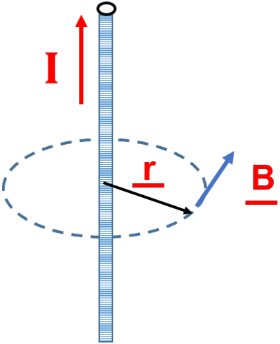
14.2 Example : B-field inside and outside a cylindrical wire

1. Outside the wire (this should be obvious $\cdots$)

- Ampere's Law $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I$
- Amperean path $\rightarrow$ circle of radius r :
  On this path $\underline{\mathbf{B}} \parallel \underline{\mathbf{d}\ell}$ and $|\underline{\mathbf{B}}|$ is constant
- $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = B \cdot 2\pi r = \mu_0 I$

$\rightarrow$ $\boxed{B = \frac{\mu_0 I}{2\pi r}}$ for an infinite wire

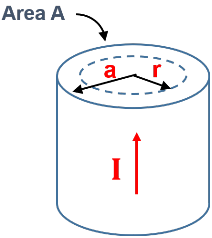
(much easier than using Biot - Savart !)



Cylindrical wire continued

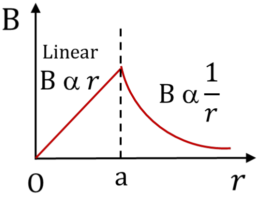
2. Inside the wire

- Current evenly distributed throughout cylinder $\rightarrow J = I/A = \frac{I}{\pi a^2}$
- Ampere's Law for field at radius r
  $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I = \mu_0 \int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$



$$B \cdot 2\pi r = \mu_0 \int_0^r \frac{I}{\pi a^2} 2\pi r' dr'$$

$$= \mu_0 I \underbrace{\left(\frac{\pi r^2}{\pi a^2}\right)}_{\text{ratio of areas}}$$

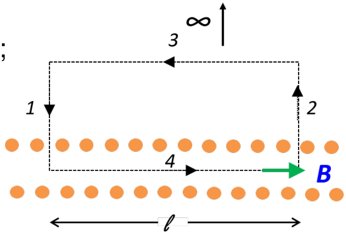


$$\rightarrow \boxed{B = \left(\frac{\mu_0 I}{2\pi a^2}\right) r} \quad \text{inside wire}$$

14.3 Example : B-field of a long solenoid

- Solenoid carrying current I
- Amperean path is a rectangle inside and outside the solenoid
- Take side 3 to ∞ (i.e. does not contribute); sides 1 & 2 cancel (due to symmetry)
- Contribution from side 4 only
  $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = B \cdot \ell = \mu_0 \, N \, I$

  where N is the number of turns within the Amperean surface



$$\rightarrow \quad \boxed{B = \mu_0 \, \frac{N}{\ell} \, I}$$

same as from Biot-Savart law as before $^{(*)}$

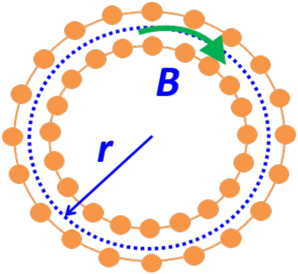
- B is uniform inside and zero outside the solenoid (if “infinite”)

(*) Note that if the coil is not “infinite”, end effects will need to be taken into account and here the field will not be uniform, i.e. Ampere’s Law will not be as useful as presented here.

14.4 Example : B-field of a toroidal coil

- Toroid has N windings of wire carrying current I
- Amperean path inside the solenoid cuts current-carrying loops N times
- $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = B \cdot 2\,\pi\, r = \mu_0\, N\, I$

  $\rightarrow \quad \boxed{B = \frac{\mu_0\, N\, I}{2\,\pi\, r}}$
- B- field is uniform in toroid and follows circular path
- B-field is zero outside the confines of the toroid



Ampere's Law summary

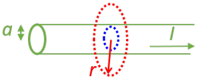
Ampere's law: $$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = \mu_0 I$$

Electric currents generate magnetic fields whose field lines form closed loops.

"Gauss' law of Magnetism": $$\oint_S \boldsymbol{B}\,.\,\boldsymbol{da} = 0$$

There are no magnetic monopoles.

Infinite straight wire.



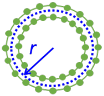
outside: $r > a$

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = B.\,2\pi r = \mu_0 I \rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

inside: $0 < r < a$

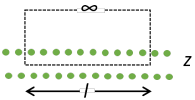
$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = B.\,2\pi r = \mu_0 I = \frac{\pi r^2}{\pi a^2} \rightarrow \boxed{B = \frac{\mu_0 I}{2\pi a^2} r}$$

Toroidal coil.



$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = B.\,2\pi r = \mu_0 N I \rightarrow \boxed{B = \frac{\mu_0 N I}{2\pi r}}$$

Infinite solenoid.

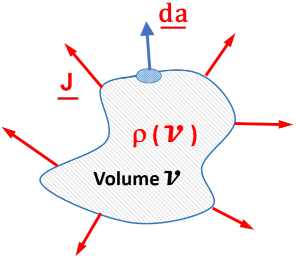


$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = B.\,l = \mu_0 N I \rightarrow \boxed{B = \mu_0 I \frac{N}{l}}$$

14.5 Conservation of charge

- Consider a volume $\mathcal{V}$ bounded by a surface S.

- The integral of current density flowing out (or into) the surface $\underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$ is equal to the charge lost by the volume [per unit time].



- $\int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = I = -\frac{dQ}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V}$
  Statement of the conservation of charge
- Use the divergence theorem on the LHS

  $\int_{\mathcal{V}} \underline{\nabla} \cdot \underline{\mathbf{J}}\, d\mathcal{V} = -\frac{d}{dt} \int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V}$

This gives the *continuity equation* $\rightarrow$ $\boxed{\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{d}{dt}(\rho)}$

(mathematical statement of charge conservation)

14.6 Current density and Ohm's Law

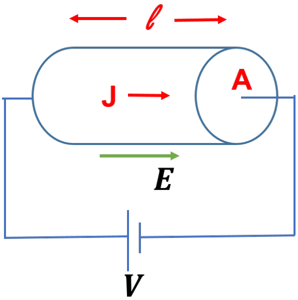
- Ohm's Law $V = I\,R$

  $$V = E\,\ell$$
  $$I = J\,A$$
  $$\rightarrow \; E\,\ell = J\,A\,R$$

- This gives Ohm's Law in terms of current density: $\rightarrow$ $\boxed{J = \frac{\ell}{R\,A}\,E}$

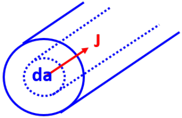
- Conductivity $\sigma = \frac{\ell}{R\,A}$

  Resistivity $\rho = 1/\sigma$

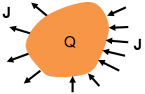


Summary : charge conservation & the continuity equation

Define current density **J**: $\mathbf{J} = \dfrac{d\mathbf{I}}{da_{\perp}}$



For a closed volume, the net current entering must be equal to the rate in change of charge inside the volume (charge conservation):



Continuity Equation:

$$\boxed{\oint_S \mathbf{J} \cdot d\mathbf{a} = I = -\frac{\partial Q}{\partial t} \quad \longleftrightarrow \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$$

In the limit of electro/magneto-statics:

$$\underbrace{\frac{\partial \mathbf{J}}{\partial t} = 0 \quad \text{steady currents}}_{\text{constant B-fields}} \qquad \underbrace{\frac{\partial \rho}{\partial t} = 0 \quad \text{stationary charges}}_{\text{constant E-fields}} \quad \xrightarrow{\text{C E}} \quad \nabla \cdot \mathbf{J} = 0$$