CP2 ELECTROMAGNETISM

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LECTURE 12:

BIOT SAVART LAW & THE MAGNETIC DIPOLE

Neville Harnew[1]

University of Oxford

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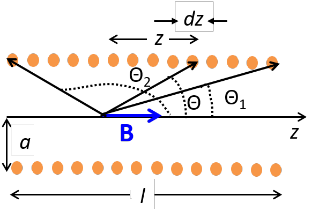
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 12. BIOT SAVART LAW & THE MAGNETIC DIPOLE

12.1 Example : B-field of a solenoid

Calculate the B-field due to a solenoid with current I, radius a, length ℓ with N turns. Sum over all contributions from all loops at a distance z (integrate from θ_1 to θ_2).



- Contribution from one element dz: $dB = \frac{\mu_0}{2a}\sin^3\theta\, dI$ where $dI = I\,(\frac{N}{\ell})\,dz$ along the axis of the solenoid.
- $\tan\theta = \frac{a}{z} \;\rightarrow\; z = \frac{\cos\theta}{\sin\theta}\,a$

  $\rightarrow\; dz = -a\,\frac{1}{\sin^2\theta}\,d\theta$

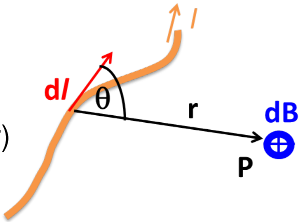
- $B = -\int_{\theta_1}^{\theta_2}\frac{\mu_0}{2a}\sin^3\theta\,\frac{IN}{\ell}\left(a\,\frac{1}{\sin^2\theta}d\theta\right) = -\frac{\mu_0 I\,N}{2\ell}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$
- Hence $B = \frac{\mu_0 I\,N}{2\ell}(\cos\theta_2 - \cos\theta_1)$
- For a long coil $\theta_1 = 0,\; \theta_2 = \pi \;\rightarrow\;$ $B = -\mu_0\,I\,\frac{N}{\ell}$

  (sign depends on direction of current $\rightarrow$ RH screw rule)

12.2 Biot-Savart Law in terms of current density

- The Biot-Savart Law :

$$\underline{\mathbf{dB}} = \mu_0 \, I \, \frac{\underline{\mathbf{d\ell}} \times \hat{\mathbf{r}}}{4\pi \, r^2}$$



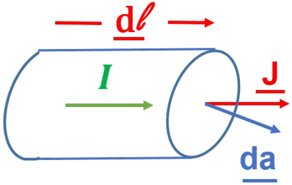
- Define *current density* $\underline{\mathbf{J}}$:

$$I = \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$$

$\underline{\mathbf{J}}$ is the current per unit area (a vector)

$$\underline{\mathbf{J}} = \frac{dI}{da_\perp} \times \left(\frac{\underline{\mathbf{d\ell}}}{|\underline{\mathbf{d\ell}}|} \right)$$

$da_\perp$ is the area perpendicular to the flow of current



- Also since $\underline{\mathbf{J}} \parallel \underline{\mathbf{d\ell}}$

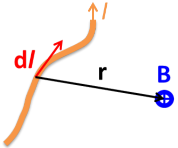
$$I \, \underline{\mathbf{d\ell}} = (\underline{\mathbf{J}} \cdot \underline{\mathbf{da}}) \, \underline{\mathbf{d\ell}} = \underline{\mathbf{J}} \, (\underline{\mathbf{da}} \cdot \underline{\mathbf{d\ell}}) = \underline{\mathbf{J}} \, d\mathcal{V}$$

- Hence

$$\underline{\mathbf{B}} = \int_{\mathcal{V}} \mu_0 \, \frac{\underline{\mathbf{J}} \times \hat{\mathbf{r}}}{4\pi \, r^2} \, d\mathcal{V}$$

Biot-Savart Law in terms of current density $\underline{\mathbf{J}}$ integrated over volume $\mathcal{V}$

Biot-Savart Law summary

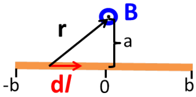


The magnetic flux density **B** created by a current loop is given by:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int I\,\frac{\mathrm{d}l \times \hat{\mathbf{r}}}{r^2} \quad \overset{\text{or}}{\longleftrightarrow} \quad \mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\times\mathbf{r}}{r^3}\,\mathrm{d}^3 r$$

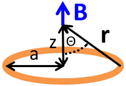
Biot-Savart Law

Straight wire.



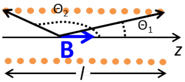
$$\mathbf{B} = \frac{\mu_0 I}{2\pi a}\,\frac{1}{\left(a^2/b^2+1\right)^{1/2}}$$

Circular loop.



$$\mathbf{B} = \frac{\mu_0 I a^2}{2\sqrt{z^2+a^2}^{\,3}} = \frac{\mu_0 I}{2a}\sin^3\theta$$

Solenoid.



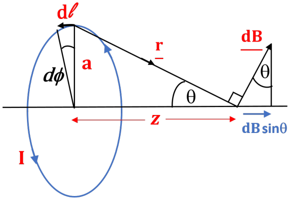
$$\mathbf{B} = \frac{\mu_0 I N}{2l}\left(\cos\theta_2 - \cos\theta_1\right)$$

12.3 The magnetic dipole

A small current loop defines a *magnetic dipole*

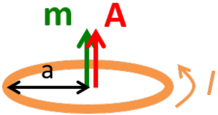
- Re-visit the field due to a circular current loop :

$$\underline{\mathbf{B}} = \frac{\mu_0\, I\, a^2}{2\left(z^2+a^2\right)^{\frac{3}{2}}}\,\underline{\hat{\mathbf{z}}}$$



In terms of loop area: $\underline{\mathbf{B}} = \dfrac{2\,\mu_0 I\,(\pi\, a^2)}{4\pi\, r^3}\,\underline{\hat{\mathbf{z}}}$

- Compare this with the on-axis field of the *electric* dipole (i.e. for $\theta = 0$) which has the same form :



Electric dipole : $\quad \underline{\mathbf{E}}_{\mathbf{r}} = \dfrac{2qd\cos\theta}{4\pi\epsilon_0 r^3}\,\underline{\hat{\mathbf{r}}} \;\rightarrow\; \underline{\mathbf{E}}_{\mathbf{z}} = \dfrac{2p}{4\pi\epsilon_0 r^3}\,\underline{\hat{\mathbf{z}}} \quad (p = q\,d)$

- Define $(I\,\pi\, a^2) = I\,A$ as the *magnetic dipole moment m*

Magnetic dipole moment $\underline{\mathbf{m}} = I\,\underline{\mathbf{A}}$
= [Current] $\times$ [Area bounded by the loop]