CP2 ELECTROMAGNETISM

https://users.physics.ox.ac.uk/~harnew/lectures/

LECTURE 10:

CAPACITANCE, ENERGY & MAGNETOSTATICS

Neville Harnew[1]

University of Oxford

HT 2022

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

OUTLINE : 10. CAPACITANCE, ENERGY & MAGNETOSTATICS

10.1 Force between capacitor plates (2 cases)

10.2 Energy density of the electric field

10.3 Example : hollow spherical shell

10.4 Principle of superposition for energy density

10.5 Origins of magnetism

10.6 Magnetostatics terminology

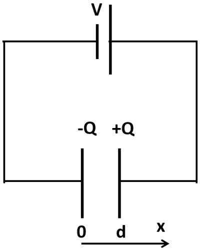
10.7 Forces on current-carrying wires in magnetic fields

10.8 The Lorenz force

10.9 Example : measuring B field

10.1 Force between capacitor plates (2 cases)

- Capacitor plates are oppositely charged $\rightarrow$ an attractive force F exists between them.
- By pulling the plates apart we perform work on the capacitor / battery system



Work done in pulling apart : $W = -\int \underline{\mathbf{F}} \cdot \underline{\mathbf{dx}}$

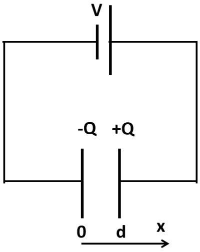
Energy stored in capacitor : $U_C = \frac{1}{2} Q^2 / C$

Energy stored in the battery : $U_B = V\,Q$

1. Pull apart at constant charge: battery disconnected, $dU_B = 0$
   - Force between plates : $F = -\frac{\partial U_C}{\partial x}\Big|_{Q\ const.} = -\frac{1}{2} Q^2 \frac{\partial}{\partial x}\left(\frac{1}{C}\right)$
   - For a parallel plate capacitor $\frac{1}{C} = \frac{x}{\epsilon_0 A}$
   - Hence $F = -\frac{1}{2} Q^2 \frac{1}{\epsilon_0 A}$
   - Mechanical work required to move plates from separation d_1 to d_2 : $W = -\int_{d_1}^{d_2} \underline{\mathbf{F}} \cdot \underline{\mathbf{dx}} = \frac{1}{2} Q^2 \frac{1}{\epsilon_0 A} (d_2 - d_1)$

Force between capacitor plates continued

2. Plates pulled apart at constant voltage (which is supplied by the battery)



- $F = -\frac{\partial U_{total}}{\partial x}\Big|_{V\ const.} = -\frac{\partial}{\partial x}(\underbrace{\frac{1}{2}V^2 C}_{\text{capacitor}} - \underbrace{V^2 C}_{\text{battery}})$
- $F = \frac{1}{2}V^2 \frac{\partial C}{\partial x}$ where $C = \epsilon_0 A/x$

$$F = -\frac{1}{2}V^2\,\epsilon_0\,A/x^2$$

- Mechanical work required to move plates from separation d_1 to d_2 : $W = -\int_{d_1}^{d_2} \underline{\mathbf{F}} \cdot \underline{\mathbf{dx}}$

$$W = \frac{1}{2}V^2\,\epsilon_0\,A\,(\frac{1}{d_1} - \frac{1}{d_2}) = \frac{1}{2}V^2\,(C_1 - C_2)$$

- Pulling plates apart leaves the capacitance lowered, charge returns to the battery, work is performed on the capacitor/battery system.

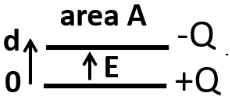
10.2 Energy density of the electric field

- Consider parallel plate capacitor

  $U_C = \frac{1}{2} C V^2$; $E = \frac{V}{d}$; $C = \epsilon_0 A/d$

- Hence $U_C = \frac{1}{2}\epsilon_0 \frac{A}{d} E^2 d^2$

  $= \frac{1}{2}\epsilon_0 E^2 \underbrace{A\,d}_{\text{volume}}$



  - Energy density in between the plates :

    $U_\rho = U_C/[\text{unit volume}] = \frac{1}{2}\epsilon_0 E^2$

  - This is actually a *general result* for *any* region in space in an $\underline{\mathbf{E}}$ field. The volume can be made arbitrarily small :

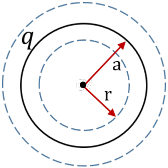
    $dU = \frac{1}{2}\epsilon_0 E^2 d\mathcal{V} \leftarrow$ volume element

  - Hence $U = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E^2 d\mathcal{V}$ over all space in the general case.

10.3 Example : hollow spherical shell

Example : Energy of hollow spherical shell carrying charge q

$$\begin{cases} E = 0 \quad \textit{for}\, 0 \le r < a\, (\textit{inside}) \\ E = \frac{q}{4\pi\epsilon_0 r^2} \quad \textit{for}\, a \le r\, (\textit{radial, as point charge}) \end{cases}$$



- $U = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E^2 d\mathcal{V}$ over all space

$$= \frac{1}{2}\epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} \frac{q^2}{16\pi^2\epsilon_0^2 r^4} \underbrace{r^2 dr \sin\theta\, d\theta\, d\phi}_{\text{volume element}}$$

$$= \frac{1}{2}\epsilon_0 4\pi \frac{q^2}{16\pi^2\epsilon_0^2}\int_a^{\infty}\frac{1}{r^2}\,dr$$

$$U = \frac{q^2}{8\pi\epsilon_0}\frac{1}{a}$$

- Alternative approach : energy required to bring up charge dq from infinity against potential $V(q)$ is $dW = V(q)\,dq$

$$W = \int_0^q V(q')\,dq' = \int_0^q \frac{q'}{4\pi\epsilon_0}\frac{1}{a}\,dq' = \frac{q^2}{8\pi\epsilon_0}\frac{1}{a}$$

which is the same result as above.

10.4 Principle of superposition for energy density

Question: does the principle of superposition apply to energy density?

- Principle of superposition : $\underline{\mathbf{E}} = \underline{\mathbf{E}}_\mathbf{1} + \underline{\mathbf{E}}_\mathbf{2}$
- $U = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E^2 \, d\mathcal{V} = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} (\underline{\mathbf{E}}_\mathbf{1} + \underline{\mathbf{E}}_\mathbf{2})^2 \, d\mathcal{V}$

$$= \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E_1^2 \, d\mathcal{V} + \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E_2^2 \, d\mathcal{V} + \epsilon_0 \int_{\mathcal{V}} \underline{\mathbf{E}}_\mathbf{1} \cdot \underline{\mathbf{E}}_\mathbf{2} \, d\mathcal{V}$$

$$= U_1 + U_2 + \epsilon_0 \int_{\mathcal{V}} \underline{\mathbf{E}}_\mathbf{1} \cdot \underline{\mathbf{E}}_\mathbf{2} \, d\mathcal{V}$$

- Therefore the answer is no !

MAGNETOSTATICS - OVERVIEW

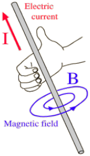
1. Introduction: Origins of Magnetism
2. Forces on Current-Carrying Wires in Magnetic Fields
3. The Biot-Savart Law (B-fields of Wires, Solenoids, etc.)
4. Magnetic Dipoles
5. Ampere's Law & Gauss' Law of Magnetostatics
6. Current Density and the Continuity Equation

} Problem Set 3

10.5 Origins of magnetism

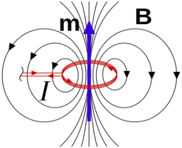
Minerals found in ancient Greek city Magnesia ("Magnetite", Fe_3O_4) attract small metal objects.

Materials containing certain atoms such as Iron (Fe), Cobalt (Co), Nickel (Ni) can exhibit "permanent" magnetic dipoles.



Forces exist between pairs of current-carrying wires (attractive for current flowing in the same, repulsive for current flowing in opposite directions).

An electric current through a wire creates a magnetic field whose field lines loop around the wire.



Magnetic field lines form closed loops. They do not originate from "magnetic monopoles".

The fundamental generators of magnetic fields are dipoles that may result from electrical current loops or inherent material properties such as aligned angular momenta of charged particles.

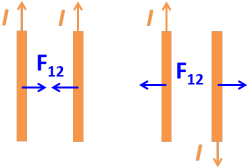
10.6 Magnetostatics terminology

- Magnetic flux density $\underline{\mathbf{B}}$ (“B-field”) [B] = T (Tesla)
- Magnetic field (strength) $\underline{\mathbf{H}} = \frac{1}{\mu_0}\underline{\mathbf{B}}$ (in non magnetic materials) $[H] = A\, m^{-1}$

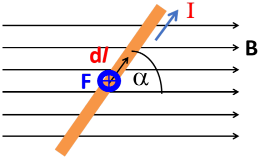
  where $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2} (permeability of free space)

10.7 Forces on current-carrying wires in magnetic fields

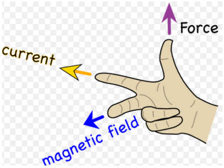
Experimental observations :



1. Two wires attract (repel) one another if they carry current in the same (opposite) directions.

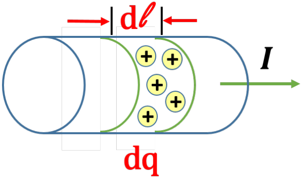


2. A current-carrying wire in a magnetic field, flux density *B*, experiences a force with :
   - $F \propto B$
   - $F \propto I$ (current in wire)
   - $F \propto \ell$ (length of wire)
   - $F \propto \sin\alpha$ (α is angle between the direction of $\underline{\mathbf{B}}$ and $\underline{\mathbf{I}}$)
   - $\underline{\mathbf{F}}$ is oriented perpendicular to both $\underline{\mathbf{B}}$ and the wire
   - $dF = I\,B\,d\ell\,\sin\alpha \;\rightarrow\; \underline{\mathbf{dF}} = I\,\underline{\mathbf{d\ell}} \times \underline{\mathbf{B}}$



10.8 The Lorenz force

- Force on current-carrying wire in a B-field: $\underline{\mathbf{dF}} = I\,\underline{\mathbf{d}\ell} \times \underline{\mathbf{B}}$



- Zoom into a wire segment, assume it's the (+) charge moving ("conventional" current)
  - $I = \frac{dq}{dt}$ and $|\underline{\mathbf{v}}| = \left|\frac{\mathbf{d}\ell}{dt}\right|$ (average velocity of charge)

    $\rightarrow\ I = \frac{dq}{d\ell}\cdot\frac{d\ell}{dt} = v\,\frac{dq}{d\ell}$: Vectorizing $\ I\,\underline{\mathbf{d}\ell} = \underline{\mathbf{v}}\,dq$

    $\rightarrow\ \underline{\mathbf{dF}} = dq\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$
  - Lorenz force $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$
  - Any charge q moving with velocity $\underline{\mathbf{v}}$ in a magnetic flux density $\underline{\mathbf{B}}$ experiences a Lorenz force $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$ perpendicular to both
  - Work done on the moving charge $dW = -\underline{\mathbf{F}}\cdot\underline{\mathbf{d}\ell} = -q\,(\underline{\mathbf{v}} \times \underline{\mathbf{B}})\cdot\underline{\mathbf{v}}\,dt = 0$
  - Magnetic fields do no work

10.9 Example : measuring $\underline{\mathbf{B}}$ field

From torque on a wire loop carrying current I in field $\underline{\mathbf{B}}$:

- From diagram, torque on coil about O when $|\theta| > 0$:

  $\underline{\mathbf{T}} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$ from current in sides b (sides a, forces $\underline{\mathbf{F}}'$ cancel)
- $|\underline{\mathbf{T}}| = 2 \times \frac{a}{2} \sin\theta \underbrace{I\, b\, B}_{\text{Force}}$ ($\underline{\mathbf{F}}$ is $\perp$ to I is $\perp$ to $\underline{\mathbf{B}}$)
- $|\underline{\mathbf{T}}| = I\, B\, A \sin\theta$ (A is the area of the loop)
- Measure $T \rightarrow$ obtain B