CP2 ELECTROMAGNETISM

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LECTURE 13:

MAGNETIC DIPOLES & THE DIVERGENCE OF B

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 13. MAGNETIC DIPOLES & THE DIVERGENCE OF B

13.1 Magnetic dipole components

13.2 Torque on a magnetic dipole in a B-field

13.3 Energy of a magnetic dipole in a B-field

13.4 Divergence of B

13.5 Divergence of B from the Biot-Savart Law

13.1 Magnetic dipole components

Magnetic dipole moment $\underline{\mathbf{m}} = I\,\underline{\mathbf{A}}$
= [Current] $\times$ [Area bounded by the loop]

- Electric dipole field

$$\left.\begin{aligned} E_r &= \frac{2\,p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\ E_\phi &= 0 \\ p &= q\,d \end{aligned}\right\}$$

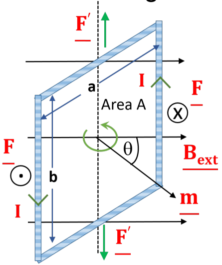
- Magnetic dipole field

$$\left\{\begin{aligned} B_r &= \frac{2\,\mu_0 m\cos\theta}{4\pi\,r^3} \\ B_\theta &= \frac{\mu_0 m\sin\theta}{4\pi\,r^3} \\ B_\phi &= 0 \\ m &= I\,A \end{aligned}\right.$$

The magnetic and electric dipole components have exactly the same form. (The exact derivation of the magnetic field components are beyond the scope of this course.)

13.2 *Torque on a magnetic dipole in a* $\underline{\mathbf{B}}$*-field*

Calculate the torque on a current loop placed in an external magnetic field:



- Net force on the whole loop :
  $\underline{\mathbf{F}} = \oint_{loop} I\, \underline{\mathbf{d\ell}} \times \underline{\mathbf{B}}_{\mathbf{ext}} = 0$

  (since equal and opposite forces from opposite elements $\underline{\mathbf{d\ell}}$ cancel pairwise)

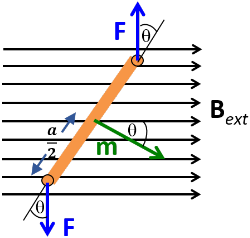
- From before, there is a torque on the current loop : $|\,\underline{\mathbf{T}}\,| = 2 \times \frac{a}{2} \times I\, B_{ext}\, b \sin\theta$

  $|\,\underline{\mathbf{T}}\,| = I\, B_{ext}\, A \sin\theta \;\rightarrow\; \underline{\mathbf{T}} = I\, \underline{\mathbf{A}} \times \underline{\mathbf{B}}_{ext}$

- Torque on the magnetic dipole

  $\underline{\mathbf{T}} = \underline{\mathbf{m}} \times \underline{\mathbf{B}}_{\mathbf{ext}}$ $(*)$

  Compare with the torque on an electric dipole $\underline{\mathbf{T}}_{\mathbf{elec}} = \underline{\mathbf{p}} \times \underline{\mathbf{E}}_{\mathbf{ext}}$



(*) This has been done for a rectangular shape. But note that this is a general result for *any* shape. Any current loop can be built from infinitesimal rectangular loops with all the internal currents cancelling, yet contributing to the overall moment.

Torque on a magnetic dipole, continued

- Do the explicit calculation for a circular current loop:
- Only the vertical component of $\underline{\mathbf{d}\ell}$ results in a torque $\;\rightarrow\; d\ell\, \sin\alpha$
- Torque due to facing elements $\underline{\mathbf{d}\ell}$:

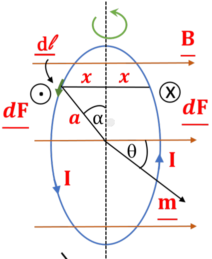
  $|d\underline{\mathbf{T}}| = 2|\underline{\mathbf{x}} \times d\underline{\mathbf{F}}| = 2x(I\, d\ell B \sin\alpha)\sin\theta$
- $x = a\sin\alpha\;;\; \ell = a\alpha \;\rightarrow\; d\ell = a\, d\alpha$

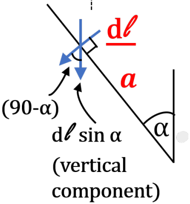
  $|d\underline{\mathbf{T}}| = 2\,(I\, a^2 \sin^2\alpha)\, B \sin\theta\, d\alpha$
- Hence

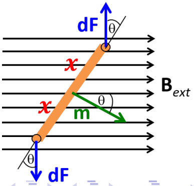
  $|\underline{\mathbf{T}}| = I\, a^2 B \sin\theta \int_0^{\pi} (1 - \cos 2\alpha)\;\, d\alpha$

  $= \pi a^2\, I\, B\, \sin\theta = I\, A\, B\, \sin\theta = mB\, \sin\theta$
- Result : torque on the magnetic dipole

  $\underline{\mathbf{T}} = \underline{\mathbf{m}} \times \underline{\mathbf{B}}_{\mathbf{ext}}$

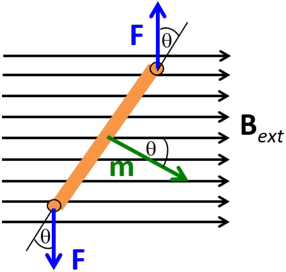






13.3 Energy of a magnetic dipole in a $\mathbf{B}$-field

The energy of a magnetic dipole placed in an magnetic field $\mathbf{B}_{\mathbf{ext}}$ is equal to the work done in rotating dipole into its position:



- Work to rotate dipole through angle $d\theta$
  $dW = T\, d\theta$
- Zero energy usually chosen at $\theta = \pi/2$
- $W = \int_{\pi/2}^{\theta} m\, B_{ext} \sin\theta'\, d\theta'$

  $= -\left[m\, B_{ext} \cos\theta\right]_{\pi/2}^{\theta}$
- Energy of the magnetic dipole

$$W = -m\, B_{ext} \cos\theta = -\mathbf{m} \cdot \mathbf{B}_{\mathbf{ext}}$$

[minimum at $\theta = 0$, maximum at $\theta = \pi$]

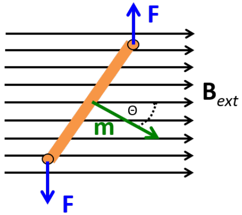
Compare with the energy of an electric dipole

$$W_{elec} = -\mathbf{p} \cdot \mathbf{E}_{\mathbf{ext}}$$

Magnetic dipole summary

Magnetic dipole moment *m* of a current loop = current × area of the loop: $\mathbf{m} = I\,\mathbf{A}$

Magnetic flux density of a magnetic dipole: $B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3}$ $B_\theta = \mu_0 \frac{m\sin\theta}{4\pi r^3}$ $B_\phi = 0$



Torque on a magnetic dipole in an external magnetic field $\mathbf{B}_{ext}$:

$$\mathbf{T} = I\,\mathbf{A} \times \mathbf{B}_{ext} = \mathbf{m} \times \mathbf{B}_{ext}$$

Energy of a magnetic dipole in an external magnetic flux density $\mathbf{B}_{ext}$:

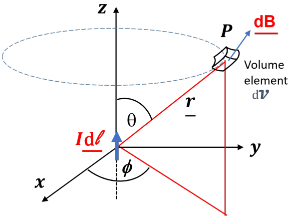
$$W = -mB_{ext}\cos\theta = -\mathbf{m}\cdot\mathbf{B}_{ext}$$

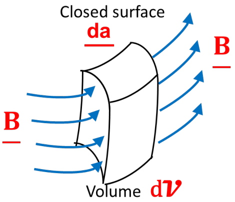
13.4 Divergence of $\underline{\mathbf{B}}$

- Place a current element $I\,\underline{\mathbf{d}\ell}$ at the origin pointing along the z-axis
- The Biot-Savart Law gives the field at point $P \rightarrow$ $\underline{\mathbf{dB}} = \mu_0\, I\, \frac{\underline{\mathbf{d}\ell}\times\hat{\underline{\mathbf{r}}}}{4\pi\, r^2}$
- $\underline{\mathbf{dB}}$ is perpendicular to $\underline{\mathbf{r}}$ and $\underline{\hat{\mathbf{z}}}$
- Rotate $\underline{\mathbf{r}}$ around ϕ, and it can be seen the lines of $\underline{\mathbf{B}}$ are *circles* in planes perpendicular to $\underline{\mathbf{d}\ell}$ and centred on it

  $\rightarrow$ the net outward flux of $\underline{\mathbf{B}}$ due to $\underline{\mathbf{d}\ell}$ through the surface of the volume element $d\mathcal{V}$ is zero
  - *Any* volume can be made up of volume elements as $d\mathcal{V}$
  - Hence $\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0$ $\rightarrow$ no magnetic monopoles.
  - Divergence Theorem : $\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = \int_{\mathcal{V}} (\underline{\nabla} \cdot \underline{\mathbf{B}})\, d\mathcal{V} \rightarrow$

    $\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$

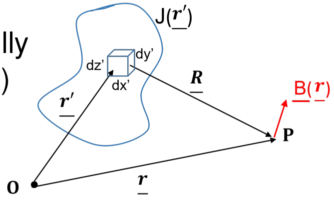




13.5 Divergence of $\underline{\mathbf{B}}$ from the Biot-Savart Law

Calculate $\underline{\mathbf{B}}$-field at point P due to a current density $\underline{\mathbf{J}}$.

$\underline{\mathbf{R}} = \underline{\mathbf{r}} - \underline{\mathbf{r}}' = (x - x', y - y', z - z')$



- $\underline{\mathbf{B}} = \int_{\mathcal{V}} \mu_0 \frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \hat{\underline{\mathbf{R}}}}{4\pi R^2} d\mathcal{V}'$
  where $d\mathcal{V}' = dx' dy' dz'$. (Note carefully the primed and unprimed coordinates)
- $\underbrace{\underline{\nabla} \cdot \underline{\mathbf{B}}}_{\text{w.r.t. } \underline{\mathbf{r}}} = \frac{\mu_0}{4\pi} \int \underline{\nabla} \cdot \left(\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right) d\mathcal{V}'$
- Using the product rule :
  - $\underline{\nabla} \cdot \left(\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right) = \frac{\hat{\underline{\mathbf{R}}}}{R^2} \cdot \underbrace{\left(\underline{\nabla} \times \underline{\mathbf{J}}(\underline{\mathbf{r}}') \right)}_{=0 \text{ (because } \underline{\mathbf{J}}(\underline{\mathbf{r}}') \text{ does not depend on } \underline{\mathbf{r}})} - \underline{\mathbf{J}}(\underline{\mathbf{r}}') \cdot \left(\underline{\nabla} \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right)$
  - $\underline{\nabla} \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} = \underline{\nabla} \times \frac{\underline{\mathbf{R}}}{R^3} = \frac{1}{R^3} \underbrace{(\underline{\nabla} \times \underline{\mathbf{R}})}_{=0} + \underbrace{\underbrace{\underline{\nabla} \left(\frac{1}{R^3} \right)}_{\text{Vector along } \underline{\mathbf{R}}} \times \underline{\mathbf{R}}}_{=0}$
  - Hence $\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$ and $\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0$