CP2 ELECTROMAGNETISM

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LECTURE 11:

MAGNETOSTATICS & THE BIOT-SAVART LAW

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 11. MAGNETOSTATICS & THE BIOT-SAVART LAW

11.1 The Biot-Savart Law for calculating magnetic fields

11.2 Example : the B-field of a straight wire

11.3 Example : force between 2 current-carrying wires

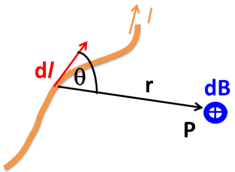
11.4 Example : B-field of a circular current loop

11.1 The Biot-Savart Law for calculating magnetic fields

The Biot-Savart is here taken as an empirical starting point for calculation of magnetic fields, but can be derived from Maxwell's equations and the magnetic potential (see later).

- The Biot-Savart Law states the field at point P:

$$\underline{\mathbf{dB}} = \mu_0\, I\, \frac{\underline{\mathbf{d}\ell} \times \underline{\hat{\mathbf{r}}}}{4\pi\, r^2}$$

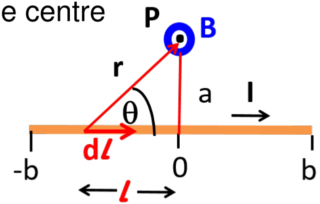


- $\mu_0 = 4\pi \times 10^{-7}$ $\mathrm{N\,A^{-2}}$ permeability of free space
- $\underline{\mathbf{dB}}$ is the magnetic flux density contribution at P
- I is the current flowing through element $\underline{\mathbf{d}\ell}$
- $\underline{\mathbf{r}}$ is the vector connecting $\underline{\mathbf{d}\ell}$ and P
- $\underline{\mathbf{dB}}$ is oriented perpendicular to $\underline{\mathbf{r}}$ and the current

Then integrate $\underline{\mathbf{dB}}$ to get *total* field from a circuit which has current

11.2 Example : the B-field of a straight wire

Calculate the B-field due to a straight wire with current I, length $2b$, at a distance a from the centre

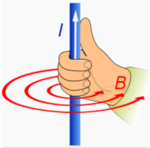


- At point P : $\mathbf{\underline{dB}} = \mu_0 \, I \, \frac{\mathbf{\underline{d\ell}} \times \mathbf{\underline{\hat{r}}}}{4\pi \, r^2}$
- Use $r^2 = a^2 + \ell^2$ and

  $|\mathbf{\underline{d\ell}} \times \mathbf{\underline{\hat{r}}}| = d\ell \sin\theta = d\ell \, \frac{a}{r}$

  $\rightarrow \; dB = \frac{\mu_0 \, I}{4\pi} \frac{a}{(a^2+\ell^2)^{\frac{3}{2}}} \, d\ell$

- Direction given by right hand screw rule.



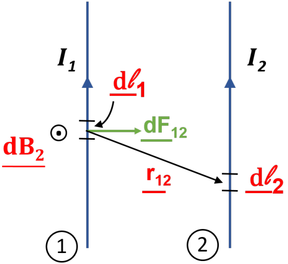
  - $B = \frac{\mu_0 \, I}{4\pi} \int_{-b}^{b} \frac{a}{(a^2+\ell^2)^{\frac{3}{2}}} \, d\ell$

    $\rightarrow \; B = \frac{\mu_0 \, I}{4\pi} \left[\frac{\ell / a}{(a^2+\ell^2)^{\frac{1}{2}}} \right]_{-b}^{b} \rightarrow$ $B = \frac{\mu_0 \, I \, b}{2\pi \, a} \frac{1}{(a^2+b^2)^{\frac{1}{2}}}$

  - For an infinite straight wire ($b \rightarrow \infty$) $B = \frac{\mu_0 \, I}{2\pi \, a}$

11.3 Example : force between 2 current-carrying wires

Two wires: force on small element of wire 1 from magnetic field of small element of wire 2

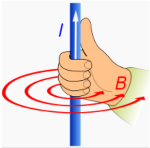


- $\underline{\mathbf{dF}}_{12} = I_1\, \underline{\mathbf{d}\ell}_1 \times \underline{\mathbf{dB}}_2$
- At point on wire 1, magnetic field element $\underline{\mathbf{dB}}_2$ from wire 2 :

  $$\underline{\mathbf{dB}}_2 = \frac{\mu_0\, I_2}{4\pi\, r_{21}^3}\, \underline{\mathbf{d}\ell}_2 \times \underline{\mathbf{r}}_{21}$$

  $$\rightarrow\ \underline{\mathbf{dF}}_{12} = \frac{\mu_0\, I_1\, I_2}{4\pi\, r_{12}^3}\, [-\underline{\mathbf{d}\ell}_1 \times (\underline{\mathbf{d}\ell}_2 \times \underline{\mathbf{r}}_{12})]$$

  (negative since $\underline{\mathbf{r}}_{12} = -\underline{\mathbf{r}}_{21}$)



- Force between 2 current-carrying wires :

$$\underline{\mathbf{F}}_{12} = \int_{\ell_1} \int_{\ell_2} \frac{\mu_0\, I_1\, I_2}{4\pi\, r_{12}^3}\, [\underline{\mathbf{d}\ell}_2 \times (\underline{\mathbf{d}\ell}_1 \times \underline{\mathbf{r}}_{12})]$$

$\cdots$ *and if the wires are parallel and infinite*

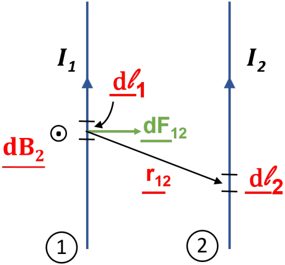
If wires are infinite, separated by distance a, currents I_1 and I_2

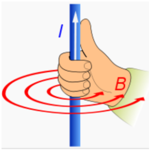
- $\underline{\mathbf{dF}}_{12} = I_1\, \underline{\mathbf{d}\ell}_1 \times \underline{\mathbf{B}}_2$
- From BS Law, from before, $|\underline{\mathbf{B}}_2| = \dfrac{\mu_0 I_2}{2\pi a}$
- Force on element $\underline{\mathbf{d}\ell}_1$:

  $|\underline{\mathbf{dF}}_{12}| = I_1\, |\underline{\mathbf{d}\ell}_1| \frac{\mu_0 I_2}{2\pi a}$ towards wire 2
- Due to the symmetry, force on every element is the same along the wire
- Hence *force per unit length* on wire 1 :

$$\frac{|\underline{\mathbf{dF}}_{12}|}{|\underline{\mathbf{d}\ell}_1|} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

(and note that $\underline{\mathbf{dF}}_{12} = -\underline{\mathbf{dF}}_{21}$)



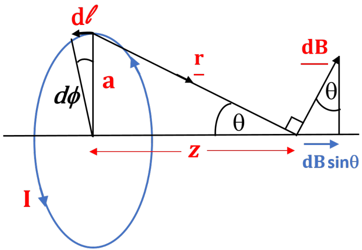


11.4 Example : B-field of a circular current loop

Calculate the B-field due to a circular wire with current I, radius a, at a distance z along its axis from the centre

- Field due to $\underline{\mathbf{d}\ell}$: $\underline{\mathbf{dB}} = \mu_0 \, I \, \frac{\underline{\mathbf{d}\ell} \times \underline{\hat{\mathbf{r}}}}{4\pi \, r^2}$
- $|\underline{\mathbf{d}\ell} \times \underline{\hat{\mathbf{r}}}| = d\ell$, since $\underline{\mathbf{r}} \perp \underline{\mathbf{d}\ell}$
- Components of $\underline{\mathbf{dB}}$ perpendicular to z-axis cancel due to symmetry $\rightarrow$ field is along the z-axis

  $\rightarrow \;\; B = \int dB \sin\theta = \int \frac{a}{r} \, dB$



- $B = \int \frac{\mu_0 \, I}{4\pi \, r^2} \, \frac{a}{r} \, d\ell$ along $\underline{\hat{\mathbf{z}}}$
  - a and r both constant for given point. $\int d\ell = 2\pi \, a$
  - Hence $$B = \frac{\mu_0 \, I \, a^2}{2(z^2 + a^2)^{\frac{3}{2}}}$$
  - Or since $\sin\theta = \frac{a}{\sqrt{(z^2 + a^2)}}$, $B = \frac{\mu_0 \, I}{2a} \sin^3\theta$