CP2 ELECTROMAGNETISM

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LECTURE 13:

MAGNETIC DIPOLES & THE DIVERGENCE OF B

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 13. MAGNETIC DIPOLES & THE DIVERGENCE OF B

13.1 Magnetic dipole components

13.2 Torque on a magnetic dipole in a B-field

13.3 Energy of a magnetic dipole in a B-field

13.4 Divergence of B

13.5 Divergence of B from the Biot-Savart Law

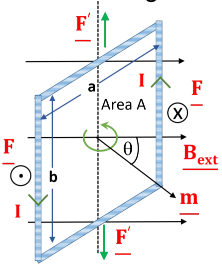
13.1 Magnetic dipole components

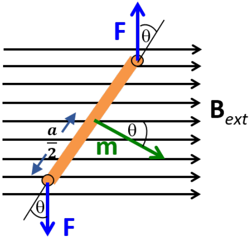
Magnetic dipole moment $\underline{\mathbf{m}} = I\,\underline{\mathbf{A}}$
= [Current] $\times$ [Area bounded by the loop]

The magnetic and electric dipole components have exactly the same form. (The exact derivation of the magnetic field components are beyond the scope of this course.)

13.2 Torque on a magnetic dipole in a $\underline{\mathbf{B}}$-field

Calculate the torque on a current loop placed in an external magnetic field:





(*) This has been done for a rectangular shape. But note that this is a general result for *any* shape. Any current loop can be built from infinitesimal rectangular loops with all the internal currents cancelling, yet contributing to the overall moment.

Torque on a magnetic dipole, continued

- Do the explicit calculation for a circular current loop:
- Only the vertical component of $\underline{\mathbf{d}\ell}$ results in a torque $\rightarrow\ d\ell\,\sin\alpha$
- Torque due to facing elements $\underline{\mathbf{d}\ell}$:

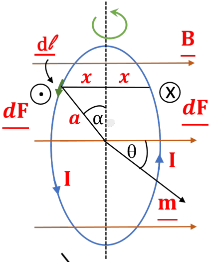
  $|d\underline{\mathbf{T}}| = 2|\underline{\mathbf{x}} \times d\underline{\mathbf{F}}| = 2x(I\,d\ell B \sin\alpha)\sin\theta$
- $x = a\sin\alpha\ ;\ \ell = a\alpha\ \rightarrow\ d\ell = a\,d\alpha$

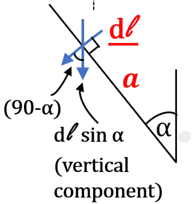
  $|d\underline{\mathbf{T}}| = 2\,(I\,a^2 \sin^2\alpha)\,B\sin\theta\,d\alpha$
- Hence

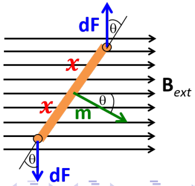
  $|\underline{\mathbf{T}}| = I\,a^2 B\sin\theta \int_0^{\pi} (1 - \cos 2\alpha)\ d\alpha$

  $= \pi a^2\,I\,B\,\sin\theta = I\,A\,B\,\sin\theta = mB\,\sin\theta$
- Result : torque on the magnetic dipole

  $\underline{\mathbf{T}} = \underline{\mathbf{m}} \times \underline{\mathbf{B}}_{\mathbf{ext}}$

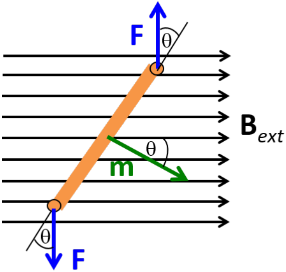






13.3 Energy of a magnetic dipole in a $\underline{\mathbf{B}}$-field

The energy of a magnetic dipole placed in an magnetic field $\underline{\mathbf{B}}_{\mathbf{ext}}$ is equal to the work done in rotating dipole into its position:

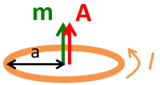


Compare with the energy of an electric dipole

$$W_{elec} = -\underline{\mathbf{p}} \cdot \underline{\mathbf{E}}_{\mathbf{ext}}$$

Magnetic dipole summary

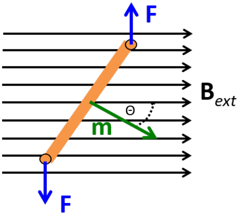
Magnetic dipole moment *m* of a current loop = current × area of the loop: $\mathbf{m} = I\,\mathbf{A}$



$A = \pi a^2$

Magnetic flux density of a magnetic dipole:

$$B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3} \qquad B_\theta = \mu_0 \frac{m\sin\theta}{4\pi r^3} \qquad B_\phi = 0$$



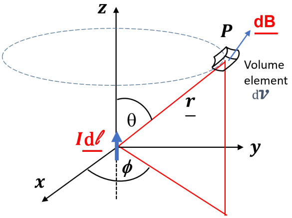
Torque on a magnetic dipole in an external magnetic field $\mathbf{B}_{ext}$:

$$\mathbf{T} = I\,\mathbf{A} \times \mathbf{B}_{ext} = \mathbf{m} \times \mathbf{B}_{ext}$$

Energy of a magnetic dipole in an external magnetic flux density $\mathbf{B}_{ext}$:

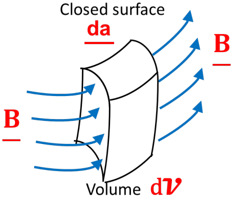
$$W = -mB_{ext}\cos\theta = -\mathbf{m}\cdot\mathbf{B}_{ext}$$

13.4 Divergence of $\underline{\mathbf{B}}$



- Place a current element $I\,\underline{\mathbf{d}\ell}$ at the origin pointing along the z-axis
- The Biot-Savart Law gives the field at point $P \rightarrow\ \underline{\mathbf{dB}} = \mu_0\, I\, \frac{\underline{\mathbf{d}\ell}\times\underline{\hat{\mathbf{r}}}}{4\pi\, r^2}$
- $\underline{\mathbf{dB}}$ is perpendicular to $\underline{\mathbf{r}}$ and $\underline{\hat{\mathbf{z}}}$
- Rotate $\underline{\mathbf{r}}$ around ϕ, and it can be seen the lines of $\underline{\mathbf{B}}$ are *circles* in planes perpendicular to $\underline{\mathbf{d}\ell}$ and centred on it

  $\rightarrow$ the net outward flux of $\underline{\mathbf{B}}$ due to $\underline{\mathbf{d}\ell}$ through the surface of the volume element $d\mathcal{V}$ is zero



  - *Any* volume can be made up of volume elements as $d\mathcal{V}$
  - Hence $\oint_S \underline{\mathbf{B}}\cdot\underline{\mathbf{da}} = 0$ $\rightarrow$ no magnetic monopoles.
  - Divergence Theorem : $\oint_S \underline{\mathbf{B}}\cdot\underline{\mathbf{da}} = \int_{\mathcal{V}} (\underline{\nabla}\cdot\underline{\mathbf{B}})\, d\mathcal{V} \rightarrow$

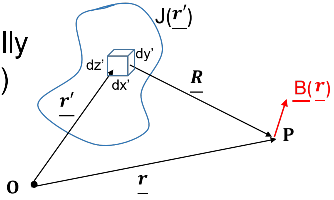
$$\underline{\nabla}\cdot\underline{\mathbf{B}} = 0$$

13.5 Divergence of $\underline{\mathbf{B}}$ from the Biot-Savart Law

Calculate $\underline{\mathbf{B}}$-field at point P due to a current density $\underline{\mathbf{J}}$.

$\underline{\mathbf{R}} = \underline{\mathbf{r}} - \underline{\mathbf{r}}' = (x - x', y - y', z - z')$

- $\underline{\mathbf{B}} = \int_{\mathcal{V}} \mu_0 \frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \hat{\underline{\mathbf{R}}}}{4\pi R^2} d\mathcal{V}'$
  where $d\mathcal{V}' = dx'dy'dz'$. (Note carefully the primed and unprimed coordinates)



- $\underbrace{\underline{\nabla} \cdot \underline{\mathbf{B}}}_{\text{w.r.t. } \underline{\mathbf{r}}} = \frac{\mu_0}{4\pi} \int \underline{\nabla} \cdot \left(\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right) d\mathcal{V}'$
- Using the product rule :
  - $\underline{\nabla} \cdot \left(\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right) = \frac{\hat{\underline{\mathbf{R}}}}{R^2} \cdot \underbrace{\left(\underline{\nabla} \times \underline{\mathbf{J}}(\underline{\mathbf{r}}') \right)}_{=0} - \underline{\mathbf{J}}(\underline{\mathbf{r}}') \cdot \left(\underline{\nabla} \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right)$
    = 0 (because $\underline{\mathbf{J}}(\underline{\mathbf{r}}')$ does not depend on $\underline{\mathbf{r}}$)
  - $\underline{\nabla} \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} = \underline{\nabla} \times \frac{\underline{\mathbf{R}}}{R^3} = \frac{1}{R^3} \underbrace{\left(\underline{\nabla} \times \underline{\mathbf{R}} \right)}_{=0} + \underbrace{\underbrace{\underline{\nabla} \left(\frac{1}{R^3} \right)}_{\text{Vector along } \underline{\mathbf{R}}} \times \underline{\mathbf{R}}}_{=0}$
  - Hence $\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$ and $\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0$