CP2 PRELIMS LECTURES

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ELECTROMAGNETISM

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$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

[1] With thanks to Prof Laura Herz

CP2 ELECTROMAGNETISM LECTURES

Lecture 1

Introduction to the Course

1.1 Syllabus of the Course

1. Electrostatics

Coulomb's law. The electric field E and potential due to a point charge and systems of point charges, including the electric dipole. The couple and force on, and the energy of, a dipole in an external electric field. Energy of a system of point charges; energy stored in an electric field. Gauss' Law; the E field and potential due to surface and volume distributions of charge (including simple examples of the method of images), no field inside a closed conductor. Force on a conductor. The capacitance of parallel-plate, cylindrical and spherical capacitors, energy stored in capacitors.

2. Magnetostatics

The forces between wires carrying steady currents. The magnetic field B, Ampere's law, Gauss' Law ("no magnetic monopoles"), the Biot-Savart Law. The B field due to currents in a long straight wire, in a circular loop (on axis only) and in straight and toroidal solenoids. The magnetic dipole; its B field. The force and couple on, and the energy of, a dipole in an external B field. Energy stored in a B field. The force on a charged particle in E and B fields.

3. Induction

Electromagnetic induction, the laws of Faraday and Lenz. EMFs generated by an external, changing magnetic field threading a circuit and due to the motion of a circuit in an external magnetic field, the flux rule. Self and mutual inductance: calculation for simple circuits, energy stored in inductors. The transformer.

4. Electromagnetic waves

Charge conservation, Ampere's law applied to a charging capacitor, Maxwell's addition to Ampere's law ("displacement current"). Maxwell's equations for fields in a vacuum (rectangular coordinates only). Plane electromagnetic waves in empty space: their speed; the relationships between **E**, **B** and the direction of propagation.

1.2 Structure of the Course

1. Electrostatics

Charges create "electric fields" which represent the resulting force experienced by a small test charge.

$$\oint_S \mathbf{E} \, . \, \mathbf{da} = \frac{Q}{\varepsilon_0} \quad \text{(closed surface)}$$

GAUSS LAW

2. Magnetostatics

Electrical currents create "magnetic fields" which create forces on moving test charges. There are no magnetic monopoles.

AMPERE'S CURRENT LAW

$$\frac{1}{\mu_0} \oint \boldsymbol{B} \, . \, \boldsymbol{dl} = I$$

$$\oint_S \boldsymbol{B} \, . \, \boldsymbol{da} = 0 \quad \text{(closed surface)}$$

3. Induction

A time-varying magnetic flux through an area creates an electromotive force along the area's rim.

$$\oint \mathbf{E} \, . \, \boldsymbol{dl} = -\frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{B} \, . \, \mathbf{da}$$

FARADAY / LENS LAW

4. Electromagnetic waves

A time-varying electric flux through an area creates an magnetic field along the area's rim.

$$\frac{1}{\mu_0} \oint \mathbf{B} \, . \, \boldsymbol{dl} = I + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{E} \, . \, \mathbf{da}$$

Electromagnetic wave propagation

1.3 Book list

Introductory undergraduate textbooks on electromagnetism:

D. J. Griffiths, *Introduction to Electromagnetism*
Pearson, 4th edition, ISBN: 978 0 321 84781 2

I. S. Grant and W. R. Phillips, *Electromagnetism*
John Wiley, 2nd edition, ISBN: 978 0 471 92712 9

E. M. Purcell and D. J. Morin, *Electricity and Magnetism*
Pearson, 4th edition, ISBN: 978 1 107 01402 2

P. Lorrain, D. R. Corson and F. Lorrain, Fundamentals of Electromagnetic Phenomena
Freeman, ISBN: 978 0 716 73568 7

Also of interest:

W. J. Duffin, *Electricity and Magnetism*
Duffin Publishing (out of print)

Feynman, Leighton, Sands, *The Feynman Lectures on Physics, Vol II*
ISBN: 978 0 465 02382 0

W. G. Rees, *Physics by Example*
Cambridge University Press, ISBN: 978 0 521 44975 5

Electrostatics - problem sheets

1.1. Introduction: Properties of charge; Coulomb's law

1.2. The Principle of Superposition

1.3. The Electric Field and Electrostatic Potential

1.4. Assemblies of discrete charges; multipoles

1.5. Continuous charge distributions

→ Problem Set 1 (1.1–1.5)

1.6. Gauss' law

1.7. Poisson and Laplace equations

1.8. The Method of Image Charges

→ Problem Set 2 (1.6–1.8)

1.9. Capacitance and Energy of the Electric Field

→ Problem Set 3 (1.9)

1.4 Electromagnetism through the years

Electrostatics

Ancient Greece: rubbing amber against fur allows it to attract other light substances such as dust or papyrus

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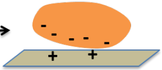


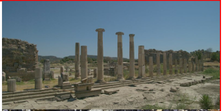
Greek word for "amber": *ἤλεκτρον* (*elektron*)

Magnetostatics

Magnesia (ancient Greek city in Ionia, today in Turkey): Naturally occurring minerals were found to attract metal objects (first references ~600BC).

Crystals are referred to as: Iron ore, Lodestone, Magnetite, Fe_3O_4

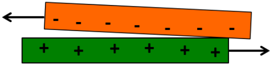
Use of Lodestone compass for navigation in medieval China



Electromagnetism through the years

17th century AD to mid 18th century:

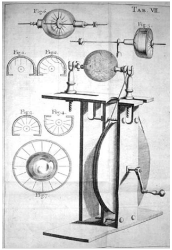
Dominated by "frictional electrostatics" :



- When two different materials are brought into contact, charge flows to equalize their electro-chemical potentials, bonds form across surface
- Separating them may lead to charge remaining unequally distributed when bonds are broken
- Rubbing enhances effect through repeat contact

Focus on "electrostatic generators" – today's van de Graaff Generators:

Machines involved frictional passage of "positive" materials such as hair, silk, fur, leather against "negative" materials such as amber, sulfur

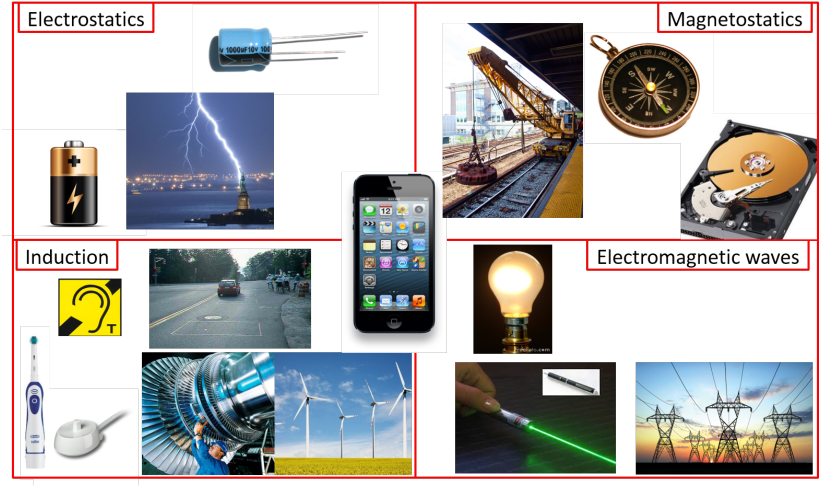


Electromagnetism through the years

From late 18th century: *Rapid progress on both fundamental science and technology:*

- 1784: Charles-Augustin de Coulomb uses "torsion balance" to show that forces between two charged spheres vary with the square of the inverse distance between them.
- 1800: Alessandro Volta constructs the first electrochemical battery (zinc/copper/sulfuric acid) allowing high-density electrical energy storage
- 1821: André-Marie Ampère investigates attractive and repulsive forces between current-carrying wires
- 1831-55: Michael Faraday discovers electromagnetic induction by experimenting with two co-axial coils of wire, wound around the same bobbin.
- 1830ies: Heinrich Lenz shows that induced currents have a direction that opposes the motions that produce them
- 1831: first commercial telegraph line, from Paddington Station to West Drayton
- 1864: James Clerk Maxwell introduces unified theory of electromagnetism, including a link to light waves
- 1887: Heinrich Hertz demonstrates the existence of electromagnetic waves in space
- Late 19th century: development of "wireless telegraphy" – radio!

Electromagnetism in everyday life



1.5 Summary of the properties of charge

- Both positive and negative charge exists (triboelectric experiments showed electrostatic attraction and repulsion)

- Charge is quantized (Millikan experiment, 1913): e=1.602×10^{-19} As

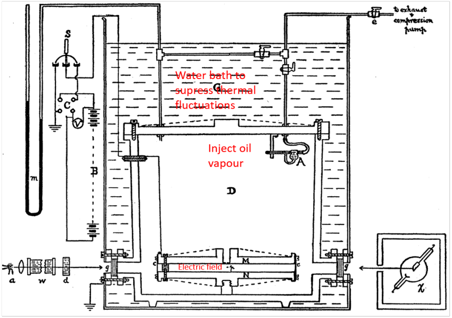
- Coulomb's law (1785): the force between two point charges varies with the square of their inverse distance:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

- Superposition: The force between two point charges varies linearly with the amount of each charge, hence the forces resulting from individual charges superimpose in an assembly of charges: $\mathbf{F} = \Sigma\, \mathbf{F}_i$

1.6 Properties of charge: Millikan Experiment

Millikan experiment, Phys. Rev. 2, 109–143 (1913):



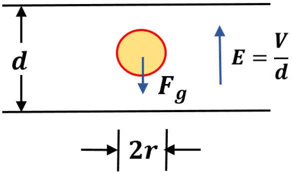
- Millikan oil drop experiment : observe small oil drops inside a parallel plate capacitor.
- Oil drops became electrically charged through friction with the nozzle as they are sprayed (or alternatively ionize with X-rays).
- Oil drop soon reaches terminal velocity due to friction with air.

Properties of charge: quantization

Stokes' Law : retarding frictional force on sphere moving in viscous fluid $\rightarrow$ $F_{Stokes} = 6\pi\eta r v_T$

[η = dynamic viscosity, r = sphere radius, v_T = terminal velocity]

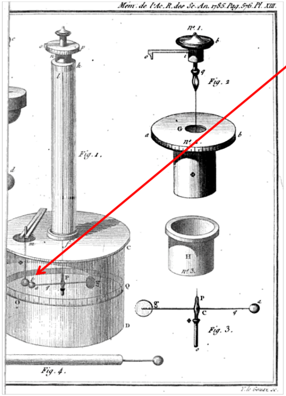
1. No E-field $F_g = F_{stokes}$: particle moving with V_T (measure)
   - $mg = \frac{4\pi}{3} r^3(\rho_{oil} - \rho_{air})g = 6\pi\eta r v_T$
   - Determine $r = \sqrt{\frac{9\eta v_T}{2(\rho_{oil}-\rho_{air})g}}$ and hence $F_g = 18\pi\eta v_T\sqrt{\frac{\eta v_T}{2(\rho_{oil}-\rho_{air})g}}$



2. Ramp E-field until particle levitates ($v = 0$, $F_{total} = 0$)
   - $F_g = qE$ $\rightarrow$ determine q
   - Millikan found : $q = Ne$ (N an integer) with $e = 1.592 \times 10^{-19}$C
   - Charge is quantized

1.7 Properties of charge: Coulomb's Law

Coulomb's Torsion Balance experiment, *Histoire de l'Academie Royale des Science*, p. 569-577 (1785):



Measure force between two charged spheres through torsion force on wire:

He found: $$F \propto \frac{1}{r^2}$$

Coulomb's law: $$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

$$\varepsilon_0 = 8.854 \times 10^{-12}\frac{As}{Vm}$$

The relative strength of the Coulomb force

- Coulomb 1785 : Magnitude of the force between two point charges q_1, q_2

  $F_{elec} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$

- Newton 1665 : Magnitude of the force between two point masses m_1, m_2

  $F_{grav} = G\frac{m_1 m_2}{r^2}$

- For two protons :

  $\frac{F_{grav}}{F_{elec}} = G \times 4\pi\epsilon_0 (\frac{m_p}{e})^2$

  $= 8 \times 10^{-37}$!!

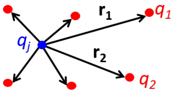
- The electrostatic force is many magnitudes stronger than the gravitational force.

1.8 The Principle of Superposition

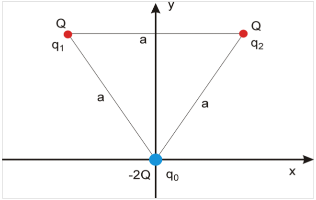
The principle of superposition states that, for all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually.

The force on charge q_j originating from all other charges q_i is given by:

$$\mathbf{F_j} = \frac{q_j}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i}{r_{ji}^2}\hat{\mathbf{r}}_{\mathbf{ji}}$$



Example 1: force on charge -2Q resulting from two charges Q in the corners of a triangle:



$$\mathbf{F} = \frac{\sqrt{3}Q^2}{2\pi\varepsilon_0 a^2}\hat{\mathbf{e}}_{\mathbf{y}}$$

Principle of Superposition

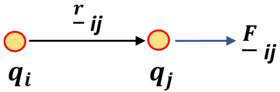
- Start with two charges q_i and q_j separated by $\underline{\mathbf{r}}_{\mathbf{ij}}$
  $\underline{\mathbf{F}}_{\mathbf{ij}}$ is the force on q_j due to q_i

  $$\rightarrow \ \underline{\mathbf{F}}_{\mathbf{ij}} = \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}^2}\,\hat{\underline{\mathbf{r}}}_{\mathbf{ij}} \quad \text{where} \quad \hat{\underline{\mathbf{r}}}_{\mathbf{ij}} = \underline{\mathbf{r}}_{\mathbf{ij}}/|\underline{\mathbf{r}}_{\mathbf{ij}}|$$

- Next go to three charges : total force on charge q_0

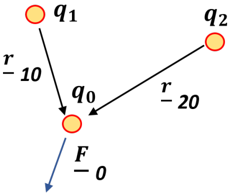
  $$\underline{\mathbf{F}}_{\mathbf{0}} = \underline{\mathbf{F}}_{\mathbf{10}} + \underline{\mathbf{F}}_{\mathbf{20}}$$

  $$\underline{\mathbf{F}}_{\mathbf{0}} = \frac{q_0 q_1}{4\pi\epsilon_0 r_{10}^2}\,\hat{\underline{\mathbf{r}}}_{\mathbf{10}} + \frac{q_0 q_2}{4\pi\epsilon_0 r_{20}^2}\,\hat{\underline{\mathbf{r}}}_{\mathbf{20}}$$



- In general :

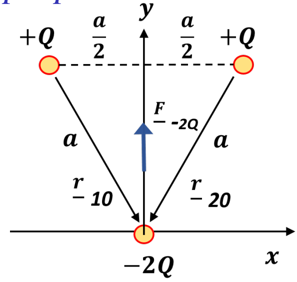
  $$\underline{\mathbf{F}}_{\mathbf{j}} = \frac{1}{4\pi\epsilon_0}\sum_{i(i\neq j)} \frac{q_i q_j}{r_{ij}^2}\,\hat{\underline{\mathbf{r}}}_{\mathbf{ij}}$$



- Principle of Superposition also works for $\dfrac{\underline{\mathbf{F}}_{\mathbf{j}}}{q_j} = \dfrac{1}{4\pi\epsilon_0}\sum_{i(i\neq j)} \dfrac{q_i}{r_{ij}^2}\,\hat{\underline{\mathbf{r}}}_{\mathbf{ij}}$

  This is a vector field that only depends on the distribution of other charges : the *electic field* generated by the other charges.

Example 1 : principle of superposition



- Three charges arranged at corners of an equilateral triangle, $+Q$ and $+Q$ on top, $-2Q$ on the bottom. Calculate the force on $-2Q$.
- $\underline{\mathbf{F}}_{-2\mathbf{Q}} = \frac{-2Q \cdot Q}{4\pi\epsilon_0 r_{10}^2}\,\hat{\underline{\mathbf{r}}}_{10} + \frac{-2Q \cdot Q}{4\pi\epsilon_0 r_{20}^2}\,\hat{\underline{\mathbf{r}}}_{20}$

  $\rightarrow\ \underline{\mathbf{F}}_{-2\mathbf{Q}} = \frac{-Q^2}{2\pi\epsilon_0 a^2}(\hat{\underline{\mathbf{r}}}_{10} + \hat{\underline{\mathbf{r}}}_{20})$
- Now $\hat{\underline{\mathbf{r}}}_{10} = \frac{1}{\mathbf{a}}\begin{pmatrix} +a/2 \\ -\sqrt{a^2 - a^2/4} \end{pmatrix}$; $\hat{\underline{\mathbf{r}}}_{20} = \frac{1}{\mathbf{a}}\begin{pmatrix} -a/2 \\ -\sqrt{a^2 - a^2/4} \end{pmatrix}$
- $\hat{\underline{\mathbf{r}}}_{10} + \hat{\underline{\mathbf{r}}}_{20} = \frac{1}{\mathbf{a}}\begin{pmatrix} +a/2 - a/2 \\ -\sqrt{3}a/2 - \sqrt{3}a/2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$
- Hence $\underline{\mathbf{F}}_{-2\mathbf{Q}} = \frac{\sqrt{3}Q^2}{2\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- This is as expected : there is only a y component of $\underline{\mathbf{F}}$ due to symmetry.

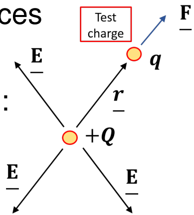
Lecture 2

The Electric Field and Potential

2.1 The Electric Field

- The electric field at point $\underline{\mathbf{r}}$, generated by a distribution of charges q_i is defined as the force per unit charge that a test charge would experience if placed at $\underline{\mathbf{r}}$.

  $\rightarrow$ a point test charge q due to a field $\underline{\mathbf{E}}$ experiences a force $\underline{\mathbf{F}} = q \cdot \underline{\mathbf{E}} = \frac{q \cdot Q}{4\pi\epsilon_0 r^2}\,\underline{\hat{\mathbf{r}}}$



- Electric field due to a point charge Q at the origin: always points away from $+$ charge (radial)

  $\underline{\mathbf{E}} = \underline{\mathbf{F}}/q = \frac{Q}{4\pi\epsilon_0 r^2}\,\underline{\hat{\mathbf{r}}}$

- The principle of superposition holds for the electric field : the electric field generated by a distribution of charges is equal to the sum of the electric fields generated by the individual charges.

2.2 The Electrostatic Potential

- Work done to move a point test charge q from A to B

  $W_{AB} = -\int_{\underline{\mathbf{r}}_\mathbf{A}}^{\underline{\mathbf{r}}_\mathbf{B}} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}}\ell = -q \int_{\underline{\mathbf{r}}_\mathbf{A}}^{\underline{\mathbf{r}}_\mathbf{B}} \underline{\mathbf{E}} \cdot \underline{\mathbf{d}}\ell$
- The electrostatic potential difference between A and B is defined as the the work done to move a unit charge between A and B
- Since the work done is related linearly to the electric field, the principle of superposition also holds for the work done.
- Note that *any* field configuration can be made up of the sum of infinitesimal point charges. So we can now check that the work done by moving a test charge through the electric field of a *point* charge does not depend on the path taken.

The Electrostatic Potential

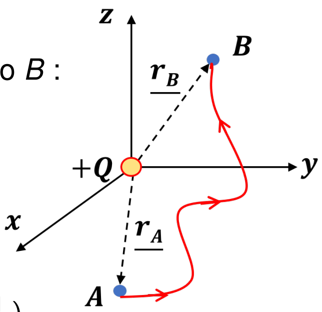
For a point charge charge Q :

- $\underline{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\underline{\hat{\mathbf{r}}}$
- Work done to move charge q from A to B :

  $W_{AB} = -q\int_A^B \underline{\mathbf{E}}\cdot\underline{\mathbf{d}\ell}$
- In spherical coordinates :

  $\underline{\mathbf{d}\ell} = dr\,\underline{\hat{\mathbf{r}}} + r\,d\theta\underline{\hat{\theta}} + r\sin\theta d\phi\underline{\hat{\phi}}$
- Hence $\underline{\mathbf{E}}\cdot\underline{\mathbf{d}\ell} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}dr$

  $W_{AB} = -q\int_A^B \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = q\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$



- Hence energy required to move test charge from A to B depends only on initial and final radial separation, and independent of path.
- Electric field is conservative

2.3 The Potential Difference

Define electrostatic potential difference

$$V_{AB} = \frac{W_{AB}}{q} = -\int_A^B \underline{\mathbf{E}} \cdot \underline{\mathbf{d}}\ell = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

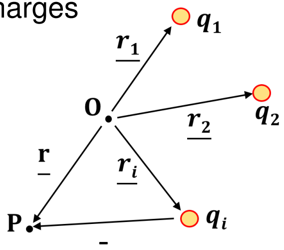
- The potential of a point charge Q at a *general* point $\underline{\mathbf{r}}$ is given by : $V(\underline{\mathbf{r}}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r_A}\right)$
  here the second term is a constant (which is often set to zero by taking $V(r \to \infty) = 0$)
- Again, since $\underline{\mathbf{E}}$ and V are linearly related, the Principle of Superposition also holds for V.

Potential at point P due to an assembly of charges

- $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_\mathbf{i}|} +$ constant

The field due to the assembly :

- $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{(\underline{\mathbf{r}}-\underline{\mathbf{r}}_\mathbf{i})^2}\widehat{\underline{\mathbf{r}}-\underline{\mathbf{r}}_\mathbf{i}}$
  where $\widehat{\underline{\mathbf{r}}-\underline{\mathbf{r}}_\mathbf{i}} = \frac{\underline{\mathbf{r}}-\underline{\mathbf{r}}_\mathbf{i}}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_\mathbf{i}|}$



Summary of Relationship between Electric Field and Potential

The electric field **E** at a point **r**, generated by a distribution of charges q_i, is equal to the force **F** per unit charge q that a small test charge q would experience if it was placed at **r**:

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q}$$

The electric potential V at a point **r** is the energy W required per unit charge q to move a small test charge q from a reference point to **r**. For a system of charges:

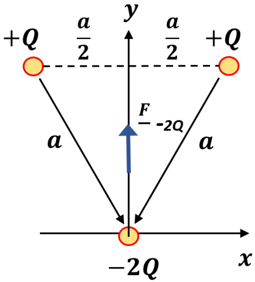
$$V(\mathbf{r}) = \frac{W(\mathbf{r})}{q}$$

The electric field and potential are related through:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \quad \longleftrightarrow \quad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

2.4 Calculating the field from the potential

Revisit example 1 : Calculate the electric field generated by charges $+Q$ and $+Q$, located on an equilateral triangle, and felt by "test charge" $-2Q$ at the origin.



- Reminder from before, force on $-2Q$

$$\underline{\mathbf{F}}_{-\mathbf{2Q}} = \frac{\sqrt{3}Q^2}{2\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- Field $\underline{\mathbf{E}}$ at $\underline{\mathbf{r}} = (0,0,0)$ = force/unit charge

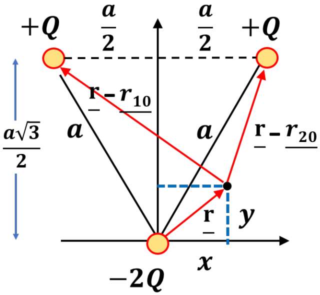
$$\underline{\mathbf{E}} = \frac{1}{-2Q}\frac{\sqrt{3}Q^2}{2\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\frac{\sqrt{3}Q}{4\pi\epsilon_0 a^2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- But now adopt a different approach : derive the electric field from $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = -\underline{\nabla} V$ and evaluate $\underline{\mathbf{E}}$ at the origin.

1. Calculate the potential due to Q and Q at a position $\underline{\mathbf{r}}$

- $$V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_{10}|} + \frac{Q}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_{20}|}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{\left[(a/2+x)^2+(a\sqrt{3}/2-y)^2+z^2\right]^{1/2}} + \frac{1}{\left[(a/2-x)^2+(a\sqrt{3}/2-y)^2+z^2\right]^{1/2}}\right\}$$



2. Derive $\underline{\mathbf{E}}(\underline{\mathbf{r}})$ from $V(\underline{\mathbf{r}})$:

$$\underline{\mathbf{E}} = -\underline{\nabla} V = -\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} V$$

- $$\underline{\mathbf{E}} = -\frac{Q}{4\pi\epsilon_0}\times$$

$$\left\{\frac{-1/2}{\left((a/2+x)^2+(a\sqrt{3}/2-y)^2+z^2\right)^{3/2}}\begin{pmatrix} 2(a/2+x) \\ -2(a\sqrt{3}/2-y) \\ 2z \end{pmatrix} + \frac{-1/2}{\left((a/2-x)^2+(a\sqrt{3}/2-y)^2+z^2\right)^{3/2}}\begin{pmatrix} -2(a/2-x) \\ -2(a\sqrt{3}/2-y) \\ 2z \end{pmatrix}\right\}$$

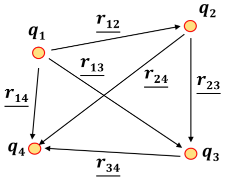
3. Calculate the field at position $\underline{\mathbf{r}} = (0, 0, 0)$

- $$\underline{\mathbf{E}} = +\frac{Q}{4\pi\epsilon_0} \times \left\{ \frac{1}{(a^2/4+3a^2/4)^{3/2}} \begin{pmatrix} a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} + \frac{1}{(a^2/4+3a^2/4)^{3/2}} \begin{pmatrix} -a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} \right\}$$
  $$= \frac{Q}{4\pi\epsilon_0 a^3} \left\{ \begin{pmatrix} a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -a/2 \\ -\sqrt{3}a/2 \\ 0 \end{pmatrix} \right\}$$
- Hence $\underline{\mathbf{E}} = -\frac{Q\sqrt{3}}{4\pi\epsilon_0 a^2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

  Which is identical to the previous calculation using vector sum over fields.

2.5 Energy of a system of charges

- Calculate the energy to bring i charges up from infinity whilst keeping all the other charges fixed in space



$$U = \text{the first charge } q_1 : \text{none}$$

$$+ \text{the second charge } q_2 : q_2\left(\frac{q_1}{4\pi\epsilon_0 r_{12}}\right)$$

$$+ \text{the third charge } q_3 : q_3\left(\frac{q_1}{4\pi\epsilon_0 r_{13}} + \frac{q_2}{4\pi\epsilon_0 r_{23}}\right)$$

$$+ \text{the fourth charge } q_4 : q_4\left(\frac{q_1}{4\pi\epsilon_0 r_{14}} + \frac{q_2}{4\pi\epsilon_0 r_{24}} + \frac{q_3}{4\pi\epsilon_0 r_{34}}\right)$$

- $+$ etc, up to the i^{th} charge
- Compare to W, the sum over potential energies experienced by *each* charge from *all other* charges:

$$W = \sum_i q_i \sum_{j(\neq i)} \frac{q_j}{4\pi\epsilon_0 r_{ij}}$$

In matrix form :

- $U =$

$$(q_1\ q_2\ q_3 \cdots q_{i-1} q_i) \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1/r_{12} & 0 & \cdots & 0 & 0 \\ 1/r_{13} & 1/r_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/r_{1,i-1} & 1/r_{2,i-1} & \cdots & 0 & 0 \\ 1/r_{1i} & 1/r_{2i} & \cdots & 1/r_{i-1,i} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \cdots \\ q_{i-1} \\ q_i \end{pmatrix}$$

and where

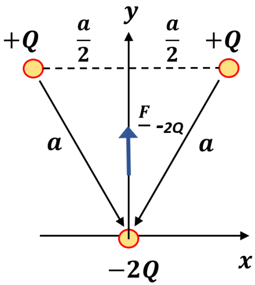
$W =$

$$(q_1\ q_2\ q_3 \cdots q_{i-1} q_i) \begin{pmatrix} 0 & 1/r_{21} & \cdots & 1/r_{i-1,1} & 1/r_{i1} \\ 1/r_{12} & 0 & \cdots & 1/r_{i-1,2} & 1/r_{i2} \\ 1/r_{13} & 1/r_{23} & \cdots & 1/r_{i-1,3} & 1/r_{i3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/r_{1,i-1} & 1/r_{2,i-1} & \cdots & 0 & 1/r_{i,i-1} \\ 1/r_{1i} & 1/r_{2i} & \cdots & 1/r_{i-1,i} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \cdots \\ q_{i-1} \\ q_i \end{pmatrix}$$

- Hence $U = \frac{1}{2} W = \sum_i \frac{1}{2} q_i V_i$ where $V_i = \sum_{j(\neq i)} \frac{q_j}{4\pi\epsilon_0 r_{ij}}$
- The energy U required to assemble a system of charges from infinity (whilst keeping all charges fixed in space) is half the energy of the sum W over potential energies experienced by each charge from all other charges.

Energy to assemble the system in Example 1

1. Charge $-2Q$ in potential of Q & Q
   $$q_1 V_1 = -2Q\left(\frac{Q}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0 a}\right) = -\frac{Q^2}{\pi\epsilon_0 a}$$
2. Charge Q in potential of Q & $-2Q$
   $$q_2 V_2 = Q\left(\frac{Q}{4\pi\epsilon_0 a} + \frac{-2Q}{4\pi\epsilon_0 a}\right) = -\frac{Q^2}{4\pi\epsilon_0 a}$$
3. Charge Q in potential of $-2Q$ & Q
   $$q_2 V_2 = q_3 V_3 = -\frac{Q^2}{4\pi\epsilon_0 a} \quad \text{(symmetry)}$$



- $U = \frac{1}{2}\sum q_i V_i = \frac{1}{2} \times \frac{-Q^2}{\pi\epsilon_0 a}(1 + 2 \times \frac{1}{4})$
  $= -\frac{3Q^2}{4\pi\epsilon_0 a}$
- Negative, since predominantly attractive forces.

2.6 Summary: assembly of discrete charge systems

The Electric field **E** and Potential V of a distribution of point charges q_i placed at positions $\mathbf{r_i}$ are:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{(\mathbf{r}-\mathbf{r}_i)^2}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}$$

The energy U required to assemble a system of point charges q_i by bringing them to positions $\mathbf{r_i}$ from infinity is given by:

$$U = \frac{1}{8\pi\varepsilon_0}\sum_i q_i \sum_{i\neq j}\frac{q_j}{r_{ji}} = \frac{1}{2}\sum_i q_i V_i$$

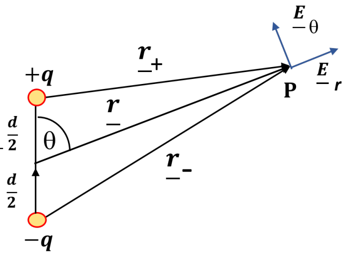
where V_i is the potential experienced by q_i at $\mathbf{r_i}$ from all *other* charges q_j.

Lecture 3

Electric Multipoles

3.1 The potential due to an electric dipole

- Two charges $+q$ and $-q$ separated by (small) distance d
- Define *dipole moment* : $\underline{\mathbf{p}} = q\,\underline{\mathbf{d}}$
- Potential at P : $V = \frac{q}{4\pi\epsilon_0 r_+} - \frac{q}{4\pi\epsilon_0 r_-}$
- Cosine rule : $r_{+/-} = \sqrt{r^2 + (d/2)^2 \mp dr\cos\theta}$



- $V = \frac{q}{4\pi\epsilon_0 r}\left(\frac{1}{\sqrt{1+(d/2r)^2-(d/r)\cos\theta}} - \frac{1}{\sqrt{1+(d/2r)^2+(d/r)\cos\theta}}\right)$
- Look at the field $d << r$:

  Expand : $\frac{1}{\sqrt{1+x}} = (1 - \frac{1}{2}x + \frac{1}{2!}(-\frac{1}{2})(-\frac{3}{2})x^2 + \cdots)$,
  retain terms only up to first order of d/r
- $V = \frac{q}{4\pi\epsilon_0 r}\left((1 + (d/2r)\cos\theta) - (1 - (d/2r)\cos\theta)\right)$

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\underline{\mathbf{p}}\cdot\underline{\mathbf{r}}}{4\pi\epsilon_0 r^3}$$

3.2 The field of an electric dipole

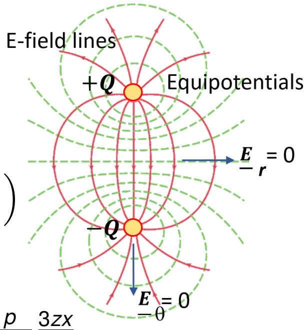
Use $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = -\underline{\nabla} V$; $\quad V = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}$

1. Spherical polar coordinates $\quad \underline{\nabla} \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2qd\cos\theta}{4\pi\epsilon_0 r^3} = \frac{2\underline{\mathbf{p}}\cdot\underline{\mathbf{r}}}{4\pi\epsilon_0 r^4}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{qd\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$



2. Cartesian coordinates $\quad \underline{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\underline{\mathbf{p}}\cdot\underline{\mathbf{r}} = pz \;;\; r = (x^2+y^2+z^2)^{\frac{1}{2}} = \frac{z}{\cos\theta}$$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{pz}{4\pi\epsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}}\right) = \frac{p}{4\pi\epsilon_0}\frac{3zx}{r^5}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{p}{4\pi\epsilon_0}\frac{3zy}{r^5}$$

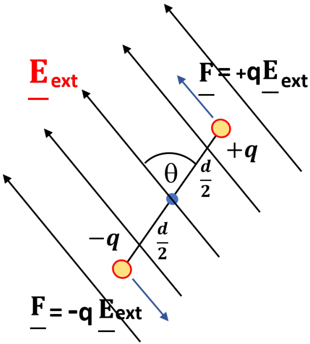
$$E_z = -\frac{\partial V}{\partial z} = -\frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) = \frac{p}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)$$

3.3 The torque on a dipole in an external E-field

- Torque (couple) on the dipole :
  $\underline{\mathbf{T}} = \sum_i \underline{\mathbf{r}}_\mathbf{i} \times \underline{\mathbf{F}}_\mathbf{i}$
- Taking moments about the centre point between the charges :

  $\underline{\mathbf{T}} = 2\left((\underline{\mathbf{d}}/2) \times q\underline{\mathbf{E}}_\mathbf{ext}\right) = q\underline{\mathbf{d}} \times \underline{\mathbf{E}}_\mathbf{ext}$
- $\underline{\mathbf{T}} = \underline{\mathbf{p}} \times \underline{\mathbf{E}}_\mathbf{ext}$
- Magnitude of torque from cross product :

  $|\underline{\mathbf{T}}| = |\underline{\mathbf{p}}|\,|\underline{\mathbf{E}}_\mathbf{ext}|\,\sin\theta$
- There is *only* a couple : no translational force.

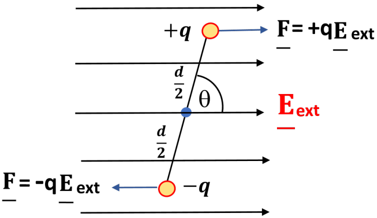


3.4 The energy of a dipole in an external $\mathbf{E}$-field

- Calculate the work done by an applied force to rotate the dipole from angle $\pi/2$ to θ

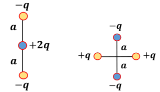
  (take $\theta = \pi/2$ as the zero of potential energy)
- $W = \int_{\frac{\pi}{2}}^{\theta} T d\theta' = \int_{\frac{\pi}{2}}^{\theta} p\, E_{ext} \sin\theta' d\theta'$
- $W = [-p\, E_{ext} \cos\theta']_{\pi/2}^{\theta}$

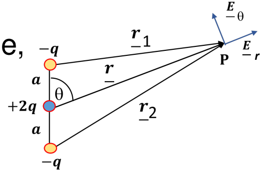
  $= -p\, E_{ext} \cos\theta$
- Hence potential energy of $\underline{\mathbf{p}}$ in $\underline{\mathbf{E}}_{\mathbf{ext}}$:



$$U = -p\, E_{ext} \cos\theta = -\underline{\mathbf{p}} \cdot \underline{\mathbf{E}}_{\mathbf{ext}}$$

3.5 The quadrupole potential





- Two charge configurations of the quadrupole, which both look identical at long distance
- Cosine rule : $r_{1/2} = \sqrt{r^2 + a^2 \mp 2ar\cos\theta}$
- Potential at P :

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} - \frac{q}{r\sqrt{1+(a/r)^2-2(a/r)\cos\theta}} - \frac{q}{r\sqrt{1+(a/r)^2+2(a/r)\cos\theta}}\right]$$

- Expand : $\frac{1}{r\sqrt{1+x}} = \frac{1}{r}\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right)$

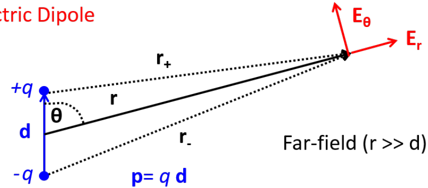
  Retain only up to powers of $(a/r)^2$

- $$V = \frac{1}{4\pi\epsilon_0 r}\left[2q - q\left\{1 - \frac{1}{2}(\frac{a}{r})^2 + \frac{a}{r}\cos\theta + \frac{3}{8}(\frac{a}{r})^2 4\cos^2\theta\right\}\right] - \left[q\left\{1 - \frac{1}{2}(\frac{a}{r})^2 - \frac{a}{r}\cos\theta + \frac{3}{8}(\frac{a}{r})^2 4\cos^2\theta\right\}\right]$$
  $$= \frac{1}{4\pi\epsilon_0 r}\left[+(\frac{a}{r})^2 - \frac{3}{4}(\frac{a}{r})^2 4\cos^2\theta\right]$$

- Quadrupole potential : $V = \frac{q\,a^2}{4\pi\epsilon_0 r^3}(1 - 3\cos^2\theta)$

Summary: Electric dipole and quadrupole

Electric Dipole

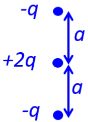


Far-field (r >> d) potential: $$V = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}$$

Far-field (r >> d) electric field: $$\mathbf{E}_r = \frac{2\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^4} \quad \mathbf{E}_\theta = \frac{qd\sin\theta}{4\pi\varepsilon_0 r^3} \quad \mathbf{E}_\phi = 0$$

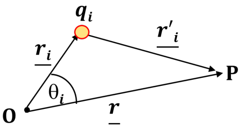
Energy of dipole in external electric field $\mathbf{E}_{ext}$: $$W = -\mathbf{E}_{\text{ext}}\cdot\mathbf{p}$$

Electric Quadrupole:



$$V = \frac{qa^2}{4\pi\varepsilon_0 r^3}(1 - 3\cos^2\theta)$$

3.6 The general multipole expansion



- Potential at P : $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\underline{\mathbf{r}}'_{\mathbf{i}}|}$

  where $\underline{\mathbf{r}}'_{\mathbf{i}} = \underline{\mathbf{r}} - \underline{\mathbf{r}}_{\mathbf{i}}$

- Cosine rule : $r'_i = \sqrt{r^2 + r_i^2 - 2\, r\, r_i \cos\theta_i}$

  $$= r\sqrt{1 - 2\frac{r_i \cos\theta_i}{r} + \frac{r_i^2}{r^2}} \equiv r\sqrt{1+x}$$

- For points P far from the charge assembly

  $r_i << r \;\; \rightarrow x << 1$

- Expand : $\frac{1}{r\sqrt{1+x}} = \frac{1}{r}\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right)$

  Retain only up to powers of $(r_i/r)^2$

- $\frac{1}{|\underline{\mathbf{r}}'_{\mathbf{i}}|} \approx \frac{1}{r}\left[1 + \frac{r_i\cos\theta_i}{r} - \frac{r_i^2}{2r^2} + \frac{3}{2}\frac{r_i^2}{r^2}\cos^2\theta_i + \cdots\right]$

  $$= \frac{1}{r} + \frac{r_i\cos\theta_i}{r^2} + \frac{r_i^2}{r^3}\frac{1}{2}\left(3\cos^2\theta_i - 1\right) + \cdots$$

The general multipole expansion

- $$V(\underline{\mathbf{r}}) = \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r}}_{\text{monopole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i r_i \cos\theta_i}{r^2}}_{\text{dipole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_i \frac{\frac{1}{2} q_i r_i^2 \left(3\cos^2\theta_i - 1\right)}{r^3}}_{\text{quadrupole term}} + \cdots$$
- So any assembly of charges can be described in terms of the sum over contributions from multipoles
- The n-th multipole potential falls off with $1/r^n$

Lecture 4

Continuous Charge Distributions

4.1 Continuous Charge Distributions

- Reminder : the potential at P , position vector $\underline{\mathbf{r}}$, due to assembly of charges :

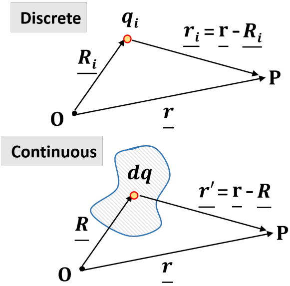
  $V(\underline{\mathbf{r}}) = \sum_i V_i(q_i) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\underline{\mathbf{r_i}}|}$

  (where $\underline{\mathbf{r_i}} = \underline{\mathbf{r}} - \underline{\mathbf{R_i}}$).

- And the field : $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{(\underline{\mathbf{r_i}})^2} \frac{\underline{\mathbf{r_i}}}{|\underline{\mathbf{r_i}}|}$

- For a continuous charge distribution $\sum_i V_i \rightarrow \int dV$

  Hence $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}|}$

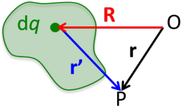


- Integrate over all infinitesimal dq over the charge distribution, noting that $q \equiv q(\underline{\mathbf{R}})$
- Alternatively $V(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})\, d\mathcal{V}}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}|}$ over volume $\mathcal{V}$, where $\rho(\underline{\mathbf{R}})$ is the charge density.
- Similarly for the electric field $\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})(\underline{\mathbf{r}} - \underline{\mathbf{R}})}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}|^3} d\mathcal{V}$

Continuous Charge Distributions

Continuous charge distribution:

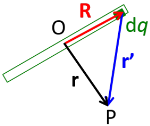
$$\sum V_i \rightarrow \int \mathrm{d}V$$



$$V = \int \frac{\mathrm{d}q}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{R}|}$$

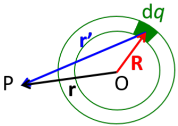
Line charge:

$dq = \lambda\, dl$



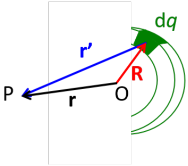
Surface charge:

$dq = \sigma\, dA$



Volume charge:

$dq = \rho\, dV$

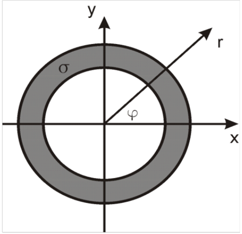


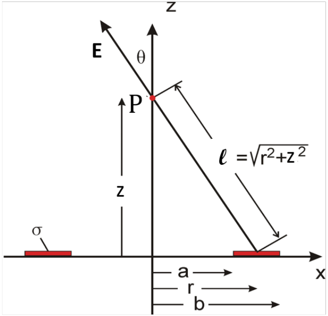
Choose a convenient origin O suiting the geometry of the charge distribution!

Adopt the notation: λ = charge density for 1D distribution, σ = charge density for 2D, ρ = charge density for 3D

4.2 Example 1 : uniformly charged annulus

Uniformly charged ring.





$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right]$$

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right]\hat{\mathbf{z}}$$

Uniformly charged annulus

Annulus contains charge q. Calculate the potential V on the annulus axis at a distance z above its centre. Note the radial symmetry.

- Charge density σ . Charge dq contained in infinitesimally thin ring of radius r :

  $\rightarrow \quad dq$ = area $\times$ charge density $= 2\pi r\, dr\, \sigma$

- Potential at P : $\quad V = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{dq}{\ell(r)}$

$$V = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{2\pi r\, dr\, \sigma}{\sqrt{r^2+z^2}}$$

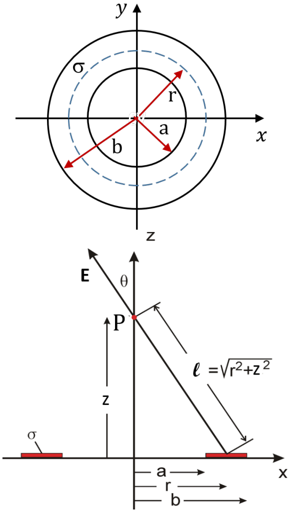
$$= \frac{\sigma}{2\epsilon_0} \int_a^b \frac{r\, dr}{\sqrt{r^2+z^2}}$$

$$V = \frac{\sigma}{2\epsilon_0} \sqrt{r^2+z^2}\Big|_a^b$$

$$= \frac{\sigma}{2\epsilon_0} \left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2} \right]$$

- From symmetry, E field points along z-axis: $\underline{\mathbf{E}} = -\underline{\hat{\mathbf{z}}}\, \frac{\partial}{\partial z} V(z)$

$$\underline{\mathbf{E}} = \frac{\sigma}{2\epsilon_0} \left\{ \frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}} \right\} \underline{\hat{\mathbf{z}}}$$

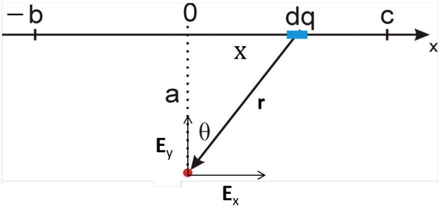


Special cases

1. $a = 0$ (disk)
   - $V = \frac{\sigma}{2\epsilon_0}\left[\sqrt{b^2+z^2} - z\right]$
   - $\underline{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{b^2+z^2}}\right]\hat{\underline{\mathbf{z}}}$
2. Disk ($a = 0$) for $z >> b$ (far away)
   - $V = \frac{\sigma}{2\epsilon_0}\left[z\sqrt{(b/z)^2+1} - z\right]$
     Use $\sqrt{1+(b/z)^2} \approx 1 + \frac{1}{2}(b/z)^2 + \cdots$
   - $V = \frac{\sigma}{2\epsilon_0}(z + \frac{b^2}{2z} - z) = \frac{\sigma b^2}{4\epsilon_0 z}$. But $\sigma = \frac{q}{\pi b^2}$:
   - Hence $V = \frac{q}{4\pi\epsilon_0 z}$ (point charge)
   - Using same method : $\underline{\mathbf{E}} = \frac{q}{4\pi\epsilon_0 z^2}\hat{\underline{\mathbf{z}}}$
3. Disk ($a = 0$) for $z << b$ (close to plate)
   - $V = \frac{\sigma}{2\epsilon_0}b$ & $\underline{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\hat{\underline{\mathbf{z}}}$ ("infinite" charged plane)

4.3 Example 2 : uniformly charged rod

Uniformly charged rod.

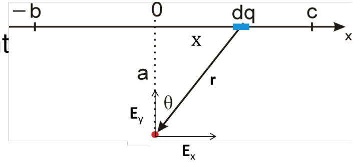


$$\mathbf{E}_x = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+c^2}} - \frac{1}{\sqrt{a^2+b^2}}\right]$$

$$\mathbf{E}_y = \frac{-\lambda}{4\pi\varepsilon_0 a}\left[\frac{c}{\sqrt{a^2+c^2}} + \frac{b}{\sqrt{a^2+b^2}}\right]$$

Uniformly charged rod

- Calculate the field $\underline{\mathbf{E}}$ at a distance a from a uniformly charged rod, with length between coordinates $-b$ and c.
- Charge dq contained in a small element dx , where $dq = \lambda dx$:
- $r = \sqrt{a^2 + x^2}$, $\hat{\underline{\mathbf{r}}} = \frac{1}{r}\begin{pmatrix} -x \\ -a \end{pmatrix}$



- $\underline{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \int_{-b}^{c} \frac{\hat{\underline{\mathbf{r}}}}{r^2} dq$

  Integrating components:

- $E_x = \frac{1}{4\pi\epsilon_0} \int_{-b}^{c} \frac{-x\lambda}{(a^2+x^2)^{\frac{3}{2}}} dx = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{a^2+c^2}} - \frac{1}{\sqrt{a^2+b^2}} \right]$

  $E_y = \frac{1}{4\pi\epsilon_0} \int_{-b}^{c} \frac{-a\lambda}{(a^2+x^2)^{\frac{3}{2}}} dx \quad \left[\text{use} \quad \frac{d}{dx} \frac{x}{a^2(a^2+x^2)^{\frac{1}{2}}} = \frac{1}{(a^2+x^2)^{\frac{3}{2}}} \right]$

- $E_y = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{-x/a}{(a^2+x^2)^{\frac{1}{2}}} \right]_{-b}^{c} = \frac{-\lambda}{4\pi\epsilon_0 a} \left[\frac{c}{\sqrt{a^2+c^2}} + \frac{b}{\sqrt{a^2+b^2}} \right]$
- $E_z = 0 \quad$ (symmetry)

Special cases

1. $b = c = \ell/2$
   - $E_x = 0$ (cancellation by symmetry)
   - $E_y = -\frac{\lambda}{4\pi\epsilon_0 a}\frac{\ell}{\sqrt{a^2+(\ell/2)^2}}$
2. $b = c \to \infty$
   - $E_x = 0$ (symmetry)
   - $E_y = -\frac{\lambda}{2\pi\epsilon_0 a}$ (radial field)

   - Note that this configuration is most easily solved via Gauss Law (see next lecture)

Lecture 5

Gauss Law

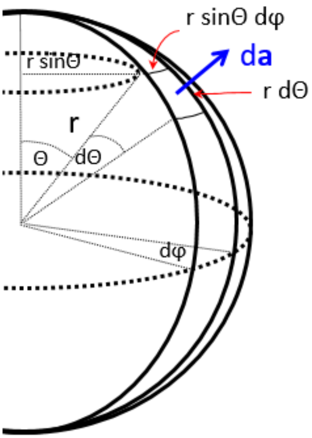
5.1 Introduction to solid angles

- Consider an element of area on a sphere. Define a vector of surface element $\underline{\mathbf{da}}$ normal to the surface :
- $\underline{\mathbf{da}} = (r \sin\theta \, d\phi) \times (r d\theta) \, \underline{\hat{\mathbf{r}}}$

  $\underline{\mathbf{da}} = r^2 \underbrace{\sin\theta \, d\theta \, d\phi} \, \underline{\hat{\mathbf{r}}}$
- Define $d\Omega = \sin\theta \, d\theta \, d\phi$ as a *solid angle* element.

  (note that $d\Omega$ is *independent* of r)
- Hence :

  $\int_{surface} d\Omega = \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi$

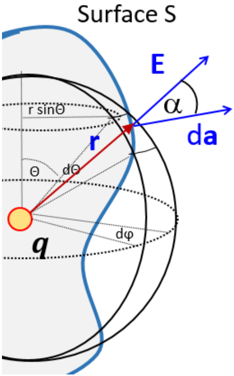


5.2 Gauss' Law

Calculate the *flux* $d\Phi_E = \underline{\mathbf{E}} \cdot \underline{\mathbf{da}}$ through an infinitesimal area $\underline{\mathbf{da}}$ of surface S at a distance r away from a point charge q

- $d\Phi_E = \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot da \cos\alpha$
- Note that $(da \cos\alpha)$ is the surface element $\underline{\mathbf{da}}$ of S resolved onto the sphere centred on charge q
- Hence $d\Phi_E = \left(\frac{q}{4\pi\epsilon_0}\frac{\hat{\underline{\mathbf{r}}}}{r^2}\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{\mathbf{r}}}\right)$

  $= \frac{q}{4\pi\epsilon_0} \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega}$ independent of r



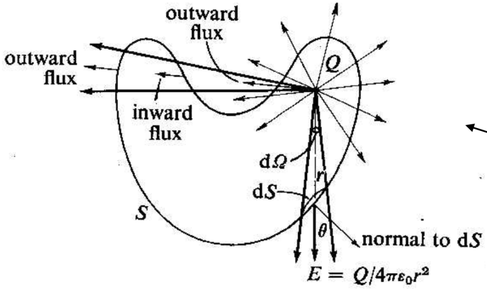
  - Therefore for any *closed* surface

    $\oint_{\substack{closed\\surface}} \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{q}{\epsilon_0} \oint \frac{d\Omega}{4\pi}$. Hence $\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{q}{\epsilon_0}$
  - It does not matter WHERE q is inside the surface for this to hold (because flux $d\Phi_E$ is independent of r) !

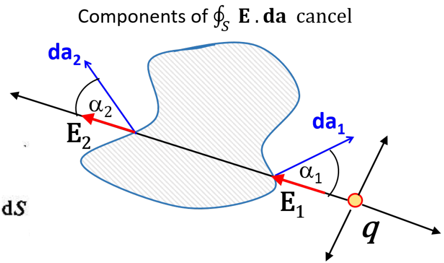
Gauss' Law

$$\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \begin{cases} \frac{q}{\epsilon_0} & \textit{if q is INSIDE closed surface} \\ 0 & \textit{if q is OUTSIDE closed surface} \end{cases}$$

INSIDE



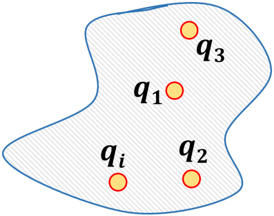
OUTSIDE



5.3 Gauss' Law for a collection of charges

- $\oint \underline{\mathbf{E_i}} \cdot \underline{\mathbf{da}} = \frac{q_i}{\epsilon_0}$ for any charge enclosed
- Apply the principle of superposition

  $\oint \sum_i \underline{\mathbf{E_i}} \cdot \underline{\mathbf{da}} = \frac{\sum_i q_i}{\epsilon_0}$ ($\sum_i \underline{\mathbf{E_i}}$ is the sum of field components on the surface)

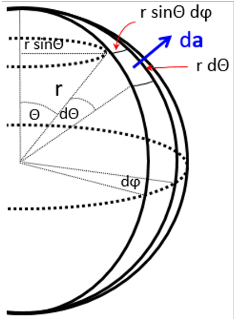


- Gauss Law : $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{Q_\mathcal{V}}{\epsilon_0}$

  $\underline{\mathbf{E}}$ is the field at surface S

  $Q_\mathcal{V} = \sum_i q_i$ is the total charge in the volume $\mathcal{V}$ enclosed by surface S
- For a continuous charge distribution, density ρ : $Q_\mathcal{V} = \int_\mathcal{V} \rho(\underline{\mathbf{r}}) d\mathcal{V}$
  - Gauss Law allows finding the total charge enclosed inside a closed surface if the field is known on the surface, and vice versa
  - Allows a straightforward calculation of field for symmetrical charge configurations

Gauss Law : summary



Area and Solid angle elements:

da = r^2 sinΘ dΘ dφ = r^2 dΩ

Calculate electric field flux dΦ through area **da** for a point charge q_i a distance *r* away from **da**:

$$d\Phi = \mathbf{E}_i \,.\, \mathbf{da} = \frac{q_i}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\cdot\hat{\mathbf{r}}\, r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{q_i}{4\pi\varepsilon_0}\underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega}$$

Independent of r !

Integrate over a closed surface:

$$\oint_S \mathbf{E_i} \,.\, \mathbf{da} = \frac{q_i}{\varepsilon_0}\frac{\oint d\Omega}{4\pi} = \begin{cases} \dfrac{q_i}{\varepsilon_0} & \text{if } q_i \text{ is enclosed} \\ 0 & \text{if } q_i \text{ is not enclosed} \end{cases}$$

Principle of superposition →

Gauss' Law;

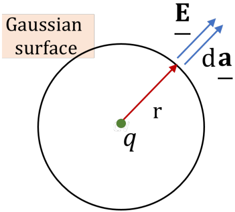
$$\oint_S \mathbf{E} \,.\, \mathbf{da} = \frac{Q}{\varepsilon_0}$$

E-field on closed surface

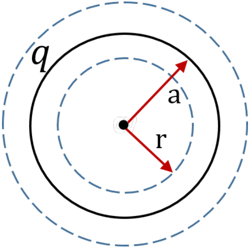
total charge enclosed

5.4 Example : Spherically symmetric charge distributions

1. Point charge
   - By symmetry : $\underline{\mathbf{E}} = E\,\hat{\underline{\mathbf{r}}}$
     $d\underline{\mathbf{a}} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\underline{\mathbf{r}}} = r^2\, d\Omega\, \hat{\underline{\mathbf{r}}}$
   - $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \oint_S r^2 d\Omega = E 4\pi r^2 = \frac{q}{\epsilon_0}$
   - $E = \frac{q}{4\pi\epsilon_0 r^2}$ as expected



2. Hollow sphere, radius a, with q evenly distributed on surface.
   - Inside sphere ($r < a$):
     $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 4\pi r^2 = \frac{q_{enclosed}}{\epsilon_0} = 0$
   - No charge enclosed $\rightarrow$ $E = 0$
   - Outside sphere ($r > a$):
     $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 4\pi r^2 = \frac{q_{enclosed}}{\epsilon_0} = \frac{q}{\epsilon_0}$
   - $E = \frac{q}{4\pi\epsilon_0 r^2}$ as for point charge



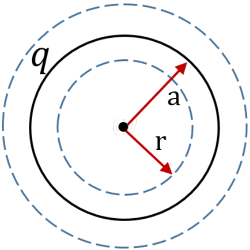
Lecture 6

Gauss Law Examples

6.1 Gauss theorem : uniform volume charge

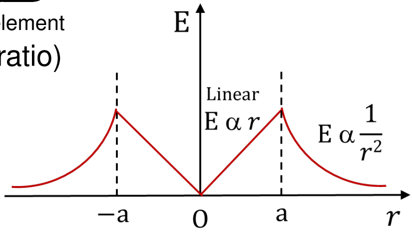
Sphere with uniform volume charge density

- $\rho = \begin{cases} \frac{q}{(4/3)\,\pi\, a^3} & \textit{for } 0 \le r \le a \;(\textit{inside}) \\ 0 & \textit{for } a \le r \;(\textit{outside}) \end{cases}$
- $\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$
  (volume $\mathcal{V}$ bounded by surface)



- Inside sphere :
  $E \cdot 4\,\pi\, r^2 = \frac{1}{\epsilon_0} \int_0^r \frac{q}{(4/3)\,\pi\, a^3} \underbrace{4\,\pi\, r'^2 dr'}_{\text{volume element}}$

  $= \frac{q}{\epsilon_0} \int_0^r \frac{3\, r'^2}{a^3} dr' = \frac{q}{\epsilon_0} \frac{r^3}{a^3}$ (volume ratio)
- $E = \frac{q}{4\pi\epsilon_0 a^3} r$ (radial)



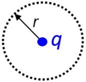
- Outside sphere :
  $\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{q}{\epsilon_0}$
- $E = \frac{q}{4\pi\epsilon_0 r^2}$ (point charge again)

Summary Gauss Law : spherical symmetry

Spherically symmetric charge distributions.

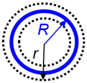
$$\oint_S \mathbf{E} \, . \, \mathbf{da} = E_r \times 4\pi r^2 = \frac{1}{\varepsilon_0} \int_V \rho \, dV \longrightarrow E_r = \frac{1}{4\pi\varepsilon_0 r^2} \int_V \rho \, dV$$

(i) point charge q:



$$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$ for any r

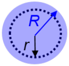
(ii) hollow sphere with q spread evenly across surface:



For $0 < r < R$ (inside sphere): $E_r = 0$

For $R < r$ (outside sphere): $E_r = \frac{q}{4\pi\varepsilon_0 r^2}$

(iii) Sphere carrying uniform volume charge ρ:



For $0 < r < R$ (inside sphere): $E_r = \frac{q}{4\pi\varepsilon_0 R^2} \frac{r}{R}$

For $R < r$ (outside sphere): $E_r = \frac{q}{4\pi\varepsilon_0 r^2}$

6.2 Gauss Theorem : Long, uniformly charged rod

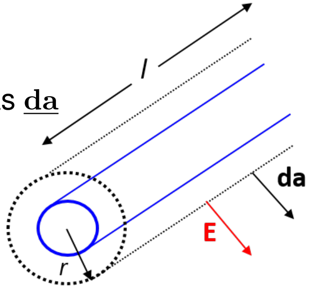
- Long, uniformly charged cylindrical rod with surface charge q
- Choose cylindrical Gaussian surface

  Symmetry : $\underline{\mathbf{E}}$ is in the same direction as $\underline{\mathbf{da}}$

  $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 2\pi\, r \cdot \ell = \frac{q}{\epsilon_0}$

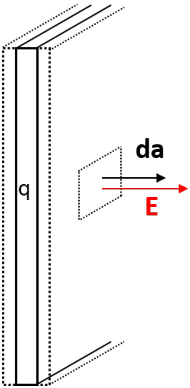
- $E = \frac{q}{\ell} \frac{1}{2\,\pi\,\epsilon_0 r} = \frac{\lambda}{2\,\pi\,\epsilon_0 r}$ (radial)

  λ is the charge per unit length



6.3 Uniformly charged infinite plate

1. Uniformly charged "infinite" plate of area A



- By symmetry : $\mathbf{E} \cdot \mathbf{da} = E \cdot da \;\; (\mathbf{E} \parallel \mathbf{da})$

  $\oint_S \mathbf{E} \cdot \mathbf{da} = E \cdot 2\,A = \frac{q}{\epsilon_0}$

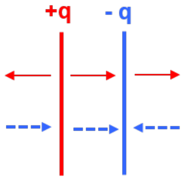
  (factor 2 due to both sides)

  $$E = \frac{1}{2\,\epsilon_0}\frac{q}{A} = \frac{\sigma}{2\,\epsilon_0}$$

  Field is uniform. σ is the charge per unit area

- As the plates become very large, the contribution from the edges become negligible

2. The capacitor



- Principle of superposition between the plates

  $E = \frac{\sigma}{2\,\epsilon_0} - \frac{-\sigma}{2\,\epsilon_0} = \frac{\sigma}{\epsilon_0}$

- Outside the plates $\quad E = \frac{\sigma}{2\,\epsilon_0} + \frac{-\sigma}{2\,\epsilon_0} = 0$

6.4 Electric field inside a conductor

Inside a conductor, one or more electrons per atom are free to move throughout the material (copper, gold, and other metals). We are considering electroSTATICS (static charge). As a result:

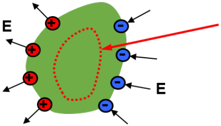
(i) **E**=0 inside a conductor (free charge moves to surface until the internal electric field is cancelled).

(ii) ρ =0 inside a conductor (from Gauss' law: **E**=0 hence ρ=0).

(iii) Therefore any net charge resides on the surface.

(iv) A conductor is an equipotential (since **E**=0, $V(\mathbf{r}_1)=V(\mathbf{r}_2)$).

(v) At the surface of a conductor, **E** is perpendicular to the surface (otherwise charges will flow until the tangential component becomes zero when equilibrium is reached).



$$\oint_S \mathbf{E} \cdot d\mathbf{a} = 0$$

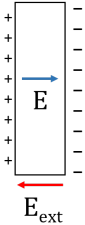
ρ=0

inside *V*

Properties of conductors

1. $\underline{\mathbf{E}} = 0$ inside a conductor

   - We are dealing with electroSTATICS - charges can move in an $\underline{\mathbf{E}}$-field !
   - They will move to the surface, creating surface charge which opposes applied field.
   - Equilibrium reached with $\underline{\mathbf{E}} = 0$ inside conductor.



2. $\rho = 0$ inside a conductor :

   - $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$
   - $\underline{\mathbf{E}} = 0$, $\rho = 0$

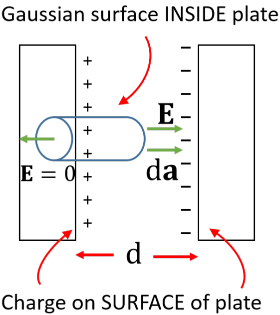
- Alternative treatment for the capacitor :

$$EA + 0 = \frac{q}{\epsilon_0} \rightarrow E = \frac{q}{A\,\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

Where the "0" term is the field *inside* the plate

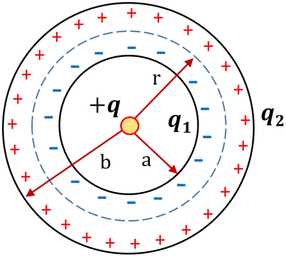
- Potential difference between plates

$V = -\int_0^d \underline{\mathbf{E}} \cdot \underline{\mathbf{d\ell}} = -Ed$



6.5 Revisit the electric field inside a hollow sphere

Consider an uncharged hollow metal sphere of finite thickness, with point charge $+q$ at its centre.



- Inside hollow :

$\oint_S \mathbf{E} \cdot \mathbf{da} = E_r \cdot 4\pi\, r^2 = \frac{q}{\epsilon_0}$

- Inside metal $\mathbf{E} = 0$:

$\oint_S \mathbf{E} \cdot \mathbf{da} = (q + q_1)/\epsilon_0 = 0$

$\rightarrow$ Inner surface charge $q_1 = -q$ must be induced on inner surface

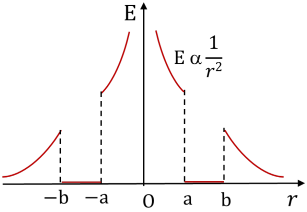
- Outside sphere :

$\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{q+q_1+q_2}{\epsilon_0}$

- Because there is no net charge on the sphere

$\rightarrow$ Outer surface charge given by $q_1 + q_2 = 0$

- $\rightarrow$ $q_2 = +q$ is induced on the outer surface



$\rightarrow$ $E_r = \frac{q}{4\pi\epsilon_0\, r^2}$

Lecture 7

Laplace & Poisson Equations

7.1 Poisson and Laplace Equations

- The expression derived previously is the “integral form" of Gauss’ Law

  $\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$ over volume $\mathcal{V}$
- We can express Gauss’ Law in differential form using the Divergence Theorem :

  $\int_{\mathcal{V}} (\underline{\nabla} \cdot \underline{\mathbf{F}}) d\mathcal{V} = \oint_S \underline{\mathbf{F}} \cdot \underline{\mathbf{da}}$ [$\underline{\mathbf{F}}$ is any general vector field.]

  Hence $\int_{\mathcal{V}} (\underline{\nabla} \cdot \underline{\mathbf{E}}) d\mathcal{V} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho \, d\mathcal{V}$
- This gives $\underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$

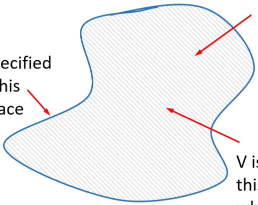
  the differential form of Gauss’s Law
- Using $\underline{\mathbf{E}} = -\underline{\nabla} V$ get *Poisson's Equation* for potential V

  $\underline{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$
- In regions where $\rho = 0$ we get *Laplace's Equation*:

  $\underline{\nabla}^2 V = 0$ (zero charge density)

7.2 Uniqueness Theorem

This states : *The solution to Laplace's equation in some volume is uniquely determined if the potential V is specified on the boundary surface S.* Why is this so?



- Suppose there are TWO solutions V_1 and V_2 to Laplace's equation for potential inside the volume
- $\underline{\nabla}^2 V_1 = 0$; $\underline{\nabla}^2 V_2 = 0$

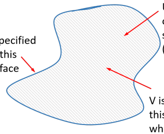
  and $V_1 = V_2$ on the boundary surface S
- Define the difference $V_3 = V_1 - V_2$

  Then $\underline{\nabla}^2 V_3 = \underline{\nabla}^2 V_1 - \underline{\nabla}^2 V_2 = 0$

  (V_3 also obeys Laplace's equation)
- But on the boundary $V_3 = V_1 - V_2 = 0$

Uniqueness Theorem continued

From the previous page :



- $\underline{\nabla}^2 V_1 = 0$ & $\underline{\nabla}^2 V_2 = 0$ with $V_1 = V_2$ on the surface
- $V_3 = V_1 - V_2$ (which $= 0$ on the surface)
- $\underline{\nabla}^2 V_3 = 0$ everywhere.
  - The $\underline{\nabla}^2$ operator is a three-dimensional second derivative of a function - when a function has an extrema, the second derivative will be negative for a maximum and positive for a minimum.
  - The fact that the second derivative is always zero therefore indicates that there are no such minima or maxima in the region of interest
  - Hence solutions to Laplace's equation do not have minima or maxima.
  - Since $V_3 = 0$ on the surface, the maximum and minimum values of V_3 must also be zero everywhere inside it.

Hence $V_3 = 0$ everywhere, and *V* **must be unique**

- Note the same applies to Poisson's equation.
- If $\underline{\nabla}^2 V_1 = -\rho/\epsilon_0$ and $\underline{\nabla}^2 V_2 = -\rho/\epsilon_0$, then $\underline{\nabla}^2 V_3 = 0$ as before.

Poisson and Laplace Equations : summary

Gauss' law: $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ Definition of Potential: $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson equation

In regions where ρ=0: $\nabla^2 V = 0$ Laplace equation

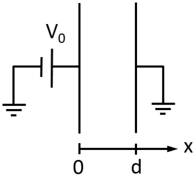
Uniqueness Theorem:
The potential V inside a volume is *uniquely* determined, if the following are specified:
(i) The charge density throughout the region
(ii) The value of V on all boundaries

7.3 Laplace equation in cartesian coordinates

Example : Solutions to Laplace's equation for a parallel plate capacitor. Symmetry suggests use of cartesian coordinates.

- $$\frac{\partial^2 V}{\partial x^2} + \underbrace{\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}_{= 0 \text{ (by symmetry)}} = 0$$

  Need to solve $\frac{\partial^2 V}{\partial x^2} = 0$



- $\frac{\partial V}{\partial x} = C_1 \;\rightarrow\; V(x) = C_1 x + C_2$
- Values on boundary defined by capacitor plates :

  $V(x=0) = V_0$ and $V(x=d) = 0$
- $x = 0\,,\; C_2 = V_0$ and

  $x = d\,,\; C_1 d + C_2 = 0 \;\rightarrow\; C_1 = -V_0/d$
- Solution : $V(x) = V_0(1 - x/d)$
- Electric field $\underline{\mathbf{E}} = -\nabla V = -\frac{\partial}{\partial x} V \,\underline{\hat{\mathbf{x}}} \;\rightarrow\; \underline{\mathbf{E}} = \frac{V_0}{d}\underline{\hat{\mathbf{x}}}$

7.4 Laplace Equation in spherical coordinates

... assuming azimuthal symmetry.

General solutions to Laplace's equation for charge distributions with azimuthal symmetry (mainly for information here : see second year).

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}}_{=0} = 0$$

Separation of variables yields the general solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

where A_l, B_l are constants determined by boundary conditions and P_l are Legendre Polynomials in cos θ, i.e.:

$P_0(\cos\theta) = 1$
$P_1(\cos\theta) = \cos\theta$
$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$
Etc ...

$$V(r,\theta) = A_0 + \frac{B_0}{r} + A_1 r\cos\theta + \frac{B_1}{r^2}\cos\theta$$

$$+ A_2 r^2 \frac{1}{2}(3\cos^2\theta - 1) + \frac{B_2}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \cdots$$

Laplace equation examples in spherical coordinates

1. Take a defined small spherical volume which contains some azimuthally symmetric charge distribution :
   - Outside the volume $\rho = 0$
   - Boundary condition on potential : $V \to 0$ as $r \to \infty$
   - Hence $A_\ell = 0$ for all ℓ
   - Retain just multipole expansion terms (monopole + dipole+ quadrupole + $\cdots$ terms)
2. Special case of spherically symmetric charge distribution inside the volume :
   - Outside the volume $\rho = 0$, $\nabla^2 V = 0$ with no θ dependence
   - $A_\ell = B_\ell = 0$ for $\ell \neq 0$
   - $V(r) = A_0 + B_0/r$ as expected from Gauss' Law

Lecture 8

Method of Images

8.1 The method of images

- The method of images is useful for calculating potentials created by charges placed in the vicinity of metal conductors.
- Replace conducting elements with imaginary charges (“image charges”) which replicate the boundary conditions of the problem on a surface.
- The Uniqueness Theorem guarantees that within the region bounded by the surface, the potential calculated for the “imagined” charge distribution is identical to that of the “real” situation.
- If a suitable replacement image charge distribution is chosen, the calculation of the potential becomes mathematically much simpler.

8.2 Example : Point charge above a metal plate

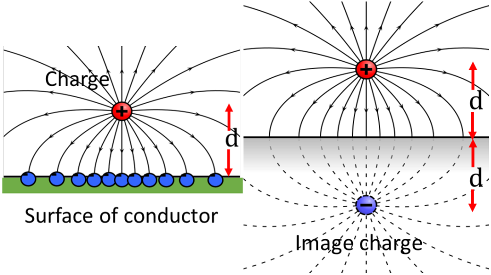
Point charge a distance d above a grounded metal plate:

- Boundary conditions

  1. At the metal surface (z = 0), the parallel component of $\underline{\mathbf{E}} = 0$. Field lines are perpendicular to the surface.

  2. Surface is an equipotential $\rightarrow \quad V = 0$

  3. Far from the charge and metal plate :
  $x^2 + y^2 + z^2 >> d^2 \quad V \rightarrow 0$



- The two configurations share the same charge distribution and boundary conditions for the upper volume half.
- The Uniqueness Theorem states that the potential in those regions must therefore be identical.
- In the upper half, the fields in both scenarios are identical.

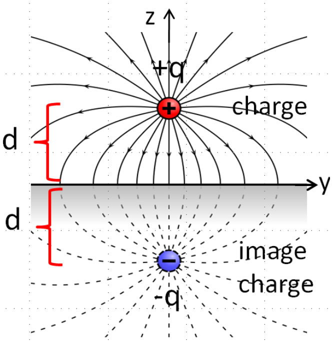
Point charge above a metal plate, continued

- Above plate : real (physical) region.
  Here find solution at point (x, y, z).
- Below plate : imagined "mirror charge"
- $V(x, y, z) =$
  $$\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x^2+y^2+(z-d)^2)}} - \frac{q}{\sqrt{(x^2+y^2+(z+d)^2)}}\right]$$
- Gives :

  1. $V = 0$ when $z = 0$
  2. $V \rightarrow 0$ for $x^2 + y^2 + z^2 >> d^2$

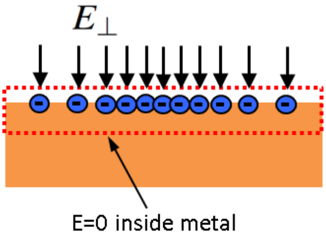
  $\rightarrow$ correct boundary conditions

  $\rightarrow$ unique solution!
- $\underline{\mathbf{E}}$ then calculated from $\underline{\mathbf{E}} = -\underline{\nabla} V$



8.3 Induced surface charge

- At the metal surface $E_{||} = 0$, $E_{\perp} = -\frac{\partial V}{\partial z}$ (z is the normal coordinate)
- Gauss Law at surface for element $\underline{\mathbf{da}}$: $\underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E_{\perp} \cdot da = \frac{q_{induced}}{\epsilon_0}$ where $\sigma_{induced} = q_{induced}/da$. No E-field in the “virtual” conductor



- So $\sigma_{induced}(x, y) = \epsilon_0 E_{\perp} = -\epsilon_0 \frac{\partial V}{\partial z}$
  - For the case of the point charge above the metal plate
  - $\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0} \left[\frac{-2\times\frac{1}{2}\times q\,(z-d)}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}} - \frac{-2\times\frac{1}{2}\times q\,(z+d)}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}} \right]$
  - $\sigma_{induced}(x, y) = -\epsilon_0 \frac{\partial V}{\partial z}|_{z=0} = -\frac{1}{2\pi} \left[\frac{qd}{(x^2+y^2+d^2)^{\frac{3}{2}}} \right]$ this is the surface charge in the $x - y$ plane
  - $\sigma_{induced}$ is negative, and largest for $x = y = 0$

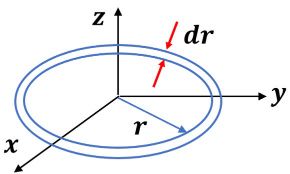
Total charge induced on the plate surface

- Now switch to polar coordinates (radial symmetry) :
- $\sigma_{induced} = -\frac{1}{2\pi}\left[\frac{qd}{(x^2+y^2+d^2)^{\frac{3}{2}}}\right] = -\frac{1}{2\pi}\left[\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}\right]$

$$q_{induced} = \int_0^\infty \sigma(2\pi r\,dr)$$

$$= \int_0^\infty \left(-\frac{1}{2\pi}\right)\left[\frac{qd}{(r^2+d^2)^{\frac{3}{2}}}(2\pi r\,dr)\right]$$

$$= \frac{qd}{\sqrt{r^2+d^2}}\Big|_0^\infty = 0 - \frac{qd}{d} = -q$$



- The total charge induced on the plate is just $-q$, as would be expected.

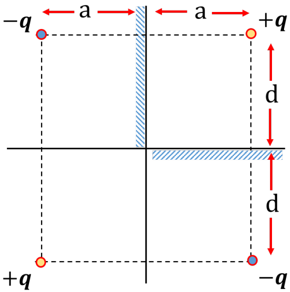
8.4 Force between the charge and the plate & energy stored

1. Force between the point charge and the plate :
   - Reduces to the case of the force between 2 point charges : $\underline{\mathbf{F}} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\,\underline{\hat{\mathbf{z}}}$
2. Energy stored in the electric field
   - Bringing in charge from infinity - but noting the separation must always be maintained at $2z$ to preserve the geometry and potential of the plane. $W = -\int_{\infty}^{d}\underline{\mathbf{F}}\cdot\underline{\mathbf{d}\ell} = +\frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{(2z)^2}\,dz$
   - $W = -\frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{(4z)}\right)\Big|_{\infty}^{d} = -\frac{1}{16\pi\epsilon_0}\frac{q^2}{d}$

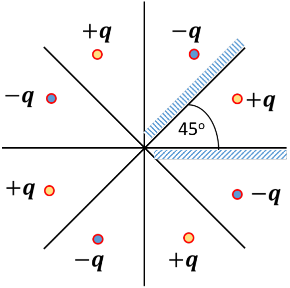
   Compare this to the case of bringing two point charges together from infinity to separation $2d$, with the first charge fixed in space.
   - Bring charge $+q$ up to d needs no work $\left(V_+ = \frac{+q}{4\pi\epsilon_0 r}\right)$
   - Second charge at $r = 2d$: $W = -q\,V_+ = -\frac{q^2}{8\pi\epsilon_0 d}$
   - This is a factor 2 greater than bringing charge up to a plate

8.5 Image charges due to a pair of plates

1. Image charges required to replicate the field due a charge *q* located between 2 grounded semi-infinite plates with a 90° angle

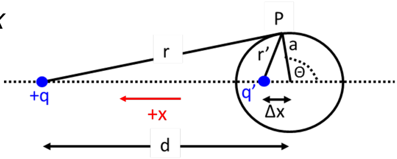


2. Image charges required to replicate the field due a charge *q* located between 2 grounded semi-infinite plates with a 45° angle



8.6 Point charge with grounded metal sphere

Look at a more complicated configuration $\cdots$ replicating the field due to a point charge outside a grounded metal sphere, radius a. Origin (0, 0, 0) is at sphere centre.



- Place image charge q' at position Δx from origin
- Potential at P on sphere :

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

where $r^2 = a^2 + d^2 + 2\,a\,d\,\cos\theta$ &
$r'^2 = a^2 + \Delta x^2 + 2\,a\,\Delta x\,\cos\theta$

- We need $V = 0$ for all points on the surface of the sphere (for all θ)

$$\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} = -\frac{q'}{\sqrt{a^2+\Delta x^2+2\,a\,\Delta x\,\cos\theta}}$$

Point charge with grounded metal sphere, continued

$$\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} = -\frac{q'}{\sqrt{a^2+\Delta x^2+2\,a\,\Delta x\,\cos\theta}}$$

- This can be solved rigorously by inputting specific values for $\cos\theta$ (eg, -1, 0, 1). However note that :
- LHS = $\frac{q}{\sqrt{a^2+d^2+2\,a\,d\,\cos\theta}} \times \underbrace{\left[\frac{a}{d}\Big/\sqrt{\frac{a^2}{d^2}}\right]}_{=1}$

  $= \frac{q\frac{a}{d}}{\sqrt{\frac{a^4}{d^2}+a^2+2\,\frac{a^3}{d}\,\cos\theta}}$
- By inspection, image charge $q' = -q\frac{a}{d}$, at position $\Delta x = \frac{a^2}{d}$
- Hence potential at any point (x, y, z) OUTSIDE the sphere

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q\,a/d}{\sqrt{(x-\frac{a^2}{d})^2+y^2+z^2}}\right]$$

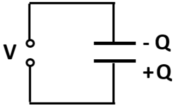
(and $V = 0$ inside)

Lecture 9

Capacitance

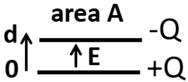
9.1 Capacitance

- Capacitors store electrostatic energy, by keeping two opposite charge accumulations on different metallic surfaces.



- Capacitance is defined as the charge that is stored per unit voltage applied between the two surfaces.

  Capacitance definition $C = \frac{\textit{Stored charge } Q}{\textit{Voltage applied}}$



- The charge is equal and opposite on both surfaces.
  - Simple example : Parallel plate capacitor
  - From before, $\underline{\mathbf{E}}$ constant between plates (Gauss) :

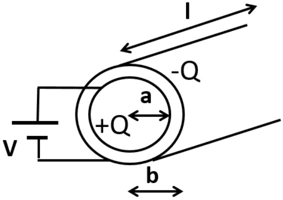
$$\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = \frac{Q}{\epsilon_0} \;\rightarrow\; E = \frac{Q}{\epsilon_0 A}$$

$$V_{+-} = -\int_0^d E \cdot dx = -\frac{Q\,d}{\epsilon_0 A} \;\rightarrow\; V_{-+} = +\frac{Q\,d}{\epsilon_0 A}$$

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

9.2 Cylindrical capacitor

- Example : coaxial cable. Battery supplies $+Q$ on the inner surface, $-Q$ is induced on the outer (Gauss)



From before, Gauss :

$\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = E \cdot 2\pi r \, \ell = \frac{Q}{\epsilon_0}$

$\rightarrow \; \underline{\mathbf{E}} = \frac{Q/\ell}{2\pi\epsilon_0 r} \, \underline{\hat{\mathbf{r}}}$ (radial) for $a \leq r \leq b$

$\rightarrow \; E = 0$ for $0 \leq r < a$ and for $r > b$

- $V_{+-} = - \int_a^b E_r \cdot dr = - \int_a^b \frac{Q/\ell}{2\pi\epsilon_0 r} \, dr$

$$= -\frac{Q/\ell}{2\pi\epsilon_0} log_e\left(\frac{b}{a}\right) \; \rightarrow \; V_{-+} = +\frac{Q/\ell}{2\pi\epsilon_0} log_e\left(\frac{b}{a}\right)$$

$$C = Q/V = \frac{2\pi\epsilon_0}{log_e(\frac{b}{a})} \times \ell$$

- Capacitance per unit length :

$$C' = C/\ell = \frac{2\pi\epsilon_0}{log_e(\frac{b}{a})}$$

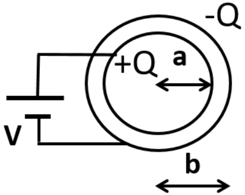
9.3 Spherical capacitor

- Example : spherical capacitor with concentric hollow spheres. Battery supplies $+Q$ on the inner sphere, $-Q$ is induced on the outer (Gauss).

From before, Gauss :

$\underline{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2}\,\underline{\hat{\mathbf{r}}}$ (radial) for $a \leq r \leq b$

$E = 0$ for $r < a$ and $r > b$



- $V_{+-} = -\int_a^b E_r \cdot dr = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$

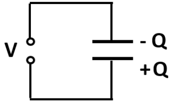
  $\rightarrow\ V_{-+} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$

- Capacitance : $C = Q/V = 4\pi\epsilon_0 \frac{ab}{b-a}$

Capacitors summary

Capacitance: Storage of energy through separation of two oppositely poled charge accumulations

Capacitance C = $\dfrac{\text{charge Q}}{\text{voltage V applied}}$



(i) parallel-plate capacitor:

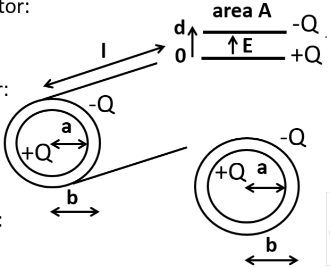
$$C = \varepsilon_0 \frac{A}{d}$$

(ii) cylindrical capacitor:

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

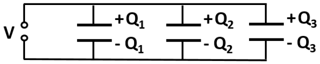
(iii) spherical capacitor:

$$C = 4\pi\varepsilon_0 \frac{a\,b}{b-a}$$

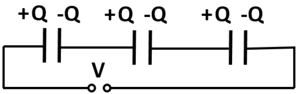


9.4 Capacitance networks

1. Capacitors in parallel
   - Voltage is the same across each capacitor.
   - Total charge :
     $Q = Q_1 + Q_2 + Q_3 + \cdots$
     $= C_1 V + C_2 V + C_3 V + \cdots$
   - Total capacitance
     $C = \frac{Q}{V} = C_1 + C_2 + C_3 + \cdots$



2. Capacitors in series
   - Charge is the same on each capacitor plate (inner plates are isolated from the outside world, with $Q_{tot} = 0$).
   - Total voltage :
     $V = V_1 + V_2 + V_3 + \cdots$
     $\frac{1}{C} = \frac{V}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q} + \frac{V_3}{Q} + \cdots$
   - $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$



9.5 Energy stored in a capacitor

- Capacitor is initially uncharged : add a small amount of charge.
- Further charge will have to be brought up against the potential created by the existing charge :

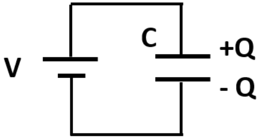
  Work done $\rightarrow \; dW = V(q)\, dq$
- Energy required to charge the capacitor to potential V_0 :

  $W = \int_0^{Q_0} V(q)\, dq$

  with $q = CV \;\rightarrow\; dq = C\, dV$
- $W = \int_0^{V_0} C\, V\, dV = \frac{1}{2}\, C\, V_0^2$
- Hence, energy stored in a capacitor with charge Q_0 and voltage V_0 :

$$U_C \equiv W = \frac{1}{2}\, C\, V_0^2 = \frac{1}{2}\, Q_0\, V_0 = \frac{1}{2}\, Q_0^2 / C$$

9.6 Changing C at constant V



- Battery maintains capacitor at constant V. What happens if C changes ?
- Energy stored in capacitor : $U_C = \frac{1}{2}\, C\, V^2$

  Change in capacitor energy : $dU_C = \frac{1}{2}\, V^2\, dC$
- Hence if C increases, U_C increases
- Since $Q = CV$, if C increases (ie. dC is positive), battery has to *supply* charge to maintain the same V. Hence charge on capacitor *increases*, and energy stored in battery *decreases*.
- Battery supplies dQ at constant $V \;\rightarrow\;$ energy change of battery is $dU_B = -V\, dQ$ (minus because battery *loses* stored energy in providing $+dQ$ to the plates of the capacitor)
- $Q = CV$, $dQ = V\, dC$, hence $dU_B = -V^2\, dC$
- This is a general result. If U_C *increases* at constant V, this is matched by a factor 2 *decrease* in battery energy.
- Cons. of energy : $dU_{total} = dU_B + dU_C = dW$, where

  $dW = -\frac{1}{2}\, V^2\, dC$ is the work done to change $C^{(*)}$.

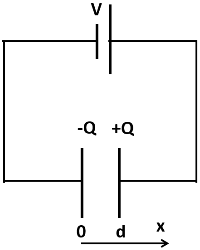
  (*) Note dC is negative if plates are pulled apart, since C decreases.

Lecture 10

Capacitance, Energy & Magnetostatitcs

10.1 Force between capacitor plates (2 cases)

- Capacitor plates are oppositely charged $\rightarrow$ an attractive force F exists between them.
- By pulling the plates apart we perform work on the capacitor / battery system



Work done in pulling apart : $W = -\int \underline{\mathbf{F}} \cdot \underline{\mathbf{dx}}$

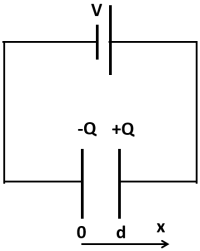
Energy stored in capacitor : $U_C = \frac{1}{2} Q^2 / C$

Energy stored in the battery : $U_B = V\,Q$

1. Pull apart at constant charge: battery disconnected, $dU_B = 0$
   - Force between plates : $F = -\frac{\partial U_C}{\partial x}|_{Q\ const.} = -\frac{1}{2} Q^2 \frac{\partial}{\partial x}(\frac{1}{C})$
   - For a parallel plate capacitor $\frac{1}{C} = \frac{x}{\epsilon_0 A}$
   - Hence $F = -\frac{1}{2} Q^2 \frac{1}{\epsilon_0 A}$
   - Mechanical work required to move plates from separation d_1 to d_2 : $W = -\int_{d_1}^{d_2} \underline{\mathbf{F}} \cdot \underline{\mathbf{dx}} = \frac{1}{2} Q^2 \frac{1}{\epsilon_0 A}(d_2 - d_1)$

Force between capacitor plates continued

2. Plates pulled apart at constant voltage (which is supplied by the battery)



- $F = -\frac{\partial U_{total}}{\partial x}|_{V\ const.} = -\frac{\partial}{\partial x}(\underbrace{\frac{1}{2}V^2 C}_{\text{capacitor}} - \underbrace{V^2 C}_{\text{battery}})$
- $F = \frac{1}{2}V^2 \frac{\partial C}{\partial x}$ where $C = \epsilon_0 A/x$

$$F = -\frac{1}{2}V^2\,\epsilon_0\,A/x^2$$

- Mechanical work required to move plates from separation d_1 to d_2 : $W = -\int_{d_1}^{d_2} \underline{\mathbf{F}} \cdot \underline{\mathbf{dx}}$

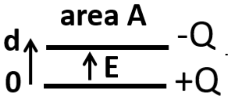
$$W = \frac{1}{2}V^2\,\epsilon_0\,A\,(\frac{1}{d_1} - \frac{1}{d_2}) = \frac{1}{2}V^2\,(C_1 - C_2)$$

- Pulling plates apart leaves the capacitance lowered, charge returns to the battery, work is performed on the capacitor/battery system.

10.2 Energy density of the electric field

- Consider parallel plate capacitor

  $U_C = \frac{1}{2} C V^2$; $E = \frac{V}{d}$; $C = \epsilon_0 A/d$



- Hence $U_C = \frac{1}{2}\epsilon_0 \frac{A}{d} E^2 d^2$

  $$= \frac{1}{2}\epsilon_0 E^2 \underbrace{A\,d}_{\text{volume}}$$

  - Energy density in between the plates :

    $$U_\rho = U_C/[\text{unit volume}] = \frac{1}{2}\epsilon_0 E^2$$

  - This is actually a *general result* for *any* region in space in an $\underline{\mathbf{E}}$ field. The volume can be made arbitrarily small :

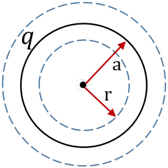
    $dU = \frac{1}{2}\epsilon_0 E^2 d\mathcal{V} \;\leftarrow\;$ volume element

  - Hence $U = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E^2 d\mathcal{V}$ over all space in the general case.

10.3 Example : hollow spherical shell

Example : Energy of hollow spherical shell carrying charge q

$$\begin{cases} E = 0 \quad for\, 0 \leq r < a\, (inside) \\ E = \frac{q}{4\pi\epsilon_0 r^2} \quad for\, a \leq r\, (radial, as\ point\ charge) \end{cases}$$



- $U = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E^2 d\mathcal{V}$ over all space

$$= \frac{1}{2}\epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_a^{\infty} \frac{q^2}{16\pi^2\epsilon_0^2 r^4} \underbrace{r^2 dr \sin\theta\, d\theta\, d\phi}_{\text{volume element}}$$

$$= \frac{1}{2}\epsilon_0 4\pi \frac{q^2}{16\pi^2\epsilon_0^2} \int_a^{\infty} \frac{1}{r^2}\, dr$$

$$U = \frac{q^2}{8\pi\epsilon_0}\frac{1}{a}$$

- Alternative approach : energy required to bring up charge dq from infinity against potential $V(q)$ is $dW = V(q)\, dq$

$$W = \int_0^q V(q')\, dq' = \int_0^q \frac{q'}{4\pi\epsilon_0}\frac{1}{a}\, dq' = \frac{q^2}{8\pi\epsilon_0}\frac{1}{a}$$

which is the same result as above.

10.4 Principle of superposition for energy density

Question: does the principle of superposition apply to energy density?

- Principle of superposition : $\underline{\mathbf{E}} = \underline{\mathbf{E}}_1 + \underline{\mathbf{E}}_2$
- $U = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E^2 \, d\mathcal{V} = \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} (\underline{\mathbf{E}}_1 + \underline{\mathbf{E}}_2)^2 \, d\mathcal{V}$

$$= \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E_1^2 \, d\mathcal{V} + \frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E_2^2 \, d\mathcal{V} + \epsilon_0 \int_{\mathcal{V}} \underline{\mathbf{E}}_1 \cdot \underline{\mathbf{E}}_2 \, d\mathcal{V}$$

$$= U_1 + U_2 + \epsilon_0 \int_{\mathcal{V}} \underline{\mathbf{E}}_1 \cdot \underline{\mathbf{E}}_2 \, d\mathcal{V}$$

- Therefore the answer is no !

MAGNETOSTATICS - OVERVIEW

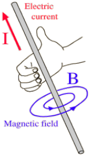
1. Introduction: Origins of Magnetism
2. Forces on Current-Carrying Wires in Magnetic Fields
3. The Biot-Savart Law (B-fields of Wires, Solenoids, etc.)
4. Magnetic Dipoles
5. Ampere's Law & Gauss' Law of Magnetostatics
6. Current Density and the Continuity Equation

} Problem Set 3

10.5 Origins of magnetism

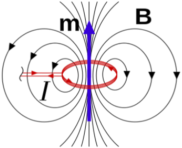
Minerals found in ancient Greek city Magnesia ("Magnetite", Fe_3O_4) attract small metal objects.

Materials containing certain atoms such as Iron (Fe), Cobalt (Co), Nickel (Ni) can exhibit "permanent" magnetic dipoles.



Forces exist between pairs of current-carrying wires (attractive for current flowing in the same, repulsive for current flowing in opposite directions).

An electric current through a wire creates a magnetic field whose field lines loop around the wire.



Magnetic field lines form closed loops. They do not originate from "magnetic monopoles".

The fundamental generators of magnetic fields are dipoles that may result from electrical current loops or inherent material properties such as aligned angular momenta of charged particles.

10.6 Magnetostatics terminology

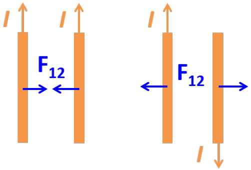
- Magnetic flux density $\underline{\mathbf{B}}$ ("B-field") [B] = T (Tesla)
- Magnetic field (strength) $\underline{\mathbf{H}} = \frac{1}{\mu_0}\underline{\mathbf{B}}$ (in non magnetic materials) $[H] = A\,m^{-1}$

  where $\mu_0 = 4\pi \times 10^{-7}$ $\mathrm{N\,A^{-2}}$ (permeability of free space)

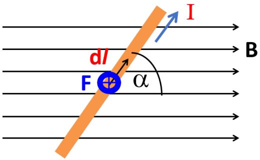
10.7 Forces on current-carrying wires in magnetic fields

Experimental observations :

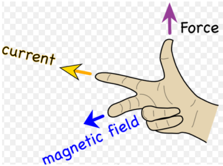
1. Two wires attract (repel) one another if they carry current in the same (opposite) directions.



2. A current-carrying wire in a magnetic field, flux density B, experiences a force with :

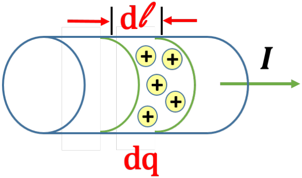


   - $F \propto B$
   - $F \propto I$ (current in wire)
   - $F \propto \ell$ (length of wire)
   - $F \propto \sin\alpha$ (α is angle between the direction of $\underline{\mathbf{B}}$ and $\underline{\mathbf{I}}$)
   - $\underline{\mathbf{F}}$ is oriented perpendicular to both $\underline{\mathbf{B}}$ and the wire
   - $dF = I\,B\,d\ell\,\sin\alpha \;\rightarrow\; \underline{\mathbf{dF}} = I\,\underline{\mathbf{d\ell}} \times \underline{\mathbf{B}}$



10.8 The Lorenz force

- Force on current-carrying wire in a B-field: $\underline{\mathbf{dF}} = I\,\underline{\mathbf{d}\ell} \times \underline{\mathbf{B}}$
- Zoom into a wire segment, assume it's the (+) charge moving ("conventional" current)



  - $I = \frac{dq}{dt}$ and $|\underline{\mathbf{v}}| = \left|\frac{\mathbf{d}\ell}{dt}\right|$ (average velocity of charge)

    $\rightarrow\ I = \frac{dq}{d\ell}\cdot\frac{d\ell}{dt} = v\,\frac{dq}{d\ell}$: Vectorizing $I\,\underline{\mathbf{d}\ell} = \underline{\mathbf{v}}\,dq$

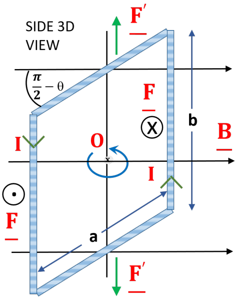
    $\rightarrow\ \underline{\mathbf{dF}} = dq\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$
  - Lorenz force $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$
  - Any charge q moving with velocity $\underline{\mathbf{v}}$ in a magnetic flux density $\underline{\mathbf{B}}$ experiences a Lorenz force $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$ perpendicular to both
  - Work done on the moving charge $dW = -\underline{\mathbf{F}}\cdot\underline{\mathbf{d}\ell} = -q\,(\underline{\mathbf{v}} \times \underline{\mathbf{B}})\cdot\underline{\mathbf{v}}\,dt = 0$
  - Magnetic fields do no work

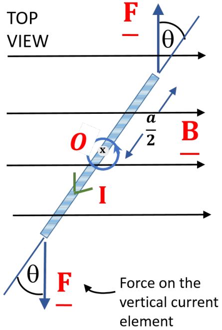
10.9 Example : measuring $\underline{\mathbf{B}}$ field

From torque on a wire loop carrying current I in field $\underline{\mathbf{B}}$:

- From diagram, torque on coil about O when $|\theta| > 0$:

  $\underline{\mathbf{T}} = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$ from current in sides b (sides a, forces $\underline{\mathbf{F}}'$ cancel)
- $|\underline{\mathbf{T}}| = 2 \times \frac{a}{2} \sin\theta \underbrace{I\, b\, B}_{\text{Force}}$ $\quad$ ($\underline{\mathbf{F}}$ is $\perp$ to I is $\perp$ to $\underline{\mathbf{B}}$)
- $|\underline{\mathbf{T}}| = I\, B\, A \sin\theta$ $\quad$ (A is the area of the loop)
- Measure T $\;\rightarrow\;$ obtain B





Lecture 11

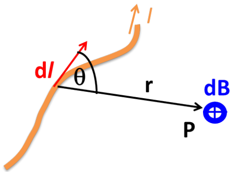
Magnitostatics & the Biot-Savart Law

11.1 The Biot-Savart Law for calculating magnetic fields

The Biot-Savart is here taken as an empirical starting point for calculation of magnetic fields, but can be derived from Maxwell's equations and the magnetic potential (see later).

- The Biot-Savart Law states the field at point *P*:

$$\underline{\mathbf{dB}} = \mu_0 \, I \, \frac{\underline{\mathbf{d}\ell} \times \underline{\hat{\mathbf{r}}}}{4\pi \, r^2}$$

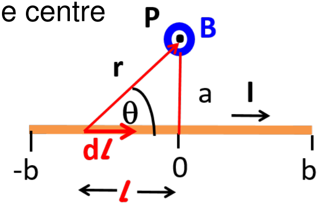


  - $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2} permeability of free space
  - $\underline{\mathbf{dB}}$ is the magnetic flux density contribution at *P*
  - I is the current flowing through element $\underline{\mathbf{d}\ell}$
  - $\underline{\mathbf{r}}$ is the vector connecting $\underline{\mathbf{d}\ell}$ and *P*
  - $\underline{\mathbf{dB}}$ is oriented perpendicular to $\underline{\mathbf{r}}$ and the current

Then integrate $\underline{\mathbf{dB}}$ to get *total* field from a circuit which has current

11.2 Example : the B-field of a straight wire

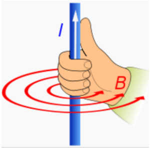
Calculate the B-field due to a straight wire with current I, length $2b$, at a distance a from the centre



- At point P : $\underline{\mathbf{dB}} = \mu_0 \, I \, \frac{\underline{\mathbf{d\ell}} \times \hat{\underline{\mathbf{r}}}}{4\pi \, r^2}$
- Use $r^2 = a^2 + \ell^2$ and

  $|\underline{\mathbf{d\ell}} \times \hat{\underline{\mathbf{r}}}| = d\ell \sin\theta = d\ell \, \frac{a}{r}$

  $\rightarrow \; dB = \frac{\mu_0 \, I}{4\pi} \frac{a}{(a^2+\ell^2)^{\frac{3}{2}}} \, d\ell$
- Direction given by right hand screw rule.

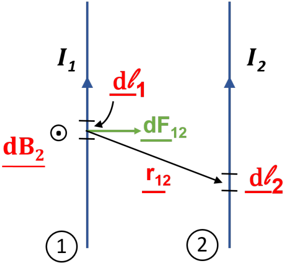


  - $B = \frac{\mu_0 \, I}{4\pi} \int_{-b}^{b} \frac{a}{(a^2+\ell^2)^{\frac{3}{2}}} \, d\ell$

    $\rightarrow \; B = \frac{\mu_0 \, I}{4\pi} \left[\frac{\ell/a}{(a^2+\ell^2)^{\frac{1}{2}}} \right]_{-b}^{b} \rightarrow$ $B = \frac{\mu_0 \, I \, b}{2\pi \, a} \frac{1}{(a^2+b^2)^{\frac{1}{2}}}$
  - For an infinite straight wire ($b \rightarrow \infty$) $B = \frac{\mu_0 \, I}{2\pi \, a}$

11.3 Example : force between 2 current-carrying wires

Two wires: force on small element of wire 1 from magnetic field of small element of wire 2

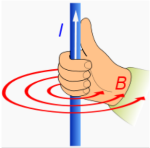


- $\underline{\mathbf{dF}}_{12} = I_1\, \underline{\mathbf{d}\ell}_1 \times \underline{\mathbf{dB}}_2$
- At point on wire 1, magnetic field element $\underline{\mathbf{dB}}_2$ from wire 2 :

  $$\underline{\mathbf{dB}}_2 = \frac{\mu_0\, I_2}{4\pi\, r_{21}^3}\, \underline{\mathbf{d}\ell}_2 \times \underline{\mathbf{r}}_{21}$$

  $$\rightarrow\ \underline{\mathbf{dF}}_{12} = \frac{\mu_0\, I_1\, I_2}{4\pi\, r_{12}^3}\, [-\underline{\mathbf{d}\ell}_1 \times (\underline{\mathbf{d}\ell}_2 \times \underline{\mathbf{r}}_{12})]$$

  (negative since $\underline{\mathbf{r}}_{12} = -\underline{\mathbf{r}}_{21}$)



- Force between 2 current-carrying wires :

$$\underline{\mathbf{F}}_{12} = \int_{\ell_1} \int_{\ell_2} \frac{\mu_0\, I_1\, I_2}{4\pi\, r_{12}^3}\, [\underline{\mathbf{d}\ell}_2 \times (\underline{\mathbf{d}\ell}_1 \times \underline{\mathbf{r}}_{12})]$$

$\cdots$ *and if the wires are parallel and infinite*

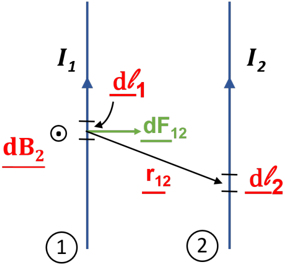
If wires are infinite, separated by distance a, currents I_1 and I_2

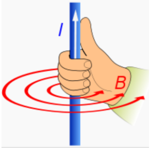
- $\underline{\mathbf{dF}}_{12} = I_1\,\underline{\mathbf{d}\ell}_1 \times \underline{\mathbf{B}}_2$
- From BS Law, from before, $|\underline{\mathbf{B}}_2| = \frac{\mu_0 I_2}{2\pi a}$
- Force on element $\underline{\mathbf{d}\ell}_1$:

  $|\underline{\mathbf{dF}}_{12}| = I_1\,|\underline{\mathbf{d}\ell}_1|\,\frac{\mu_0 I_2}{2\pi a}$ towards wire 2
- Due to the symmetry, force on every element is the same along the wire
- Hence *force per unit length* on wire 1 :

$$\frac{|\underline{\mathbf{dF}}_{12}|}{|\underline{\mathbf{d}\ell}_1|} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

(and note that $\underline{\mathbf{dF}}_{12} = -\underline{\mathbf{dF}}_{21}$)



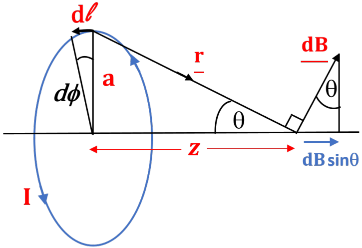


11.4 Example : B-field of a circular current loop

Calculate the B-field due to a circular wire with current I, radius a, at a distance z along its axis from the centre

- Field due to $\underline{\mathbf{d}\ell}$: $\underline{\mathbf{dB}} = \mu_0\, I\, \frac{\underline{\mathbf{d}\ell}\times\underline{\hat{\mathbf{r}}}}{4\pi\, r^2}$
- $|\underline{\mathbf{d}\ell} \times \underline{\hat{\mathbf{r}}}| = d\ell$, since $\underline{\mathbf{r}} \perp \underline{\mathbf{d}\ell}$
- Components of $\underline{\mathbf{dB}}$ perpendicular to z-axis cancel due to symmetry $\rightarrow$ field is along the z-axis

  $\rightarrow\ B = \int dB \sin\theta = \int \frac{a}{r}\, dB$



- $B = \int \frac{\mu_0\, I}{4\pi\, r^2}\, \frac{a}{r}\, d\ell$ along $\underline{\hat{\mathbf{z}}}$
  - a and r both constant for given point. $\int d\ell = 2\pi\, a$
  - Hence $$B = \frac{\mu_0\, I\, a^2}{2(z^2+a^2)^{\frac{3}{2}}}$$
  - Or since $\sin\theta = \frac{a}{\sqrt{(z^2+a^2)}}$, $B = \frac{\mu_0\, I}{2a} \sin^3\theta$

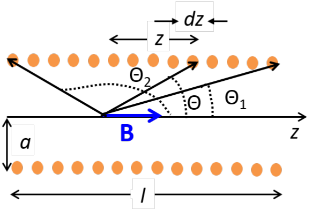
Lecture 12

The Biot Savart Law & the Magnetic Dipole

12.1 Example : B-field of a solenoid

Calculate the B-field due to a solenoid with current I, radius a, length ℓ with N turns. Sum over all contributions from all loops at a distance z (integrate from θ_1 to θ_2).

- Contribution from one element dz:
  $dB = \frac{\mu_0}{2a} \sin^3\theta \, dI$ where $dI = I\,(\frac{N}{\ell})\,dz$
  along the axis of the solenoid.



- $\tan\theta = \frac{a}{z} \;\rightarrow\; z = \frac{\cos\theta}{\sin\theta}\, a$
  $\rightarrow\; dz = -a\,\frac{1}{\sin^2\theta}\, d\theta$
- $B = -\int_{\theta_1}^{\theta_2} \frac{\mu_0}{2a} \sin^3\theta \,\frac{IN}{\ell} \left(a\,\frac{1}{\sin^2\theta} d\theta\right) = -\frac{\mu_0 I N}{2\ell} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$
- Hence $B = \frac{\mu_0 I N}{2\ell}(\cos\theta_2 - \cos\theta_1)$
- For a long coil $\theta_1 = 0,\ \theta_2 = \pi \;\rightarrow\; B = -\mu_0\, I\, \frac{N}{\ell}$

  (sign depends on direction of current $\rightarrow$ RH screw rule)

12.2 Biot-Savart Law in terms of current density

- The Biot-Savart Law :

$$\underline{\mathbf{dB}} = \mu_0\, I\, \frac{\underline{\mathbf{d\ell}} \times \hat{\mathbf{r}}}{4\pi\, r^2}$$

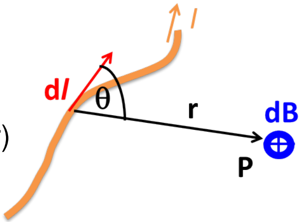
- Define *current density* $\underline{\mathbf{J}}$:

$$I = \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$$

$\underline{\mathbf{J}}$ is the current per unit area (a vector)

$$\underline{\mathbf{J}} = \frac{dI}{da_\perp} \times \left(\frac{\underline{\mathbf{d\ell}}}{|\underline{\mathbf{d\ell}}|}\right)$$

$da_\perp$ is the area perpendicular to the flow of current



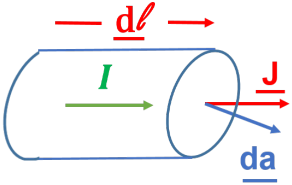
- Also since $\underline{\mathbf{J}} \parallel \underline{\mathbf{d\ell}}$

$$I\, \underline{\mathbf{d\ell}} = (\underline{\mathbf{J}} \cdot \underline{\mathbf{da}})\, \underline{\mathbf{d\ell}} = \underline{\mathbf{J}}\, (\underline{\mathbf{da}} \cdot \underline{\mathbf{d\ell}}) = \underline{\mathbf{J}}\, d\mathcal{V}$$

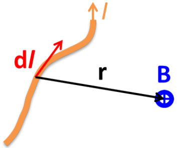
- Hence

$$\underline{\mathbf{B}} = \int_{\mathcal{V}} \mu_0\, \frac{\underline{\mathbf{J}} \times \hat{\mathbf{r}}}{4\pi\, r^2}\, d\mathcal{V}$$

Biot-Savart Law in terms of current density $\underline{\mathbf{J}}$ integrated over volume $\mathcal{V}$



Biot-Savart Law summary

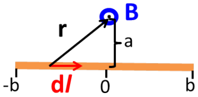


The magnetic flux density **B** created by a current loop is given by:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int I \frac{\mathrm{d}l \times \hat{\mathbf{r}}}{r^2} \quad \text{or} \quad \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \mathbf{r}}{r^3} \mathrm{d}^3 r$$

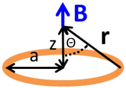
Biot-Savart Law

Straight wire.



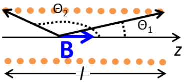
$$\mathbf{B} = \frac{\mu_0 I}{2\pi a} \frac{1}{(a^2/b^2+1)^{1/2}}$$

Circular loop.



$$\mathbf{B} = \frac{\mu_0 I a^2}{2\sqrt{z^2+a^2}^3} = \frac{\mu_0 I}{2a} \sin^3\theta$$

Solenoid.



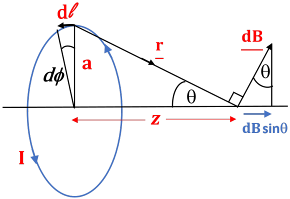
$$\mathbf{B} = \frac{\mu_0 I N}{2l} (\cos\theta_2 - \cos\theta_1)$$

12.3 The magnetic dipole

A small current loop defines a *magnetic dipole*

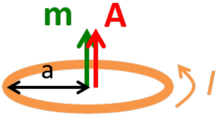
- Re-visit the field due to a circular current loop :

$$\underline{\mathbf{B}} = \frac{\mu_0\, I\, a^2}{2\left(z^2+a^2\right)^{\frac{3}{2}}}\,\underline{\hat{\mathbf{z}}}$$



In terms of loop area: $\underline{\mathbf{B}} = \frac{2\,\mu_0 I\,(\pi\, a^2)}{4\pi\, r^3}\,\underline{\hat{\mathbf{z}}}$

- Compare this with the on-axis field of the *electric* dipole (i.e. for $\theta = 0$) which has the same form :



Electric dipole : $\underline{\mathbf{E}}_{\mathbf{r}} = \frac{2qd\cos\theta}{4\pi\epsilon_0 r^3}\,\underline{\hat{\mathbf{r}}} \rightarrow \underline{\mathbf{E}}_{\mathbf{z}} = \frac{2p}{4\pi\epsilon_0 r^3}\,\underline{\hat{\mathbf{z}}} \quad (p = q\,d)$

- Define $(I\,\pi\,a^2) = I\,A$ as the *magnetic dipole moment m*

Magnetic dipole moment $\underline{\mathbf{m}} = I\,\underline{\mathbf{A}}$
= [Current] $\times$ [Area bounded by the loop]

Lecture 13

Magnetic Dipoles & the Divergence of B

13.1 Magnetic dipole components

Magnetic dipole moment $\underline{\mathbf{m}} = I\,\underline{\mathbf{A}}$
= [Current] $\times$ [Area bounded by the loop]

- Electric dipole field

$$\left.\begin{aligned} E_r &= \frac{2\,p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\ E_\phi &= 0 \\ p &= q\,d \end{aligned}\right\}$$

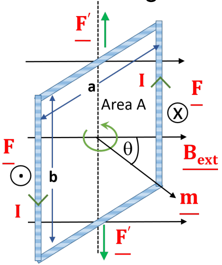
- Magnetic dipole field

$$\left\{\begin{aligned} B_r &= \frac{2\,\mu_0 m\cos\theta}{4\pi\, r^3} \\ B_\theta &= \frac{\mu_0 m\sin\theta}{4\pi\, r^3} \\ B_\phi &= 0 \\ m &= I\,A \end{aligned}\right.$$

The magnetic and electric dipole components have exactly the same form. (The exact derivation of the magnetic field components are beyond the scope of this course.)

13.2 Torque on a magnetic dipole in a $\underline{\mathbf{B}}$-field

Calculate the torque on a current loop placed in an external magnetic field:



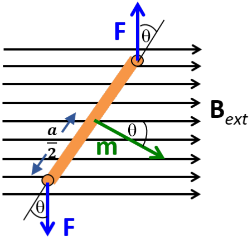
- Net force on the whole loop :

  $\underline{\mathbf{F}} = \oint_{loop} I \, \underline{\mathbf{d}\ell} \times \underline{\mathbf{B}}_{\mathbf{ext}} = 0$

  (since equal and opposite forces from opposite elements $\underline{\mathbf{d}\ell}$ cancel pairwise)

- From before, there is a torque on the current loop : $|\underline{\mathbf{T}}| = 2 \times \frac{a}{2} \times I \, B_{ext} \, b \sin\theta$

  $|\underline{\mathbf{T}}| = I \, B_{ext} \, A \sin\theta \;\rightarrow\; \underline{\mathbf{T}} = I \, \underline{\mathbf{A}} \times \underline{\mathbf{B}}_{ext}$



- Torque on the magnetic dipole

  $$\underline{\mathbf{T}} = \underline{\mathbf{m}} \times \underline{\mathbf{B}}_{\mathbf{ext}} \quad (*)$$

  Compare with the torque on an electric dipole $\underline{\mathbf{T}}_{\mathbf{elec}} = \underline{\mathbf{p}} \times \underline{\mathbf{E}}_{\mathbf{ext}}$

(*) This has been done for a rectangular shape. But note that this is a general result for *any* shape. Any current loop can be built from infinitesimal rectangular loops with all the internal currents cancelling, yet contributing to the overall moment.

Torque on a magnetic dipole, continued

- Do the explicit calculation for a circular current loop:
- Only the vertical component of $\underline{\mathbf{d}\ell}$ results in a torque $\rightarrow$ $d\ell \sin\alpha$
- Torque due to facing elements $\underline{\mathbf{d}\ell}$:

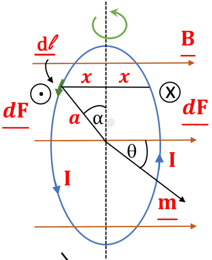
  $|d\underline{\mathbf{T}}| = 2|\underline{\mathbf{x}} \times d\underline{\mathbf{F}}| = 2x(I\, d\ell B \sin\alpha)\sin\theta$
- $x = a\sin\alpha$; $\ell = a\alpha$ $\rightarrow$ $d\ell = a\,d\alpha$

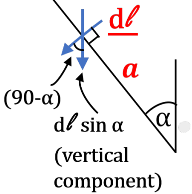
  $|d\underline{\mathbf{T}}| = 2\,(I\,a^2\sin^2\alpha)\,B\sin\theta\,d\alpha$
- Hence

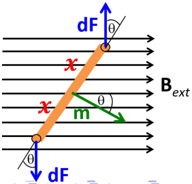
  $|\underline{\mathbf{T}}| = I\,a^2 B\sin\theta \int_0^{\pi} (1 - \cos 2\alpha)\ \ d\alpha$

  $= \pi a^2\, I\, B\, \sin\theta = I\, A\, B\, \sin\theta = mB\,\sin\theta$
- Result : torque on the magnetic dipole

  $\underline{\mathbf{T}} = \underline{\mathbf{m}} \times \underline{\mathbf{B}}_{\mathbf{ext}}$





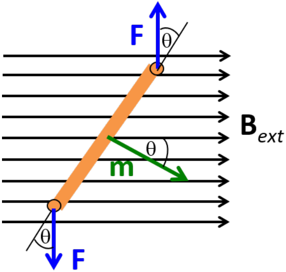


13.3 Energy of a magnetic dipole in a $\underline{\mathbf{B}}$-field

The energy of a magnetic dipole placed in an magnetic field $\underline{\mathbf{B}}_{\mathbf{ext}}$ is equal to the work done in rotating dipole into its position:

- Work to rotate dipole through angle $d\theta$
  $dW = T\,d\theta$
- Zero energy usually chosen at $\theta = \pi/2$
- $W = \int_{\pi/2}^{\theta} m\,B_{ext} \sin\theta'\,d\theta'$

  $= -\left[m\,B_{ext} \cos\theta\right]_{\pi/2}^{\theta}$
- Energy of the magnetic dipole



$$W = -m\,B_{ext} \cos\theta = -\underline{\mathbf{m}} \cdot \underline{\mathbf{B}}_{\mathbf{ext}}$$

[minimum at $\theta = 0$, maximum at $\theta = \pi$]

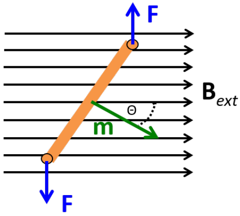
Compare with the energy of an electric dipole

$$W_{elec} = -\underline{\mathbf{p}} \cdot \underline{\mathbf{E}}_{\mathbf{ext}}$$

Magnetic dipole summary

Magnetic dipole moment *m* of a current loop = current × area of the loop: $\mathbf{m} = I\,\mathbf{A}$

Magnetic flux density of a magnetic dipole: $B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3}$ $\quad B_\theta = \mu_0 \frac{m\sin\theta}{4\pi r^3}$ $\quad B_\phi = 0$



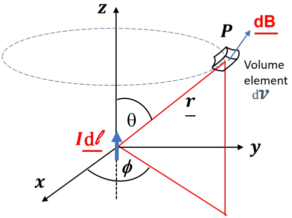
Torque on a magnetic dipole in an external magnetic field $\mathbf{B}_{ext}$:

$$\mathbf{T} = I\,\mathbf{A} \times \mathbf{B}_{ext} = \mathbf{m} \times \mathbf{B}_{ext}$$

Energy of a magnetic dipole in an external magnetic flux density $\mathbf{B}_{ext}$:

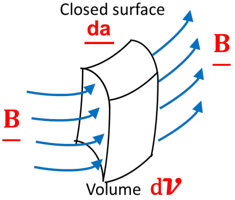
$$W = -m B_{ext} \cos\theta = -\mathbf{m} \cdot \mathbf{B}_{ext}$$

13.4 Divergence of $\underline{\mathbf{B}}$



- Place a current element $I\,\underline{\mathbf{d\ell}}$ at the origin pointing along the z-axis
- The Biot-Savart Law gives the field at point $P \rightarrow\ \underline{\mathbf{dB}} = \mu_0\, I\, \frac{\underline{\mathbf{d\ell}}\times\hat{\mathbf{r}}}{4\pi\, r^2}$
- $\underline{\mathbf{dB}}$ is perpendicular to $\underline{\mathbf{r}}$ and $\underline{\hat{\mathbf{z}}}$
- Rotate $\underline{\mathbf{r}}$ around ϕ, and it can be seen the lines of $\underline{\mathbf{B}}$ are *circles* in planes perpendicular to $\underline{\mathbf{d\ell}}$ and centred on it

  $\rightarrow$ the net outward flux of $\underline{\mathbf{B}}$ due to $\underline{\mathbf{d\ell}}$ through the surface of the volume element $d\mathcal{V}$ is zero



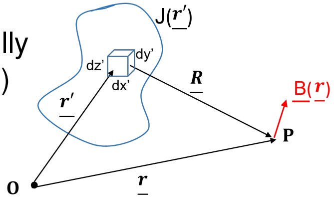
  - *Any* volume can be made up of volume elements as $d\mathcal{V}$
  - Hence $\oint_S \underline{\mathbf{B}}\cdot\underline{\mathbf{da}} = 0$ $\rightarrow$ no magnetic monopoles.
  - Divergence Theorem : $\oint_S \underline{\mathbf{B}}\cdot\underline{\mathbf{da}} = \int_{\mathcal{V}}(\underline{\nabla}\cdot\underline{\mathbf{B}})\,d\mathcal{V} \rightarrow$

    $\underline{\nabla}\cdot\underline{\mathbf{B}} = 0$

13.5 Divergence of $\underline{\mathbf{B}}$ from the Biot-Savart Law

Calculate $\underline{\mathbf{B}}$-field at point P due to a current density $\underline{\mathbf{J}}$.

$\underline{\mathbf{R}} = \underline{\mathbf{r}} - \underline{\mathbf{r}}' = (x - x', y - y', z - z')$



- $\underline{\mathbf{B}} = \int_{\mathcal{V}} \mu_0 \frac{\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \hat{\underline{\mathbf{R}}}}{4\pi R^2} d\mathcal{V}'$

  where $d\mathcal{V}' = dx'dy'dz'$. (Note carefully the primed and unprimed coordinates)
- $\underbrace{\underline{\nabla} \cdot \underline{\mathbf{B}}}_{\text{w.r.t. } \underline{\mathbf{r}}} = \frac{\mu_0}{4\pi} \int \underline{\nabla} \cdot \left(\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right) d\mathcal{V}'$
- Using the product rule :
  - $\underline{\nabla} \cdot \left(\underline{\mathbf{J}}(\underline{\mathbf{r}}') \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right) = \frac{\hat{\underline{\mathbf{R}}}}{R^2} \cdot \underbrace{\left(\underline{\nabla} \times \underline{\mathbf{J}}(\underline{\mathbf{r}}') \right)}_{=0 \text{ (because } \underline{\mathbf{J}}(\underline{\mathbf{r}}') \text{ does not depend on } \underline{\mathbf{r}})} - \underline{\mathbf{J}}(\underline{\mathbf{r}}') \cdot \left(\underline{\nabla} \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} \right)$
  - $\underline{\nabla} \times \frac{\hat{\underline{\mathbf{R}}}}{R^2} = \underline{\nabla} \times \frac{\underline{\mathbf{R}}}{R^3} = \frac{1}{R^3} \underbrace{\left(\underline{\nabla} \times \underline{\mathbf{R}} \right)}_{=0} + \underbrace{\underbrace{\underline{\nabla} \cdot \left(\frac{1}{R^3} \right)}_{\text{Vector along } \underline{\mathbf{R}}} \times \underline{\mathbf{R}}}_{=0}$
  - Hence $\underline{\nabla} \cdot \underline{\mathbf{B}} = 0$ and $\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0$

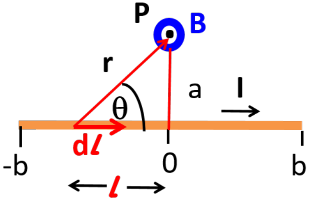
Lecture 14

Ampere's Circuital Law & Charge Conservation

14.1 Ampere's Circuital Law

- Ampere's Circuital Law can be derived formally from the Biot-Savart Law and vector calculus but is beyond the scope of this course.
- But for a special case, we return to the *B*-field due to an infinite straight wire with current I, previously derived.

$$B = \frac{\mu_0 I}{2\pi a}$$ $|\mathbf{B}|$ const. at radius a



- We can form the closed-loop integral :

$$\oint \mathbf{B} \cdot \mathbf{d}\ell = \frac{\mu_0 I}{2\pi a} \times 2\,\pi\, a = \mu_0 I$$

- This gives us *Ampere's Circuital Law* which is also applicable for the *general case* :

$$\oint \mathbf{B} \cdot \mathbf{d}\ell = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot \mathbf{da}$$ for current density $\mathbf{J}$

Note Ampere's Law needs to be amended in the presence of any time-varying electric field (see later).

Ampere's Circuital Law continued

- Ampere's Circuital Law in integral form

  $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d\ell}} = \mu_0 I = \mu_0 \int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$ for current density $\underline{\mathbf{J}}$

- Stokes Theorem : $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d\ell}} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{da}}$

  $\rightarrow \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{da}} = \mu_0 \oint_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$

- Ampere's Law in differential form :

  $\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \underline{\mathbf{J}}$

- Ampere's Law : an integral of magnetic flux density $\underline{\mathbf{B}}$ over a closed loop bounding a surface equals the current flowing through the surface.
- Allows straightforward calculations of B-fields along loops where B is constant.

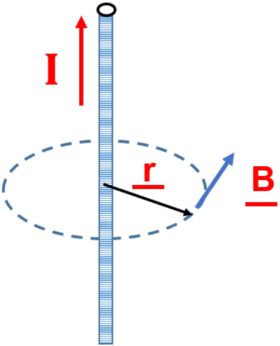
14.2 Example : B-field inside and outside a cylindrical wire

1. Outside the wire (this should be obvious $\cdots$)

- Ampere's Law $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I$
- Amperean path $\rightarrow$ circle of radius r :
  On this path $\underline{\mathbf{B}} \parallel \underline{\mathbf{d}\ell}$ and $|\underline{\mathbf{B}}|$ is constant
- $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = B \cdot 2\pi r = \mu_0 I$

$\rightarrow$ $B = \frac{\mu_0 I}{2\pi r}$ for an infinite wire

(much easier than using Biot - Savart !)



Cylindrical wire continued

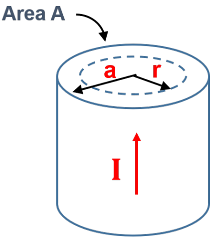
2. Inside the wire

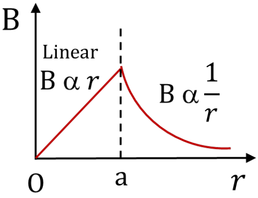
- Current evenly distributed throughout cylinder $\rightarrow\ J = I/A = \frac{I}{\pi a^2}$
- Ampere's Law for field at radius r
  $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 I = \mu_0 \int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$

$$B \cdot 2\pi r = \mu_0 \int_0^r \frac{I}{\pi a^2} 2\pi r' dr'$$

$$= \mu_0 I \underbrace{\left(\frac{\pi r^2}{\pi a^2}\right)}_{\text{ratio of areas}}$$

$$\rightarrow \quad B = \left(\frac{\mu_0 I}{2\pi a^2}\right) r \quad \text{inside wire}$$

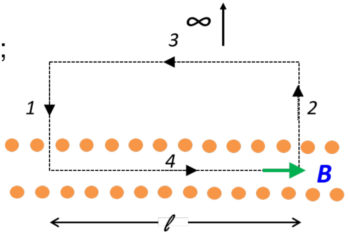




14.3 Example : B-field of a long solenoid

- Solenoid carrying current I
- Amperean path is a rectangle inside and outside the solenoid
- Take side 3 to ∞ (i.e. does not contribute); sides 1 & 2 cancel (due to symmetry)
- Contribution from side 4 only
  $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = B \cdot \ell = \mu_0 \, N \, I$

  where N is the number of turns within the Amperean surface



$$\rightarrow \quad B = \mu_0 \frac{N}{\ell} I$$

same as from Biot-Savart law as before $^{(*)}$

- B is uniform inside and zero outside the solenoid (if "infinite")

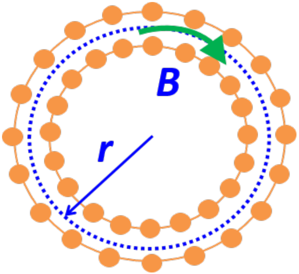
(*) Note that if the coil is not "infinite", end effects will need to be taken into account and here the field will not be uniform, i.e. Ampere's Law will not be as useful as presented here.

14.4 Example : B-field of a toroidal coil

- Toroid has N windings of wire carrying current I
- Amperean path inside the solenoid cuts current-carrying loops N times
- $\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = B \cdot 2\,\pi\,r = \mu_0\,N\,I$

$$\rightarrow \quad B = \frac{\mu_0\,N\,I}{2\,\pi\,r}$$

- B- field is uniform in toroid and follows circular path
- B-field is zero outside the confines of the toroid



Ampere's Law summary

Ampere's law:

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I$$

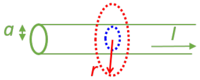
Electric currents generate magnetic fields whose field lines form closed loops.

"Gauss' law of Magnetism":

$$\oint_S \boldsymbol{B} \cdot \boldsymbol{da} = 0$$

There are no magnetic monopoles.

Infinite straight wire.



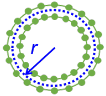
outside: $r > a$

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = B.2\pi r = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi r}$$

inside: $0 < r < a$

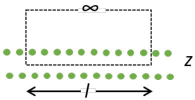
$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = B.2\pi r = \mu_0 I = \frac{\pi r^2}{\pi a^2} \rightarrow B = \frac{\mu_0 I}{2\pi a^2} r$$

Toroidal coil.



$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = B.2\pi r = \mu_0 N I \rightarrow B = \frac{\mu_0 N I}{2\pi r}$$

Infinite solenoid.



$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = B.l = \mu_0 N I \rightarrow B = \mu_0 I \frac{N}{l}$$

14.5 Conservation of charge

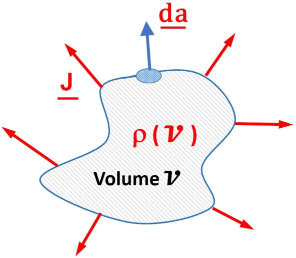
- Consider a volume $\mathcal{V}$ bounded by a surface S.

- The integral of current density flowing out (or into) the surface $\underline{\mathbf{J}} \cdot \underline{\mathbf{da}}$ is equal to the charge lost by the volume [per unit time].

- $\int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = I = -\frac{dQ}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V}$
  Statement of the conservation of charge

- Use the divergence theorem on the LHS

  $\int_{\mathcal{V}} \underline{\nabla} \cdot \underline{\mathbf{J}}\, d\mathcal{V} = -\frac{d}{dt} \int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V}$



This gives the *continuity equation* $\rightarrow$ $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{d}{dt}(\rho)$

(mathematical statement of charge conservation)

14.6 Current density and Ohm's Law

- Ohm's Law $V = I\,R$

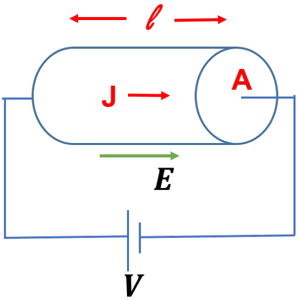
$$V = E\,\ell$$

$$I = J\,A$$

$$\rightarrow \; E\,\ell = J\,A\,R$$

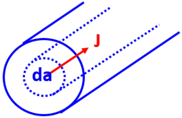
- This gives Ohm's Law in terms of current density: $\rightarrow$ $J = \frac{\ell}{R\,A}\,E$
- Conductivity $\sigma = \frac{\ell}{R\,A}$

  Resistivity $\rho = 1/\sigma$

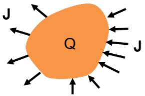


Summary : charge conservation & the continuity equation

Define current density **J**: $\mathbf{J} = \frac{d\mathbf{I}}{da_{\perp}}$



For a closed volume, the net current entering must be equal to the rate in change of charge inside the volume (charge conservation):



Continuity Equation:

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = I = -\frac{\partial Q}{\partial t} \longleftrightarrow \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

In the limit of electro/magneto-statics:

$\frac{\partial \mathbf{J}}{\partial t} = 0$ steady currents — constant B-fields

$\frac{\partial \rho}{\partial t} = 0$ stationary charges — constant E-fields

$\xrightarrow{\text{C E}}$ $\nabla \cdot \mathbf{J} = 0$

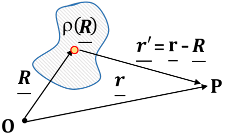
Lecture 15

Electromagnetic Induction

15.1.1 Summarizing where we are : electrostatics

1. Coulomb's Law :

$$\underline{\mathbf{E}}(\underline{\mathbf{r}}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\underline{\mathbf{R}})}{(\underline{\mathbf{r}}-\underline{\mathbf{R}})^3}(\underline{\mathbf{r}}-\underline{\mathbf{R}})\,d\mathcal{V}$$



- An electric charge generates an electric field. Electric field lines begin and end on charge or at ∞.

2. Gauss Law :

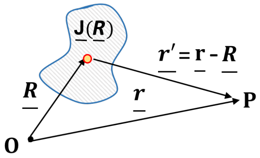
$$\underbrace{\oint_S \underline{\mathbf{E}}\cdot\underline{\mathbf{da}} = Q_{encl.}/\epsilon_0}_{\text{integral form}} \;\rightarrow\; \underbrace{\underline{\nabla}\cdot\underline{\mathbf{E}} = \rho/\epsilon_0}_{\text{differential form}}$$

3. The electric field is conservative :
   - A well-defined potential V such that $\underline{\mathbf{E}} = -\underline{\nabla}\, V$
     $\rightarrow\ \oint \underline{\mathbf{E}}\cdot\underline{\mathbf{d\ell}} = 0$ (work done is independent of path)
   - Using the vector identity : $\underline{\nabla}\times\underline{\mathbf{E}} = -\underline{\nabla}\times\underline{\nabla}\,V = 0$
   - Hence $\underline{\nabla}\times\underline{\mathbf{E}} = 0$

15.1.2 Summarizing where we are : magnetostatics

1. Biot-Savart Law :

$$\underline{\mathbf{B}}(\underline{\mathbf{r}}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\underline{\mathbf{J}}(\underline{\mathbf{R}})}{(\underline{\mathbf{r}}-\underline{\mathbf{R}})^3} \times (\underline{\mathbf{r}} - \underline{\mathbf{R}})\, d\mathcal{V}$$



- There are no magnetic monopoles. Magnetic field lines form closed loops.

2. Gauss Law of magnetostatics :

$$\underbrace{\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0}_{\text{integral form}} \rightarrow \underbrace{\underline{\nabla} \cdot \underline{\mathbf{B}} = 0}_{\text{differential form}}$$

3. Ampere's Law :
   - Magnetic fields are generated by electric currents.

$$\rightarrow \oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0\, I_{encl.} \rightarrow \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \underline{\mathbf{J}}$$

4. Continuity equation :
   - $\int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = -\frac{d}{dt}\int_{\mathcal{V}} \rho(\mathcal{V})\, d\mathcal{V} \rightarrow \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{d}{dt}(\rho)$
   (charge conserved)

Vector and scalar potential

Off syllabus, but worth a mention

Magnetic vector potential A defined through: $\mathbf{B} = \nabla \times \mathbf{A}$

Such **A** always exists because: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

Inserting into Ampere's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A})$$
$$= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

There is a certain degree of freedom in which **A** to choose – set: $\nabla \cdot \mathbf{A} = 0$

Poisson equations for magnetostatics:
(one for each **J** & **A** coordinate)

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Magnetic scalar potential V_m: $\mathbf{B} = -\mu_0 \nabla V_m$ ⟷ $V_m = -\frac{1}{\mu_0} \int_A^B \mathbf{B} \cdot d\mathbf{l}$

Caution: V_m is *pathway-dependent and not single-valued* because $\nabla \times \mathbf{B} \neq 0$.

But V_m can be used with care in simply-connected, current-free regions.

Being a scalar, V_m is mathematically easier to use than the vector potential.

15.2 Electromagnetic induction - outline

Up to now we have considered stationary charges and steady currents. We now focus on what happens when either the *E*-field or *B*-field varies with time.

1. Introduction: Electromagnetic Induction
2. Faraday's and Lenz's Laws of Induction
3. Self-Inductance and Mutual Inductance
4. The Transformer
5. Energy of the Magnetic Field
6. Charged Particles in E- and B-Fields

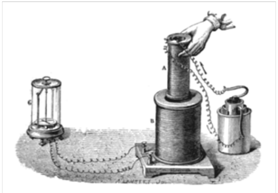
Items 1–5: Problem Set 4

Item 6: Problem Set 5

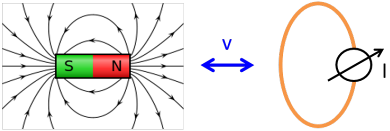
Origins of electromagnetic induction

1831: Michael Faraday carries out a series of experiments and observes:

Two coils are arranged in a way so that the magnetic flux density of coil A penetrates through coil B. He found that if the B-field in coil A is changing, this induces an electrical current in coil B.



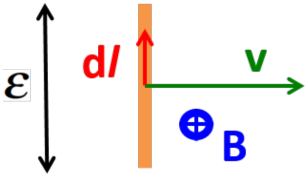
Moving a bar magnet through a circuit element (wire loop) generates a current in the circuit. Moving instead the circuit w.r.t. the bar magnet, gives the same result.



→ A change with time in the magnetic flux density through a circuit causes an "electromotive force" that moves charges along the circuit.

15.3 Faraday and Lenz's Laws of Induction

15.3.1 Electromotive force (EMF)



- Consider a wire moving with velocity $\underline{\mathbf{v}}$ through a B-field.
- Free charges in the wire experience a Lorenz force, perpendicular to $\underline{\mathbf{v}}$ & $\underline{\mathbf{B}}$:

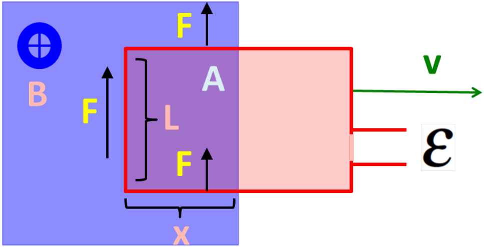
$$\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$$

  - This moves charge to one side/end of the wire, which will create an electric potential drop along the wire :
    $\mathcal{E} = \int_\ell \frac{dW}{q} = \int_\ell \frac{\underline{\mathbf{F}} \cdot \underline{\mathbf{d\ell}}}{q}$ (by definition, $V =$ work/unit charge)
  - Hence $\mathcal{E} = \int_\ell (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d\ell}}$

    $\mathcal{E}$ is the *electromotive force* (or *electromotance*) (EMF)
  - Note that $\mathcal{E}$ is *not* a force but a line integral over a force (i.e. a potential) !

15.3.2 Magnetic flux

- Now consider a wire circuit loop being pulled with velocity $\underline{\mathbf{v}}$ out of a region containing a B-field.



- EMF on vertical side :

$$\mathcal{E} = \int_\ell (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d}\ell}$$
$$= v\,B\,L$$

  - No contribution to EMF from horizontal sides
  - Define *magnetic flux* $\Phi = \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$
  - Rate of change of flux $\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = \frac{d}{dt}\int_S B\,da$ (since $\underline{\mathbf{B}}$ is $\parallel$ to $\underline{\mathbf{da}}$)
  - $\frac{d\Phi}{dt} = \frac{d}{dt}(BA) = \frac{d}{dt}(B\,Lx) = B\frac{dx}{dt}L = -v\,B\,L = -\mathcal{E}$ (negative since x *decreases* with positive v)
  - In general, $\mathcal{E}$ from magnetic flux $\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = -\mathcal{E}$

15.4 Faraday's and Lenz's Laws

- Faraday's Law

  The induced electromotance (EMF) $\mathcal{E}$ in any closed circuit is equal to (the negative of) the time rate of change of the magnetic flux Φ through the circuit.

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = -\mathcal{E}$$

- Lenz's Law

  The induced electromotance always gives rise to a current whose magnetic field opposes the original change in magnetic flux.

15.5 Faraday's Law in differential form

- Net potential around a *closed* circuit loop = 0

  $\mathcal{E} = \oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell}$, hence $V = -\mathcal{E} = -\oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell}$

- Faraday's Law in integral form

  $\mathcal{E} = \oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\frac{d}{dt} \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$

  Apply Stokes' theorem to LHS :

  $\int_S (\underline{\nabla} \times \underline{\mathbf{E}}) \cdot \underline{\mathbf{da}} = -\frac{d}{dt} \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$

- Gives Faraday's Law in differential form

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

- Any time-varying magnetic field (or change in magnetic flux) generates an electric field which results in an electric potential $\mathcal{E}$.

  (In contrast $\underline{\nabla} \times \underline{\mathbf{E}} = 0$ for electro/magnito-statics)

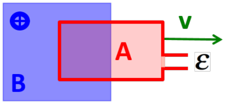
Lecture 16

Induction Examples & Self Induction

Faraday's and Lenz's Laws summary

Faraday's Law of electromagnetic induction:

The induced electromotance $\mathcal{E}$ in any closed circuit is equal to the negative of the time rate of change of the magnetic flux Φ through the circuit.



$$\mathcal{E} = \frac{d\Phi}{dt} = -\frac{d}{dt}\oint_S \boldsymbol{B} \,.\, \boldsymbol{da}$$

In terms of E- and B-fields:

Integral form:

$$\oint \boldsymbol{E} \,.\, \boldsymbol{d\ell} = -\frac{d}{dt}\int_S \boldsymbol{B} \,.\, \boldsymbol{da}$$

Differential form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Lenz's Law:

An induced electromotance always gives rise to a current whose magnetic field opposes the original change in magnetic flux.

Unit of magnetic flux Weber $[Wb] = [Tm^2] = [kg\, m^2 s^{-2} A^{-1}]$

16.1 Example : the Homopolar Generator (Faraday's disk)

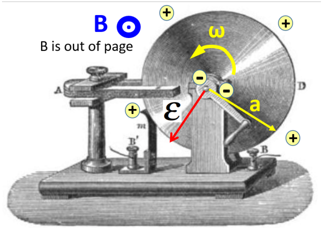
1. Determine voltage using Lorenz force

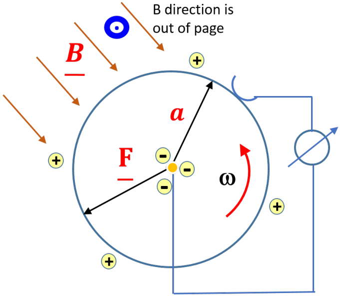
- Metal disk mechanically rotated (performing work)
- A *B*-field is present with $\underline{\mathbf{B}}$ perpendicular to the disk area.
- Voltage pick-up between the centre and rim of disk.
- EMF is *radial*, with identical potential along each circumference element, radius r

  $$\mathcal{E} = \int_{r=0}^{r=a} (\underline{\mathbf{v}} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{dr}}$$

  where $\underline{\mathbf{v}} \perp \underline{\mathbf{B}} \perp \underline{\mathbf{dr}}$ and $v = r\,\omega$

- $\mathcal{E} = \int_0^a \omega B\, r\, dr = \frac{1}{2}\omega a^2 B$





The Homopolar Generator continued

2. Determine using Faraday's Law

$$\mathcal{E} = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{da}$$

- Consider area element $\Delta A = r\,\Delta\theta\,\Delta r$

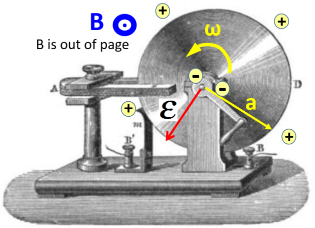
$$\frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}(r\,\Delta\theta\,\Delta r) = \omega r\,\Delta r$$

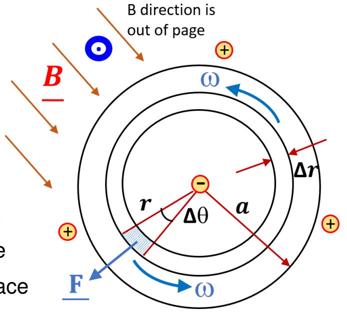
- Add up all contributions $\Delta r \to dr$

  (There is a +/- sign ambiguity depending on direction of $\mathbf{da}$. Take direction such that $\mathcal{E}$ is positive.)

- $\mathcal{E} = \int_0^a \omega B\, r\, dr = \frac{1}{2}\omega a^2 B$

  same result as before *.

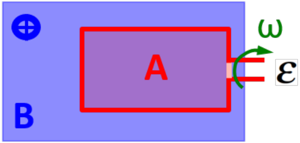




* Strictly speaking, this method from Faraday's Law is not entirely sensible since the current is continuous across the disk and $\int_S \mathbf{B}\cdot\mathbf{da}$ is in principle only applicable for a surface bounding a closed current path (see for example Griffiths).

16.2 Example : coil rotating in a B-field

Coil, N turns, rotating at angular frequency ω in a uniform B-field



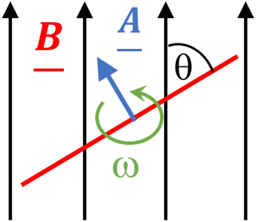
- Magnetic flux
  $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = N\,A\,B \sin\theta$

  where $\theta = \omega\, t$

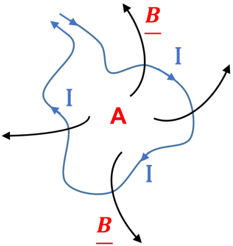
  [$\times N$ since each turn of the coil links flux]

$$\mathcal{E} = -\frac{d\Phi}{dt} = -N\,A\,B\,\omega \cos\omega\, t$$

- This is a generator/dynamo (incorporated into most aspects of electrical power generation).



16.3 Self inductance



- Take a closed-loop circuit through which current flows
- The current I has an associated magnetic field which penetrates the circuit, $B \propto I$
- If the current changes, there will be a changing B-field through the loop.
  - Faraday : The changing magnetic flux Φ induces an EMF (voltage) in the loop *itself* : $\mathcal{E} = -\frac{d\Phi}{dt}$, where $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$
  - Lenz : This EMF will act in a direction so as to oppose the change in flux which caused it
  - EMF induced $\mathcal{E} = -\frac{d\Phi}{dt}$; Note that $\Phi \propto B \propto I$
  - Define *self inductance* $L = \frac{\Phi}{I}$

    Since $\Phi \propto I$, can also be written $L = \frac{d\Phi}{dI} = \frac{d\Phi}{dt} / \frac{dI}{dt} = -\mathcal{E} / \frac{dI}{dt}$
  - L depends solely on the geometry of the circuit. (Compare with circuit theory : $V = L\,\frac{dI}{dt}$)

16.4 Example : self induction of a long coil

Calculate the self inductance of a long coil, area A, length ℓ, with N turns

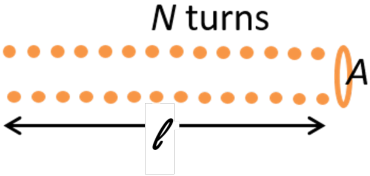
- From Ampere's law

  $B = \mu_0 \frac{N}{\ell} I$

- Magnetic flux $\Phi = \int \underline{\mathbf{B}} \cdot \underline{\mathbf{da}}$

  $\Phi = N A B = \mu_0 \frac{N^2}{\ell} A I$

  (since each of the N coils links its own flux)



- Hence $L = \frac{\Phi}{I} = \mu_0 \frac{N^2}{\ell} A$

- EMF induced in coil :

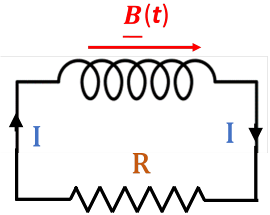
  $\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 \frac{N^2}{\ell} A \frac{dI}{dt} = -L \frac{dI}{dt}$

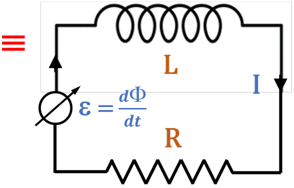
16.5 Example : long coil in varying B with resistive load

- Consider a long coil, area A, length ℓ, with N turns.
- Coil is immersed in axial time-varying magnetic field : $B(t) = B_0 \cos \omega t$
- EMF is induced in coil, coil is connected across a resistor $\rightarrow$ current will flow
- EMF induced :

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(N\,A\,B_0 \cos \omega t\right)$$

$$= N\,A\,\omega\,B_0 \sin \omega t$$





Long coil in varying B with resistive load, continued

- Self inductance of coil : $L = \mu_0 \frac{N^2}{\ell} A$
- Back EMF induced due to L opposes the changing current (Lenz)
- Ohm's Law for current flowing in the coil

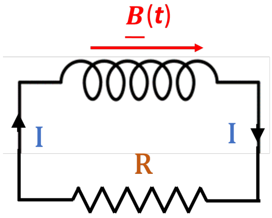
$$\underbrace{\mathcal{E}}_{\text{induced emf}} = IR + \underbrace{L\frac{dI}{dt}}_{\text{back emf}}$$

Alternatively can write $\mathcal{E} = I\,Z$
where $Z = R + j\omega L \;\rightarrow\; Z = |Z|\,e^{j\phi}$

- $\mathcal{E} = Im\left[\mathcal{E}_0\, e^{j\omega t}\right]$ where $\mathcal{E}_0 = (N\,A\,\omega\,B_0)$
- Current $I = I_0\; Im\left[e^{j(\omega t-\phi)}\right]$

where $I_0 = \mathcal{E}_0/|Z| = \mathcal{E}_0/\sqrt{R^2 + (\omega L)^2}$

and phase angle : $\tan\phi = (\omega L)/R$



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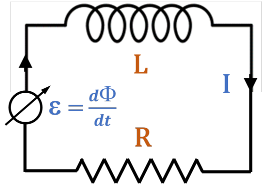
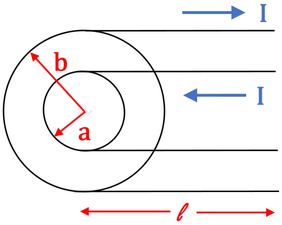


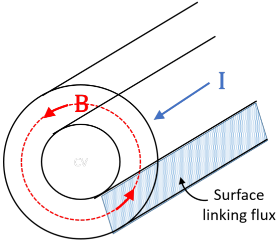
16.5 Example : self induction of a coaxial cable

Calculate the self inductance of a coaxial cable, inner/outer radii a & b, length ℓ



- From Ampere's law, for $a \geq r \geq b$: $B = \frac{\mu_0 I}{2\pi r}$
- Note that the area linking flux is *radial* : $da = \ell\, dr$
- Magnetic flux :

$$\Phi = \int_a^b \frac{\mu_0 I}{2\pi r} \ell\, dr = \frac{\mu_0 I}{2\pi} \log_e\left(\frac{b}{a}\right) \ell$$



$$L = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} \log_e\left(\frac{b}{a}\right) \ell$$

Lecture 17

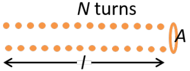
Self & Mutual Inductance

Self inductance summary

Self-inductance L is the ratio of the voltage (emf) produced in a circuit by self-induction, to the rate of change in current causing the induction.

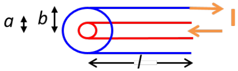
$$L = \frac{\frac{d\Phi}{dt}}{\frac{dI}{dt}} = \frac{d\Phi}{dI} = \frac{-\varepsilon}{\dot{I}}$$

Self-inductance of a long coil.



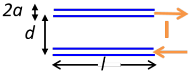
$$L = \frac{d\Phi}{dI} = \mu_0 \frac{N^2}{l} A$$

Self-inductance of a coaxial cable.



$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) l$$

Self-inductance of two parallel wires.

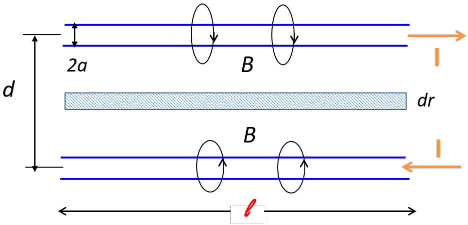


$$L = \frac{\mu_0}{\pi} \ln\left(\frac{d-a}{a}\right) l$$

Units of self inductance : the Henry $[H] \equiv [kg\, m^2 s^{-2} A^{-2}]$.
When the current changes at one ampere per second ($A\, s^{-1}$), an inductance of 1 H results in the generation of one volt (1 V) of potential difference.

17.1 Example : self inductance of two parallel wires

Calculate the self inductance of two parallel wires,radius a, separation to the centres d, and of length ℓ



- From Ampere's law, outside each wire :

  $B = \frac{\mu_0 I}{2\,\pi\,r}$

  Radial area element $\ell\,dr$
- Magnetic flux :

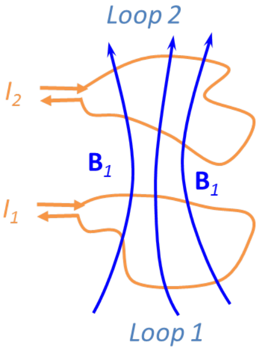
  $$\Phi = 2 \times \int_a^{d-a} B\,\ell\,dr$$

  (factor 2 because same contribution from 2 wires):

  $$= 2 \times \int_a^{d-a} \frac{\mu_0 I\,\ell}{2\,\pi}\,\frac{1}{r}\,dr \;=\; \frac{\mu_0\,\ell}{\pi} log_e\left(\frac{d-a}{a}\right)\; I$$
- $$L = \frac{\Phi}{I} = \frac{\mu_0}{\pi}\,\ell\;log_e\left(\frac{d-a}{a}\right)$$

  $$\approx \frac{\mu_0}{\pi}\,\ell\,log_e\left(\frac{d}{a}\right) \quad \text{for } a << d$$

- Current I_1 through circuit loop 1 generates magnetic field density B_1 which penetrates circuit loop 2
- A change in current I_1 will induce an EMF in circuit loop 2



- Define *mutual inductance M*

$$M_{21} = \frac{\Phi_2}{I_1} \quad ; \quad M_{12} = \frac{\Phi_1}{I_2} \quad ; \quad M_{12} = M_{21}$$

- Since $\Phi \propto I$, can also be written $M_{21} = \frac{d\Phi_2}{dI_1}$; $M_{12} = \frac{d\Phi_1}{dI_2}$

17.3 Mutual induction of two coaxial solenoids

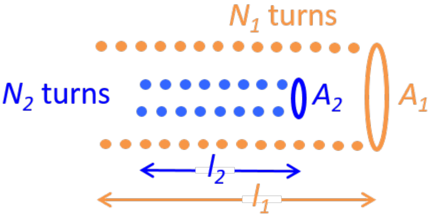
1. Current through coil 1 creates magnetic field through coil 2.

$B_1 = \mu_0 \frac{N_1}{\ell_1} I_1$

- A_2 : area of pick-up coil 2
- Flux experienced by coil 2

  $\Phi_2 = N_2 A_2 B_1 = \mu_0 \frac{N_1}{\ell_1} I_1 N_2 A_2$
- Mutual inductance :

$$M_{21} = \frac{\Phi_2}{I_1} = \mu_0 \frac{N_1 N_2}{\ell_1} A_2$$

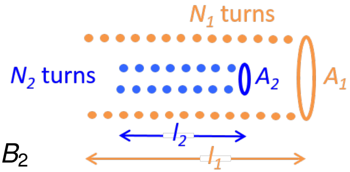


- EMF induced in coil 2 :
  - $\mathcal{E} = -\frac{d\Phi_2}{dt} = -\mu_0 \frac{N_1}{\ell_1} A_2 N_2 \frac{dI_1}{dt}$

  $\mathcal{E} = -M_{21} \frac{dI_1}{dt}$ (compare to $\mathcal{E} = -L\frac{dI}{dt}$ for self inductance)

Mutual induction of two coaxial solenoids continued

2. Current through coil 2 creates magnetic field through coil 1.



- $\Phi_1 = \int B_2 \, da_1'$ (da_1' is "effective" area)

  Now it's more complicated
  as B_2 is not uniform through coil 1 !

- Flux experienced by coil 1 $\Phi_1 = N_1' \, A_1' \, B_2$

  Overlap with volume over which B is "strongest"

  - Approximate : neglect stray fields of B_2 outside coil 2

    then $A_1' = A_2$ and $N_1' = N_1 \frac{\ell_2}{\ell_1}$ and $B_2 = \mu_0 \frac{N_2}{\ell_2} I_2$

  - Mutual inductance :

$$M_{12} = \frac{\Phi_1}{I_2} = \frac{N_1 \, (\ell_2/\ell_1) \, A_2 \, \mu_0 \, (N_2/\ell_2) \, I_2}{I_2} = \mu_0 \frac{N_1 \, N_2}{\ell_1} A_2 = M_{21}$$

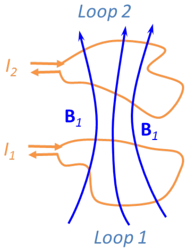
  - $M_{12} = M_{21}$ This is Neumann's theorem. (It turns out even if we had done the exact calculation the result would have been the same)

Mutual inductance summary

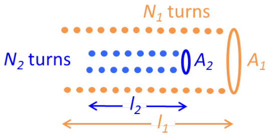
Mutual Inductance M: is the ratio of the voltage (emf) produced in a circuit by self-induction, to the rate of change in current causing the induction.

$$M_{12} = \frac{d\phi_1}{dI_2} \quad \xleftrightarrow[\text{formula}]{\text{Neumann}} \quad M_{21} = \frac{d\phi_2}{dI_1}$$

$$M_{12} = M_{21}$$



Mutual inductance of two coaxial solenoids.



$$M_{12} = \mu_0 \frac{N_1 N_2}{l_1} A_2$$

Units of mutual inductance :
again the Henry $[H] \equiv [kg\, m^2 s^{-2} A^{-2}]$.

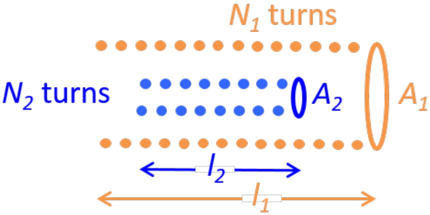
Lecture 18

Transformer & Magnetic Energy

18.1 Coaxial solenoids sharing the same area

From before : mutual inductance between coils :

$$M_{21} = M_{12} = \mu_0 \frac{N_1 N_2}{\ell_1} A_2 \ (= M)$$



- Self inductance of coils 1 & 2

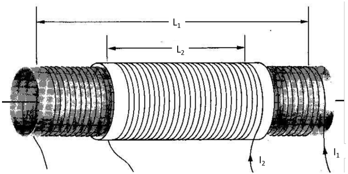
$$L_1 = \mu_0 \frac{N_1^2}{\ell_1} A_1 \quad \text{and}$$

$$L_2 = \mu_0 \frac{N_2^2}{\ell_2} A_2$$

- If $A_1 = A_2$

$$M = \left(\sqrt{\frac{\ell_2}{\ell_1}}\right) \sqrt{(L_1 L_2)}$$

If $\ell_1 = \ell_2$ then $M = \sqrt{(L_1 L_2)}$



- Hence the mutual inductance is proportional to the geometrical mean of the self inductances.

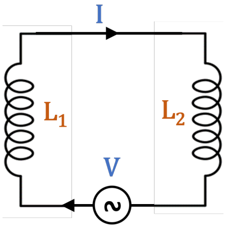
In general circuits may not be tightly coupled, hence $M = k\sqrt{(L_1 L_2)}$ where $k < 1$. k is the *coefficient of coupling*.

18.2 Inductors in series and parallel

- 1. In series with no mutual inductance between coils :

$$V = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} = (L_1 + L_2) \frac{dI}{dt}$$

$$L = L_1 + L_2$$

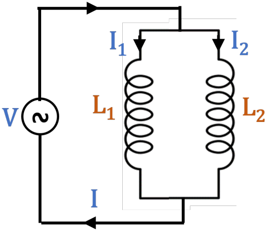


- 2. In series with mutual inductance between coils :

$$V = (L_1 + M) \frac{dI}{dt} + (L_2 + M) \frac{dI}{dt}$$

$$= (L_1 + L_2 + 2M) \frac{dI}{dt}$$

$$L = L_1 + L_2 + 2M$$



- 3. In parallel, no mutual inductance :

$$V = L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} \quad \text{where } I = I_1 + I_2$$

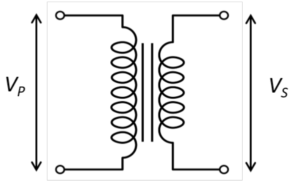
Write $V = L \frac{dI}{dt} \rightarrow V = L\left(\frac{dI_1}{dt} + \frac{dI_2}{dt}\right) = L\left(\frac{V}{L_1} + \frac{V}{L_2}\right)$

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

(with mutual inductance $L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$)

18.3 The transformer

- Primary coil creates flux $\Phi_p = A_p B_p$ per winding $\rightarrow$ secondary coil gives EMF per winding $\mathcal{E}_S = -\frac{d\Phi_S}{dt}$
- The coils are coupled : $\Phi_S = k\Phi_P$ where $k = 1$ for an ideal transformer (k depends on geometry, coupling etc.)



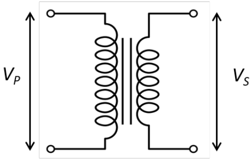
- Ratio of EMFs :

$$\mathcal{E}_P = -N_P \frac{d\Phi_P}{dt} \quad \mathcal{E}_S = -N_S \frac{d\Phi_S}{dt} \rightarrow \quad \frac{V_S}{V_P} = \underbrace{\frac{d\Phi_S}{d\Phi_P}}_{\text{k}} \times \underbrace{\frac{N_S}{N_P}}_{\text{winding ratio}}$$

  - Transformer will step up or step down applied voltage V_P by the winding ratio
  - Ideally there is no power dissipated in the transformer if coils have zero resistance

$$\rightarrow \; V_S I_S = V_P I_P \; \rightarrow \; \frac{I_S}{I_P} = \frac{V_P}{V_S} = \frac{1}{k}\frac{N_P}{N_S}$$

Transformer summary



Primary coil creates flux which permeates secondary coil, coupling their voltages:

Voltage Ratio: $$\frac{V_S}{V_P} = \frac{d\Phi_S}{d\Phi_P}\frac{N_S}{N_P}$$

Current Ratio: $$\frac{I_S}{I_P} = \frac{d\Phi_P}{d\Phi_S}\frac{N_P}{N_S}$$

18.4 Energy of the magnetic field

Consider the energy stored in an inductor L :

- Change in current results in a back EMF $\mathcal{E}$
- We need to do work to change the current : $dW = VdQ$

  Power = work per unit time = $V\frac{dQ}{dt} = VI$

  Energy expended $U = \int \underbrace{VI}_{\text{power}} dt = \int \underbrace{L\frac{dI}{dt}}_{\text{Back EMF}} I\,dt$

- $U = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I$ $\quad (L = \frac{\Phi}{I})$

  regardless of circuit / current geometry

- For a coil : $L = \mu_0 \frac{N^2}{\ell} A$ and $B = \mu_0 \frac{N}{\ell} I$ (Ampere Law)

  $\rightarrow U = \frac{1}{2}\left(\mu_0 \frac{N^2}{\ell} A\right)\left(\frac{B^2}{\mu_0^2 \frac{N^2}{\ell^2}}\right) = \frac{1}{2}\frac{B^2}{\mu_0} A\ell = \frac{1}{2}\frac{B^2}{\mu_0}\mathcal{V} \leftarrow$ volume

- In the general case : $U = \frac{1}{2\mu_0}\int B^2\,d\mathcal{V}$ over all space

Summary of energy in E and B fields

Electric field energy

- In terms of circuits :

$$U_e = \frac{1}{2}\, C\, V^2$$
$$= \frac{1}{2}\, Q\, V$$

- In terms of fields :

$$U_e = \frac{\epsilon_0}{2} \int_{all\ \ space} E^2\, d\mathcal{V}$$

Magnetic field energy

- In terms of circuits :

$$U_m = \frac{1}{2}\, L\, I^2$$
$$= \frac{1}{2}\, \Phi\, I$$

- In terms of fields :

$$U_m = \frac{1}{2\,\mu_0} \int_{all\ \ space} B^2\, d\mathcal{V}$$

Lecture 19

Motion in E & B Fields & Displacement Current

19.1 Motion of charged particles in E and B fields

- Force on a charged particle in an $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ field :

$$\underline{\mathbf{F}} = q\left(\underbrace{\underline{\mathbf{E}}}_{\text{along } \underline{\mathbf{E}}} + \underbrace{\underline{\mathbf{v}} \times \underline{\mathbf{B}}}_{\perp \text{ to both } \underline{\mathbf{v}} \text{ and } \underline{\mathbf{B}}}\right)$$

- Newton second law provides equation of motion :

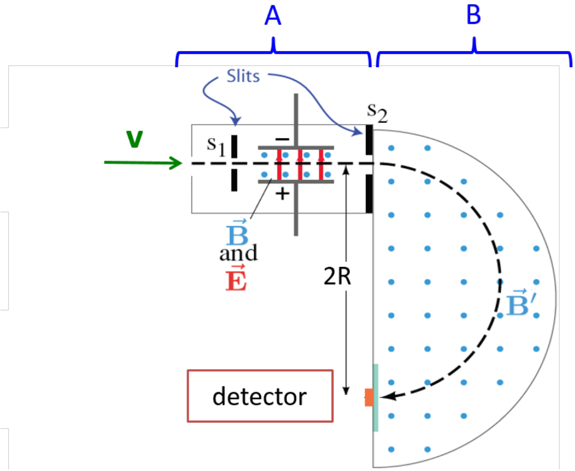
$$\underline{\mathbf{F}} = m\,\underline{\mathbf{a}} = m\,\underline{\ddot{\mathbf{r}}} = q\left(\,\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}\right)$$

- Will demonstrate with 2 examples :

  1. Mass spectrometer
  2. Magnetic lens

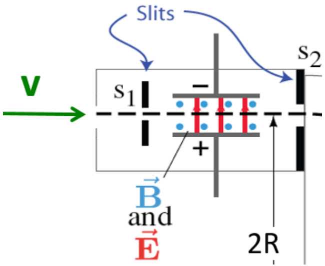
19.2 Example : the mass spectrometer

Used for detecting small charged particles (molecules, ions) by their mass m .



Stage A : The velocity filter

- The particle will pass through both slits if it experiences no net force inside the filter

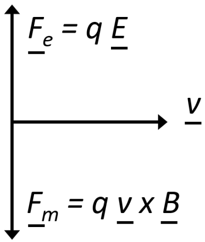


- The region has both $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields

$$\underline{\mathbf{F}} = q\,(\,\underline{\mathbf{E}} + \underline{\mathbf{v}} \times \underline{\mathbf{B}}) = 0$$

$$\rightarrow \quad \text{need } \underline{\mathbf{E}} = -\underline{\mathbf{v}} \times \underline{\mathbf{B}} \rightarrow \; v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

$$(\underline{\mathbf{E}} \perp \underline{\mathbf{v}} \,\&\, \underline{\mathbf{B}})$$



- Will filter particles with $v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$ and the spread $\pm\Delta v$ is given by the slit width

Stage B : The mass filter

- This region has only a $\underline{\mathbf{B}}$ field

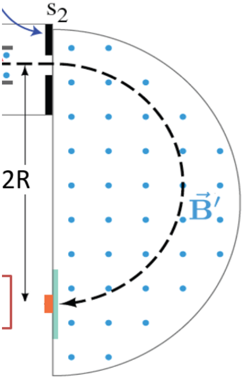
$$m\,\ddot{\underline{\mathbf{r}}} = q\,\dot{\underline{\mathbf{r}}} \times \underline{\mathbf{B}}$$

$$\text{with } \underline{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \text{ and } \dot{\underline{\mathbf{r}}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{q}{m} \begin{pmatrix} \dot{y}\,B \\ -\dot{x}\,B \\ 0 \end{pmatrix}$$

$$\rightarrow \ddot{z} = 0 \rightarrow v_z = \text{constant} \ (= 0)$$

- $$\ddot{r}^2 = \ddot{x}^2 + \ddot{y}^2 = \frac{q^2}{m^2} \underbrace{\left(\dot{x}^2 + \dot{y}^2\right)}_{v^2} B^2$$



- Circular motion in $x - y$ plane with : $\ddot{r} = \frac{q}{m} v\,B$

For circular motion $\ddot{r} = \frac{v^2}{R} \rightarrow R = \frac{mv}{qB}$

- Since q and v are constant, then $R \propto m$

Mass spectrometer summary

In the presence of both E- and B-fields, a charge experiences the force:

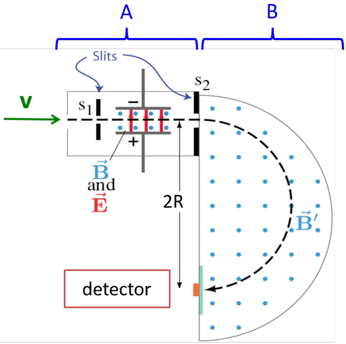
$$\mathbf{F}_{\mathrm{EM}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Mass Spectrometer.

A. velocity filter:

E&B-fields present. Charged particles pass through Stage A if their velocity equals the amplitude ratio:

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$



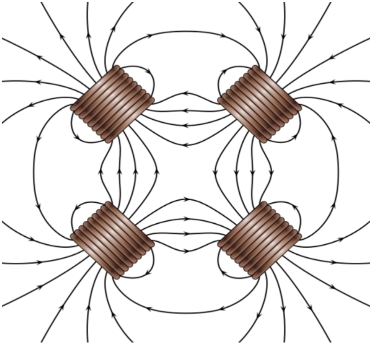
B. Filter stage:

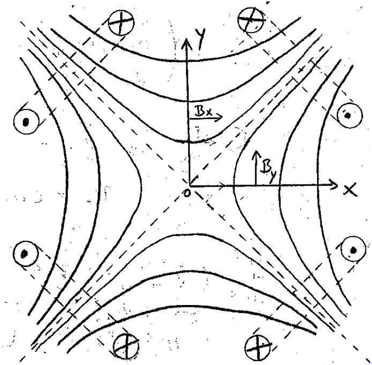
Only B-field present. Charged particles are forced on circular path with radius:

$$R = \frac{mv}{qB}$$

19.3 Example : magnetic lenses

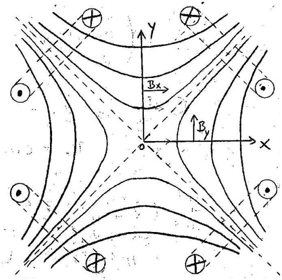
- Magnetic lenses are used for focusing and collimating charged particle beams. Used in electron microscopes, particle accelerators etc.
- Quadrupole lens : four identical coils aligned in z-direction.
- Sum of 4 dipole fields : for small values of x, y close to the axis of symmetry, $B_x \propto y$, $B_y \propto x$





Quadrupole lens

- Along x-axis : only B_y component
- Along y-axis : only B_x component
- No z-component (symmetry)
- Inside the lens, close to the z-axis



$$\underline{\mathbf{B}} = \begin{pmatrix} k\,y \\ k\,x \\ 0 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

- Equation of motion $\underline{\mathbf{F}} = q\,\underline{\mathbf{v}} \times \underline{\mathbf{B}}$

$$\mathrm{m}\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ k\,y & k\,x & 0 \end{vmatrix} = q\,k \begin{pmatrix} -x\,\dot{z} \\ y\,\dot{z} \\ x\,\dot{x} - y\,\dot{y} \end{pmatrix}$$

- Assume particle travels at a small angle wrt the z-axis :

$$\rightarrow \;\; \dot{x}, \dot{y} \approx 0 \;\rightarrow\; \ddot{z} = 0 \;\rightarrow\; \dot{z} = v = \text{constant} \;\rightarrow\; z = v\,t$$

- Equations of motion in the $x - y$ plane :

$$\ddot{x} = -\frac{q}{m} k\,v\,x \;\; \text{and} \;\; \ddot{y} = \frac{q}{m} k\,v\,y$$

Quadrupole lens continued

- Equ. of motion : $\ddot{x} = -\alpha^2 x$ & $\ddot{y} = \alpha^2 y$, where $\alpha = \sqrt{\frac{q\,k\,v}{m}}$
- Solutions : $x(t) = A\sin\alpha\,t + B\cos\alpha\,t$

  $$y(t) = C\sinh\alpha\,t + D\cosh\alpha\,t$$

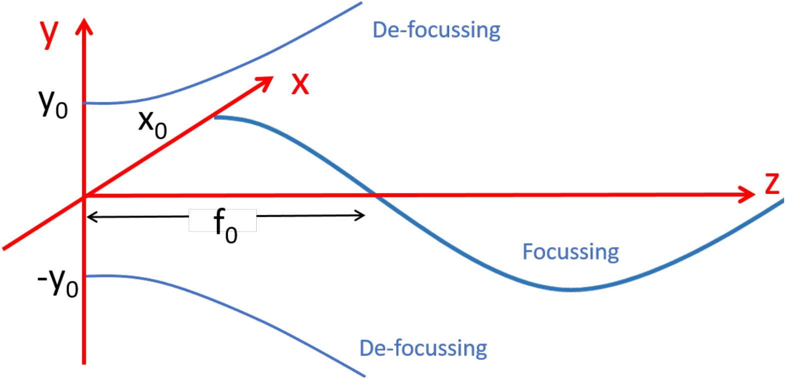
  where $\cosh y, \sinh y = (e^y \pm e^{-y})/2$
- Boundary conditions :
  At $t = 0 \;\rightarrow\; z = 0,\; x = x_0$ and $\dot{x} = 0, y = y_0$ and $\dot{y} = 0$
- Solutions : $x(t) = x_0\cos\alpha t = x_0\cos\frac{\alpha}{v}z$: focusing

  $y(t) = y_0\cosh\alpha t = y_0\cosh\frac{\alpha}{v}z$: de-focusing

  (where $t = z/v$) $\rightarrow\; x = 0$ for $\frac{\alpha}{v}z = \frac{\pi}{2} + n\,\pi$
- Focal points in z direction (x=0) at $f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$
- Use lens pair with 90° angle for collimating a charged beam

Quadrupole lens continued

The lens pulls the beam on-axis in x and removes particles deviating in y

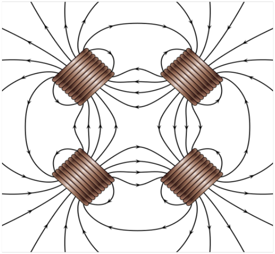


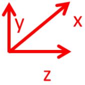
$$f_n = \frac{\pi}{2}\sqrt{\frac{m\,v}{q\,k}} + n\,\pi\sqrt{\frac{m\,v}{q\,k}}$$

Magnetic lens summary

Magnetic Lens.

$$\mathbf{B} = (\mathrm{k}\, y, \mathrm{k}\, x, 0)$$





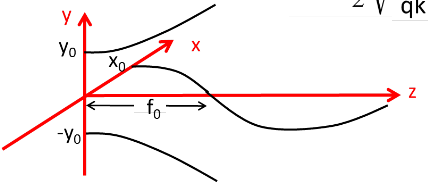
Equation of Motion: $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$

Solutions:

$$y(z) = y_0 \cosh \sqrt{\frac{q\,\mathrm{k}}{vm}} z$$ de-focusing

$$x(z) = x_0 \cos \sqrt{\frac{q\,\mathrm{k}}{vm}} z$$ focusing with

$$f_0 = \frac{\pi}{2}\sqrt{\frac{vm}{\mathrm{qk}}}$$



19.4 Electrodynamics "before Maxwell"

1. Gauss Law :

$$\oint_S \underline{\mathbf{E}} \cdot \underline{\mathbf{da}} = Q_{encl.}/\epsilon_0 \quad \rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{E}} = \rho/\epsilon_0$$

2. No magnetic monopoles :

$$\oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} = 0 \quad \rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

3. Faraday's Law :

$$\oint_\ell \underline{\mathbf{E}} \cdot \underline{\mathbf{d}\ell} = -\frac{\partial}{\partial t} \oint_S \underline{\mathbf{B}} \cdot \underline{\mathbf{da}} \quad \rightarrow \quad \underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

4. Ampere's Law :

$$\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 \, I_{encl.} \quad \rightarrow \quad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \underline{\mathbf{J}}$$

Time-varying *B*-fields generate *E*-fields. *However*, time-varying *E*-fields do not seem to create *B*-fields in this version.
Is there something wrong ?

19.5 Revisit Ampere's Law

- Ampere's Law : $\rightarrow \quad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \underline{\mathbf{J}}$

  Apply Div : $\rightarrow \quad \underbrace{\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{B}})}_{\text{always zero}} = \underbrace{\mu_0 \, \underline{\nabla} \cdot \underline{\mathbf{J}}}_{\text{not always zero !!}}$

- Recall the continuity equation :

  $$\int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{da}} = -\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho(\mathcal{V}) \, d\mathcal{V} \quad \rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$$

  [Current leaving volume through surface] = [Rate of change of charge inside volume]

- Therefore Ampere's Law in its current form violates the continuity equation (and hence charge conservation) !

- But this is not surprising since we derived Ampere's Law assuming that $\frac{\partial}{\partial t}(\rho) = 0$

  $\rightarrow$ We have to "fix" Ampere's Law !

19.6 Fixing Ampere's Law : displacement current

- Add a term to Ampere's Law to make it compatible with the continuity equation :
- $\underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\rho)$

  Apply Gauss Law $\underline{\nabla} \cdot \underline{\mathbf{E}} = \rho/\epsilon_0$

  $\rightarrow \quad \underline{\nabla} \cdot \underline{\mathbf{J}} = -\frac{\partial}{\partial t}(\epsilon_0 \underline{\nabla} \cdot \underline{\mathbf{E}}) = -\underline{\nabla} \cdot (\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t})$

  $\rightarrow \quad \underline{\nabla} \cdot (\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}) = 0$
- Implies we need to add $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ to $\underline{\mathbf{J}}$ in Ampere's law.

  $$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$$

  $\left(\epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}\right)$ is called the *displacement current* $\underline{\mathbf{J}}_D$ (but is actually a time-varying electric field)
- Time-varying $\underline{\mathbf{E}}$ fields now generate $\underline{\mathbf{B}}$ fields and vice versa. Also satisfies charge conservation.

Lecture 20

Maxwell's Equations & Electromagnetic Waves

Summary : Ampere's Law

Ampere's law does not comply with the Equation of Continuity:

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply div: $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J}$

$\nabla \cdot (\nabla \times \mathbf{B}) = 0$ always

$\mu_0 \nabla \cdot \mathbf{J}$: $= -\frac{\partial \rho}{\partial t}$, $= 0$ only for statics!

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$: displacement current $\mathbf{J_D}$

Obtain **Ampere's law with "displacement current"**:

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

Using Stokes theorem: $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot d\underline{a}$

Gives integral form : $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \underbrace{\mu_0 \int_S \underline{\mathbf{J}} \cdot d\underline{a}}_{\mu_0 I_{encl.}} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{a}$

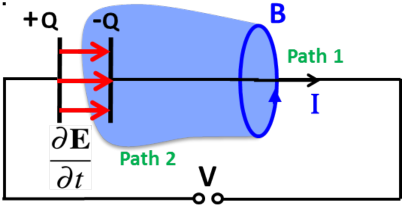
20.1 Example : Ampere's Law and a charging capacitor

- This is the first example showing why Ampere's Law fails without adding the displacement current : a straight wire, and add a capacitor into the circuit
- Before we used Ampere's Law to calculate magnetic field along Amperian loop $\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}\ell} = \mu_0 \, I_{encl.}$

- But there is not one unique path :

  (i) Path 1: the smallest area (plane surface) $\rightarrow \; I_{encl.} = I$

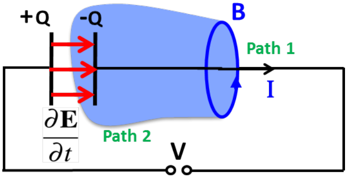
  (ii) Path 2: via a "bulged" surface that passes between the capacitor plates $\rightarrow \; I_{encl.} = 0$



- The $\underline{\mathbf{B}}$ field has to be the same no matter which path we choose
- The issue is that the $\underline{\mathbf{E}}$ field is changing in the capacitor !

A charging capacitor and Ampere's Law, continued

- Gauss Law for a parallel plate capacitor : $E = \frac{Q}{\epsilon_0 A}$
- $\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{\partial Q}{\partial t} = \frac{1}{\epsilon_0 A} I$
- Add $I_D = \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$ to Ampere's Law



- $$\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \underbrace{\mu_0 I_{encl.}}_{\text{Term 1}} + \underbrace{\mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}}_{\text{Term 2}}$$
- For the surface around the wire :

  Term 1 = $\mu_0 I$, Term 2 = 0
- For the surface around the capacitor

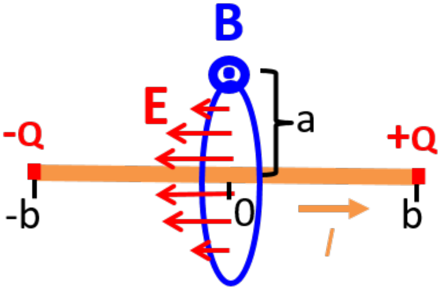
  Term 1 = 0 , Term 2 = $\mu_0 \epsilon_0 \times \frac{1}{\epsilon_0 A} I \times A = \mu_0 I$

  $\rightarrow$ RHS = $\mu_0 I$, regardless of choice of path ✓✓

  In differential form : $$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \left(\underline{\mathbf{J}} + \epsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t} \right)$$

20.2 Example : B-field of a short current-carrying wire

- Recall B-field from Biot-Savart Law $\rightarrow$ $B = \frac{\mu_0 I}{2\pi a} \frac{b}{\sqrt{b^2+a^2}}$
- Again, Ampere's law fails depending on what path we use. Need to use displacement current.
- $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = \mu_0 I_{encl.} + \mu_0 \epsilon_0 \int_S \frac{\partial \underline{\mathbf{E}}}{\partial t} \cdot d\underline{\mathbf{a}}$
- Wire is short, so charge builds up at the ends giving time-varying $\underline{\mathbf{E}}$-field

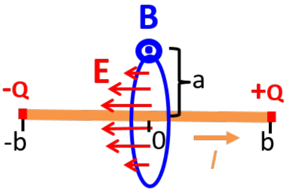


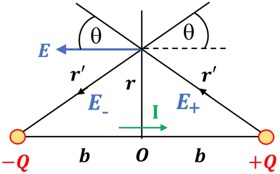
B-field of a short current-carrying wire, continued

- Integrate $\frac{\partial E}{\partial t}$ over area, radius a
- Calculate $\underline{\mathbf{E}}$-field due to two point charges at wire ends, $\pm b$

$$E(r) = -\underbrace{\frac{2Q/(4\pi\epsilon_0)}{(r^2+b^2)}}_{r'^2}\underbrace{\frac{b}{\sqrt{r^2+b^2}}}_{\cos\theta}$$

(2 field components E_+ and E_-, and note I_D and I have opposite signs)





- $I_D = \epsilon_0 \int_0^a \frac{\partial E(r)}{\partial t} 2\pi\, r\, dr = \epsilon_0 \frac{\partial Q}{\partial t} \int_0^a -\frac{b/(2\pi\epsilon_0)}{(r^2+b^2)^{\frac{3}{2}}} 2\pi\, r\, dr$
- $I_D = \frac{\partial Q}{\partial t}\left[\frac{b}{\sqrt{(r^2+b^2)}}\right]_{r=0}^{r=a} = I\left[\frac{b}{\sqrt{(a^2+b^2)}} - 1\right]$
- $\oint_C \underline{\mathbf{B}} \cdot d\underline{\ell} = B \cdot 2\pi\, a = \mu_0\, I + \mu_0\, I\left[\frac{b}{\sqrt{(a^2+b^2)}} - 1\right]$
- So : $\quad B = \frac{\mu_0\, I}{2\pi\, a}\frac{b}{\sqrt{b^2+a^2}}$ as from Biot-Savart Law ✓✓

Summary of Maxwell's Equations

$$\oint_S \mathbf{E} \cdot \mathbf{da} = \frac{Q}{\varepsilon_0} \leftrightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss' law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint \mathbf{E} \cdot \boldsymbol{dl} = -\frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{B} \cdot \mathbf{da} \leftrightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law: time-varying magnetic fields create electric fields (induction).

$$\oint_S \boldsymbol{B} \cdot \boldsymbol{da} = 0 \leftrightarrow \nabla \cdot \mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{dt}} \int_S \mathbf{E} \cdot \mathbf{da} \leftrightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Ampere's law including displacement current: electric currents and time-varying electric fields generate magnetic fields.

20.3 Electromagnetic waves in vacuum

- In the absence of electric charge or current
  $\rightarrow \quad \rho = 0$ and $\underline{\mathbf{J}} = 0$:
- Maxwell's Equations become :

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \, \epsilon_0 \, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

(note the symmetry between the $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ fields)

- Apply curl to Faraday's law :

$$\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial}{\partial t} \underline{\nabla} \times \underline{\mathbf{B}} = -\mu_0 \, \epsilon_0 \, \frac{\partial^2}{\partial t^2} \, \underline{\mathbf{E}}$$

- Use the vector identity : $\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{E}} = \underline{\nabla} \underbrace{(\underline{\nabla} \cdot \underline{\mathbf{E}})}_{=0} - \underline{\nabla}^2 \, \underline{\mathbf{E}}$

- This gives us a *wave equation* in $\underline{\mathbf{E}}$:

$$\underline{\nabla}^2 \, \underline{\mathbf{E}} - \epsilon_0 \, \mu_0 \, \ddot{\underline{\mathbf{E}}} = 0$$

Electromagnetic waves in vacuum, continued

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 0 \qquad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0$$

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \qquad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \epsilon_0\, \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

- Apply curl to Ampere's law :

  $$\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{B}} = \epsilon_0\, \mu_0\, \frac{\partial}{\partial t} \underline{\nabla} \times \underline{\mathbf{E}} = -\mu_0\, \epsilon_0\, \frac{\partial^2}{\partial t^2}\, \underline{\mathbf{B}}$$

- Use the vector identity : $\underline{\nabla} \times \underline{\nabla} \times \underline{\mathbf{B}} = \underline{\nabla} \underbrace{(\underline{\nabla} \cdot \underline{\mathbf{B}})}_{=0} - \underline{\nabla}^2\, \underline{\mathbf{B}}$

- This gives us a *wave equation* in $\underline{\mathbf{B}}$:

  $$\boxed{\underline{\nabla}^2\, \underline{\mathbf{B}} - \epsilon_0\, \mu_0\, \ddot{\underline{\mathbf{B}}} = 0}$$

  together with : $\boxed{\underline{\nabla}^2\, \underline{\mathbf{E}} - \epsilon_0\, \mu_0\, \ddot{\underline{\mathbf{E}}} = 0}$

- These equations have general solutions (in 1D) of the form:

- $E\,(x, t) = F\,(x - ct) + G\,(x + ct)$ and
  $B\,(x, t) = F'\,(x - ct) + G'\,(x + ct)$

  where F, G, F', G' are *any* functions of $(x - ct), (x + ct)$

20.4 Electromagnetic waves : 3D plane wave solutions

- Consider the simplest form of solution : 3D plane waves of the form

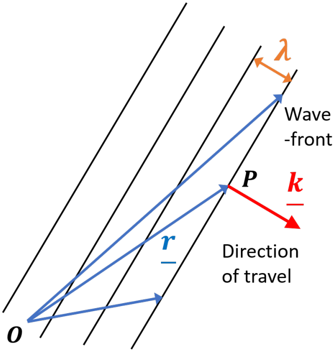
  $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ and

  $\underline{\mathbf{B}} = \underline{\mathbf{B}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$

  Real part : $Re[\underline{\mathbf{E}}] = \underline{\mathbf{E}}_0 \cos\underbrace{(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})}_{\text{phase}}$

- $\underline{\mathbf{k}}$ is in the direction normal to the wave-fronts
- All points P form a wave-front with the same phase
- Maxima are separated by the wavelength λ where $\lambda = 2\pi/k$
- Phase velocity (or propagation velocity) of wave-fronts given by $c = \omega/k$

Plane waves



Lecture 21

Electromagnetic Waves & Energy Flow

21.1 Divergence, time derivative, and curl of $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$

- The divergence of $\underline{\mathbf{E}}$: $\underline{\nabla} \cdot \underline{\mathbf{E}} = \underline{\nabla} \cdot \underline{\mathbf{E}}_0 \exp\left[i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right]$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \underline{\mathbf{E}}_0 \exp\left(i(\omega t - k_x x - k_y y - k_z z)\right)$$

$$= \left[(-i)\, k_x E_x + (-i)\, k_y E_y + (-i)\, k_z E_z\right] \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$$

$$= (-i)\, \underline{\mathbf{k}} \cdot \underline{\mathbf{E}} \quad : \quad \text{hence} \quad \underline{\nabla} \equiv -i\, \underline{\mathbf{k}}$$

- The time derivative of $\underline{\mathbf{E}}$: $\frac{\partial}{\partial t} \underline{\mathbf{E}} = \frac{\partial}{\partial t} \underline{\mathbf{E}}_0 \exp\left[i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right]$

$$= i\, \omega\, \underline{\mathbf{E}} \quad : \quad \text{hence} \quad \frac{\partial}{\partial t} \equiv i\, \omega$$

- The curl of $\underline{\mathbf{E}}$:

$$\underline{\nabla} \times \underline{\mathbf{E}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} =$$

$$(-i) \begin{pmatrix} k_y E_z - k_z E_y \\ k_z E_x - k_x E_z \\ k_x E_y - k_y E_x \end{pmatrix} = (-i)\, \underline{\mathbf{k}} \times \underline{\mathbf{E}} \quad \text{\& again} \quad \underline{\nabla} \equiv -i\, \underline{\mathbf{k}}$$

21.2 Electromagnetic waves : speed of propagation

- To get speed of propagation, substitute $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}}))$ into the wave equation $\underline{\nabla}^2 \underline{\mathbf{E}} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}$
- Use $\underline{\nabla} \equiv -i\,\underline{\mathbf{k}} \rightarrow \underline{\nabla}^2 \equiv (-i\,\underline{\mathbf{k}})^2 = -k^2$

  $$\frac{\partial}{\partial t} \equiv i\,\omega \rightarrow \frac{\partial^2}{\partial t^2} \equiv (i\,\omega)^2 = -\omega^2$$
- $-k^2 \underline{\mathbf{E}}_0 \exp(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})) = -\omega^2 \epsilon_0 \mu_0 \underline{\mathbf{E}}_0 \exp(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}}))$

  $$\rightarrow \quad k^2 = \omega^2 \epsilon_0 \mu_0$$
- Fields of this form are solutions to the wave equation with velocity of propagation :

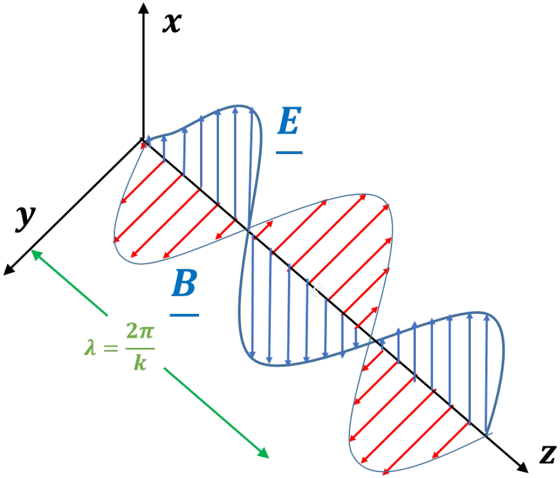
  $$c = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 \text{ m s}^{-1}$$

  i.e. the speed of light $\rightarrow$ speed of an EM wave in vacuum

21.3 Relationship between $\underline{\mathbf{E}}$ *and* $\underline{\mathbf{B}}$

- Substitute $\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \exp\left(i(\omega t - \underline{\mathbf{k}} \cdot \underline{\mathbf{r}})\right)$ into Maxwell eqn's :
  $\underline{\nabla} \cdot \underline{\mathbf{E}} = -i\, \underline{\mathbf{k}} \cdot \underline{\mathbf{E}} = 0$
  $\underline{\nabla} \cdot \underline{\mathbf{B}} = -i\, \underline{\mathbf{k}} \cdot \underline{\mathbf{B}} = 0$
  Hence $\underline{\mathbf{k}} \cdot \underline{\mathbf{E}} = 0$ and $\underline{\mathbf{k}} \cdot \underline{\mathbf{B}} = 0$
- Electric and magnetic fields in vacuum are *perpendicular* to direction of propogation $\rightarrow$ *EM waves are transverse*
- Substitute into Faraday's Law : $\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$
  $-i\, \underline{\mathbf{k}} \times \underline{\mathbf{E}} = -i\, \omega\, \underline{\mathbf{B}} \quad \rightarrow \quad \boxed{\underline{\mathbf{B}} = \frac{1}{\omega} \underline{\mathbf{k}} \times \underline{\mathbf{E}}}$
- Substitute into Ampere's Law : $\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0\, \epsilon_0\, \frac{\partial \underline{\mathbf{E}}}{\partial t}$
  $-i\, \underline{\mathbf{k}} \times \underline{\mathbf{B}} = i\, \omega\, \mu_0\, \epsilon_0\, \underline{\mathbf{E}} \quad \rightarrow \quad \boxed{\underline{\mathbf{E}} = -\frac{c^2}{\omega} \underline{\mathbf{k}} \times \underline{\mathbf{B}}}$
- $\underline{\mathbf{E}}$, $\underline{\mathbf{B}}$ & $\underline{\mathbf{k}}$ are mutually orthogonal (NB. $\underline{\mathbf{k}} \times \underline{\mathbf{B}} = kB \sin\frac{\pi}{2}\, \hat{\underline{\mathbf{E}}}$)
- $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are in phase and lie in the plane of the wavefront
- Field magnitude ratio : $\boxed{|\underline{\mathbf{E}}|/|\underline{\mathbf{B}}| = \frac{c^2}{\omega} k = c = \frac{1}{\sqrt{\mu_0\, \epsilon_0}}}$

21.4 Electromagnetic wave travelling along the z direction



$$\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 \sin\left(\omega(t - z/c)\right)\, \underline{\hat{\mathbf{x}}}$$

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}_0 \sin\left(\omega(t - z/c)\right)\, \underline{\hat{\mathbf{y}}}$$

21.5 Characteristic impedance of free space

- Take the ratio $Z = \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{H}}|}$ where $|\underline{\mathbf{H}}| = \frac{1}{\mu_0} |\underline{\mathbf{B}}|$
- Z has units $[V\, m^{-1}] / [A\, m^{-1}] =$ Ohms.
- Z is called the *characteristic impedance of free space*

$$Z = \mu_0 \frac{|\underline{\mathbf{E}}|}{|\underline{\mathbf{B}}|} = \mu_0\, c = \frac{\mu_0}{\sqrt{\mu_0\, \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\, \Omega$$

Electromagnetic waves : summary

In vacuum, free of charge or currents (ρ, **J** = 0):

$$\nabla\cdot\mathbf{E}=0 \qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B}=0 \qquad \nabla\times\mathbf{B}=\varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla^2\mathbf{E}=\varepsilon_0\mu_0\,\ddot{\mathbf{E}}$$

$$\nabla^2\mathbf{B}=\varepsilon_0\mu_0\,\ddot{\mathbf{B}}$$

Wave equations in **E, B**!

Electromagnetic waves propagate in free space:

Plane EM wave fronts: $\mathbf{E}=\mathbf{E_0}\exp\{i(\omega t-\mathbf{k}\cdot\mathbf{r})\}$ with wavelength $\lambda=\frac{2\pi}{k}$

Propagation velocity of wave fronts: $c=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon_0\mu_0}}=3\times10^8\,\mathrm{m\,s^{-1}}$

Relationship between E and B:
(in phase and mutually orthogonal with wave vector **k**) $\mathbf{B}=\frac{\mathbf{k}}{\omega}\times\mathbf{E}$ $\quad\mathbf{E}=-c^2\frac{\mathbf{k}}{\omega}\times\mathbf{B}$ $\quad\frac{|\mathbf{E}|}{|\mathbf{B}|}=c$

Impedance of free space: $Z=\frac{|\mathbf{E}|}{|\mathbf{B}|/\mu_0}=\sqrt{\frac{\mu_0}{\varepsilon_0}}=376.7\,\Omega$

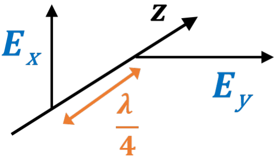
21.6 Polarisation

- Linearly (or plane) polarised wave :

  E has one specific orientation

- Circularly polarised wave :

  Two linear components of E superimposed at a right angle and phase shifted by $\pi/2$



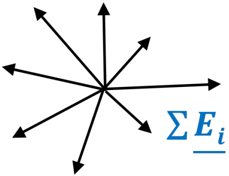
$$\begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} \sin(\omega t) + \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix} \sin(\omega t + \pi/2) = E \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}$$

- Elliptically polarised wave :

  As above but with unequal amplitudes

- Unpolarised :

  E superimposed with all orientations (with no fixed phase relationships between components)



21.7 Energy flow and the Poynting Vector

- Recall : Energy of the electric field $U_e = \int_{\mathcal{V}} \frac{1}{2}\,\epsilon_0\,\underline{\mathbf{E}}^2\, d\mathcal{V}$

  Energy of the magnetic field $U_m = \int_{\mathcal{V}} \frac{1}{2\,\mu_0}\,\underline{\mathbf{B}}^2\, d\mathcal{V}$
- Total EM energy in volume $\mathcal{V}$:

$$U = \int_{\mathcal{V}} \underbrace{\frac{1}{2}\left(\epsilon_0\,\underline{\mathbf{E}}\cdot\underline{\mathbf{E}} + \frac{1}{\mu_0}\,\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}\right)}_{\text{Energy density}} d\mathcal{V}$$

- In free space ($\underline{\mathbf{J}} = 0, \rho = 0$)

$$\underbrace{\frac{\partial \underline{\mathbf{E}}}{\partial t} = \frac{1}{\mu_0\epsilon_0}\underline{\nabla}\times\underline{\mathbf{B}}}_{\text{Ampere's Law}} \quad ; \quad \underbrace{\frac{\partial \underline{\mathbf{B}}}{\partial t} = -\underline{\nabla}\times\underline{\mathbf{E}}}_{\text{Faraday's Law}}$$

- Calculate the rate of change of energy in $\mathcal{V}$:

$$\begin{aligned}\frac{dU}{dt} &= \int_{\mathcal{V}} \left(\epsilon_0\,\underline{\mathbf{E}}\cdot\underline{\dot{\mathbf{E}}} + \frac{1}{\mu_0}\,\underline{\mathbf{B}}\cdot\underline{\dot{\mathbf{B}}}\right) d\mathcal{V} \\ &= \int_{\mathcal{V}} \left(\frac{\epsilon_0}{\mu_0\epsilon_0}\,(\underline{\mathbf{E}}\cdot\underline{\nabla}\times\underline{\mathbf{B}}) - \frac{1}{\mu_0}\,\underline{\mathbf{B}}\cdot(\underline{\nabla}\times\underline{\mathbf{E}})\right) d\mathcal{V} \\ &= -\frac{1}{\mu_0}\int_{\mathcal{V}} \underline{\nabla}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}})\, d\mathcal{V}\end{aligned}$$

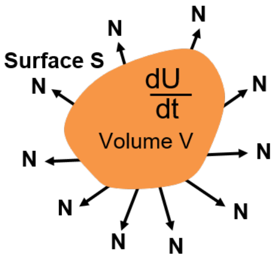
Energy flow and the Poynting Vector continued

- Energy flow out of volume $\mathcal{V}$ per unit time :

$$\frac{dU}{dt} = -\frac{1}{\mu_0}\int_{\mathcal{V}} \underline{\nabla}\cdot(\underline{\mathbf{E}}\times\underline{\mathbf{B}})\,d\mathcal{V}$$

- Apply the divergence theorem :

$$\frac{dU}{dt} = -\oint_S \underbrace{\left(\frac{1}{\mu_0}\underline{\mathbf{E}}\times\underline{\mathbf{B}}\right)}_{\text{Poynting Vector, }\underline{\mathbf{N}}}\cdot\,d\underline{\mathbf{a}}$$



$$\frac{dU}{dt} = -\oint_S \underline{\mathbf{N}}\cdot\,d\underline{\mathbf{a}} \quad \text{where} \quad \underline{\mathbf{N}} = \frac{1}{\mu_0}\underline{\mathbf{E}}\times\underline{\mathbf{B}}$$

Poynting vector $\underline{\mathbf{N}}$ is the power per unit area flowing through the surface bounded by volume $\mathcal{V}$. (It also gives the direction of flow). Units of $\underline{\mathbf{N}}$: $[W\,m^{-2}]$

- For EM waves, the intensity is the time-average of $|\underline{\mathbf{N}}|$

$$\Im = <|\underline{\mathbf{N}}|> = \frac{1}{\mu_0}E_0\,B_0\underbrace{<\cos^2(\omega t - \underline{\mathbf{k}}\cdot\underline{\mathbf{r}})>}_{1/2} = \frac{1}{2\,\mu_0\,c}E_0^2$$

21.8 Example : Poynting Vector for a long resistive cylinder

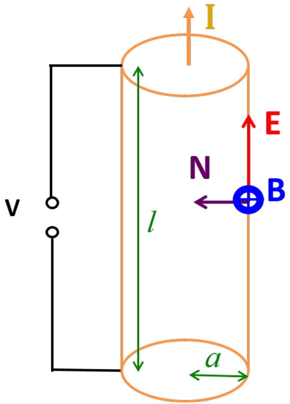
- Calculate Poynting Vector at the surface of the wire with applied potential difference V and current I

$\underline{\mathbf{N}} = \frac{1}{\mu_0}\underline{\mathbf{E}} \times \underline{\mathbf{B}}$

- Electric field along wire axis : $E = V/\ell$

  Magnetic flux density at wire surface :
  $\oint \underline{\mathbf{B}} \cdot d\underline{\ell} = B \cdot 2\pi a = \mu_0 I$

  (note that this is *tangential* - along circumference)



- $N = \frac{1}{\mu_0}\frac{V}{\ell}\frac{\mu_0 I}{2\pi a}$

  (in radial direction pointing *inwards* - i.e. wire heats up !)

- Hence $N = (V\,I)/\underbrace{2\pi\ell a}_{\text{surface area}}$

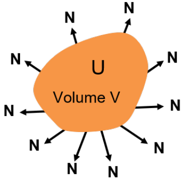
- Total power dissipated in wire : $P = \int_S \underline{\mathbf{N}} \cdot d\underline{\mathbf{a}} = V\,I$ as expected from circuit theory.

Poynting Vector : summary

Total electromagnetic energy U contained in volume V:

$$U = \int_V \frac{1}{2}\left(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \mathrm{d}V$$

energy density $U_V = \dfrac{dU}{dV}$



$$\frac{dU}{dt} = -\oint_S \mathbf{N}\,.\,\mathbf{da} \quad \text{with} \quad \mathbf{N} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

Poynting vector

Energy flow rate out of volume V

Power per unit area through area bounding V

$[N]=W/m^2$

The intensity I of an EM wave is given by the time-average over the magnitude of the Poynting vector:

$$I = \langle|\mathbf{N}|\rangle = \frac{1}{2c\mu_0}E_0^{\,2}$$