TT-2022 Revision Lecture 1 on

ELECTROMAGNETISM (CP2)

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It is recommended students attempt the question sheet

https://users.physics.ox.ac.uk/~harnew/lectures/TT22-CP2Rev.pdf

before viewing this lecture

(taken from previous years' Prelims questions)

- Electrostatics } First lecture
- Magnetostatics
- Induction
- EM Waves

(Magnetostatics, Induction, EM Waves) } Second lecture

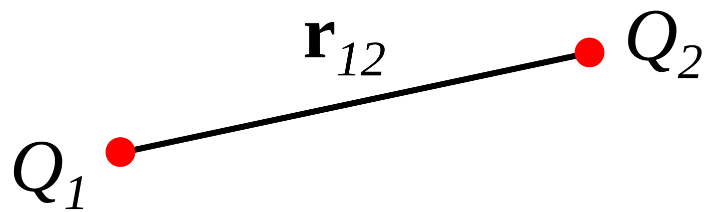
[1] With thanks to Profs Hans Kraus and Laura Hertz

1 Electrostatics

1.1. State Coulomb's Law for the force between two charges, Q_1 and Q_2. Hence show how the electric field $\mathbf{E}$ at a point $\mathbf{r}$ may be defined. What is meant by the statement that $\mathbf{E}$ is a conservative field? [4]

State Coulomb's Law. Show how **E** field may be defined. What is meant by **E** is a conservative field?

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2_{12}}\,\hat{\mathbf{r}}_{12}$$



Electric field due to single charge Q: force per unit charge

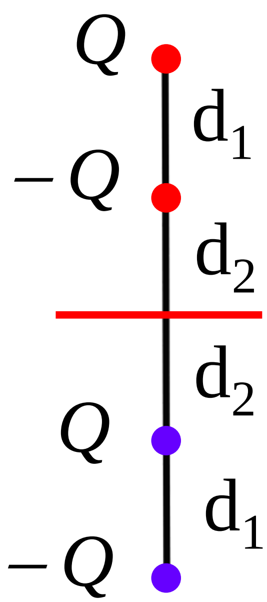
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}$$

Conservative field: $\nabla\times\mathbf{E} = 0$ and $\int \mathbf{E}\cdot d\mathbf{l}$ is path-independent. Therefore, a potential can be defined $\boldsymbol{E} = -\boldsymbol{\nabla} V$

1.2. A thundercloud and the ground below can be modelled as a charge of +40 As at a height of 10 km and a charge of −40 As at a height of 6 km above an infinite conducting plane. A person with an electrometer stands immediately below the thundercloud. What value of electric field do they measure, and what is its direction? [5]

A thundercloud with charges +40As at 10 km height and −40As at 6 km. Find the E-field on the ground.

Use method of image charges. Mirror the above to below the surface, with +40 As at depth 6 km and −40 As at depth 10 km.



$$E = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left(10^4\,\text{m}\right)^2} - \frac{1}{\left(6\times10^3\,\text{m}\right)^2} - \frac{1}{\left(6\times10^3\,\text{m}\right)^2} + \frac{1}{\left(10^4\,\text{m}\right)^2}\right]$$

$$= \frac{2\cdot 40\text{As}\quad \text{Vm}}{4\pi\cdot 8.854\times10^{-12}\,\text{As}}\left[\frac{1}{3.6\times10^7\,\text{m}^2} - \frac{1}{10^8\,\text{m}^2}\right] = 12{,}780\,\frac{\text{V}}{\text{m}}$$

Field point upwards.

1.3. An array of localised charges q_i experience potentials V_i as a result of their mutual interaction. Show that their mutual electrostatic energy, W, is given by $W = \frac{1}{2}\sum_i q_i V_i$. [6]

An array of localised charges q_i experience potentials V_i as a result of their mutual interaction. Show that their mutual electrostatic energy, W, is given by $W = \frac{1}{2}\sum_i q_i V_i$.

Potential energy of a single charge q in potential V

$$W = -\int \mathbf{F}\cdot d\mathbf{l} = -q\cdot\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} = q\cdot V(r)$$

Potential V_i due to all other charges:

$$V_i = \sum_{j\neq i} \frac{q_j}{4\pi\varepsilon_0}\cdot\frac{1}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

For total PE, sum over all charges. However, each charge appears twice:

$$\frac{1}{2}\sum_i \sum_{j\neq i}' \frac{q_i q_j}{4\pi\varepsilon_0}\cdot\frac{1}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

$$W = \tfrac{1}{2}\sum_i q_i V_i$$

Alternative: Assemble Charge Configuration explicitly

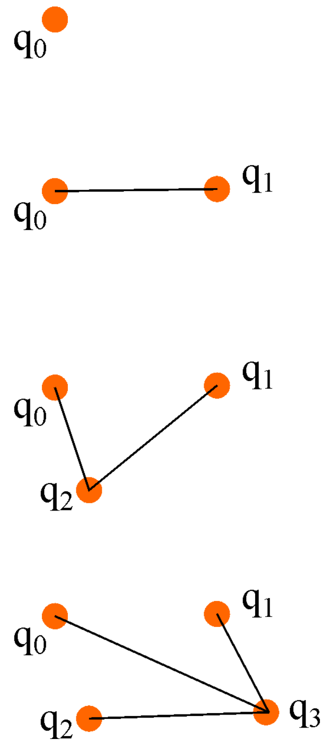
$$W = \frac{1}{4\pi\varepsilon_0}\Big[$$

No penalty for charge q_0 $\qquad 0 +$

q_1 in potential due to q_0 $\qquad + \frac{q_0 q_1}{r_{01}} +$

q_2 in potential of q_0 and q_1 $\qquad + \frac{q_0 q_2}{r_{02}} + \frac{q_1 q_2}{r_{12}} +$

q_3 in pot. of q_0, q_1 and q_2 $\qquad + \frac{q_0 q_3}{r_{03}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \Big]$



Half the links compared with:

0 1 2 3

Thus as before:

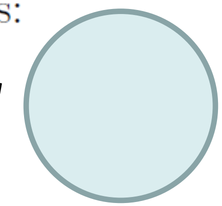
$$W = \tfrac{1}{2}\sum_i q_i V_i$$

1.4. A sphere of radius a is located at a large distance from its surroundings which define the zero of potential. It carries a total charge q. Determine the potential of its surface and the electrostatic energy of its charges in two separate situations:

(a) with the charge spread uniformly on its surface, [4]

(b) with the charge distributed uniformly within its volume. [10]

$+q$



A sphere of radius *a* is located at a large distance from its surroundings which define the zero of potential. It carries a total charge *q*. Determine the potential on its surface and the electrostatic energy : a) uniform q spread on surface.

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{r} = \frac{q}{4\pi\varepsilon_0 r}$$

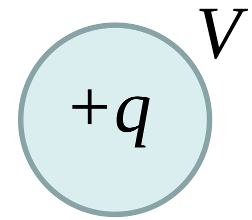
Need to sum up $\int V dq$

For shell: $V = \dfrac{q}{4\pi\varepsilon_0 a}$

$$W = \int V dq = \int \frac{q\,dq}{4\pi\varepsilon_0 a} = \frac{q^2}{8\pi\varepsilon_0 a}$$

(alternative in this simple case: $W = \dfrac{1}{2} q \dfrac{q}{4\pi\varepsilon_0 a} = \dfrac{q^2}{8\pi\varepsilon_0 a}$)

b) For uniformly distributed charged sphere:



Bring up successive shells thickness dr to radius r and potential V, and sum up all contributions to radius a

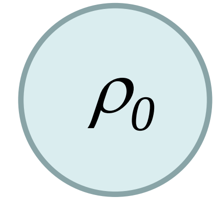
$$W = \int V \mathrm{d}q = \int V \rho \,\mathrm{d}^3 r$$

$$W = \int \frac{r^3}{a^3} \frac{q}{4\pi\varepsilon_0 r} \frac{q}{\frac{4\pi}{3} a^3} r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi \,\mathrm{d}r$$

[Fraction of charge inside r] x [Potential @ radius r] [Charge density] x [Volume element]

$$W = 3\frac{q^2}{4\pi\varepsilon_0} \int_0^a \frac{r^4}{a^6} \mathrm{d}r = \frac{3}{5} \frac{q^2}{4\pi\varepsilon_0 a}$$

1.5. Calculate the electric field strength E and the electrostatic potential V, as functions of radial distance r, for a sphere of uniform positive charge density ρ_0, of radius R, centred at the origin. Sketch graphs of E and V against r. [9]



Use Gauss' Law: $$\oiint \mathbf{E}\cdot\mathbf{dS} = \frac{1}{\varepsilon_0}\cdot\iiint \rho\, dV$$

$$\oiint E_r dS = E_r \cdot 4\pi r^2 = \tfrac{1}{\varepsilon_0}\cdot\begin{cases} \tfrac{4\pi}{3} R^3 \rho_0 & \text{for} \quad r \geq R \\ \tfrac{4\pi}{3} r^3 \rho_0 & \text{for} \quad r < R \end{cases}$$

$$E_r = \frac{\rho_0}{3\varepsilon_0}\cdot\frac{R^3}{r^2} \quad \text{for } r \geq R \qquad \text{and} \qquad E_r = \frac{\rho_0}{3\varepsilon_0}\cdot r \quad \text{for } r < R$$

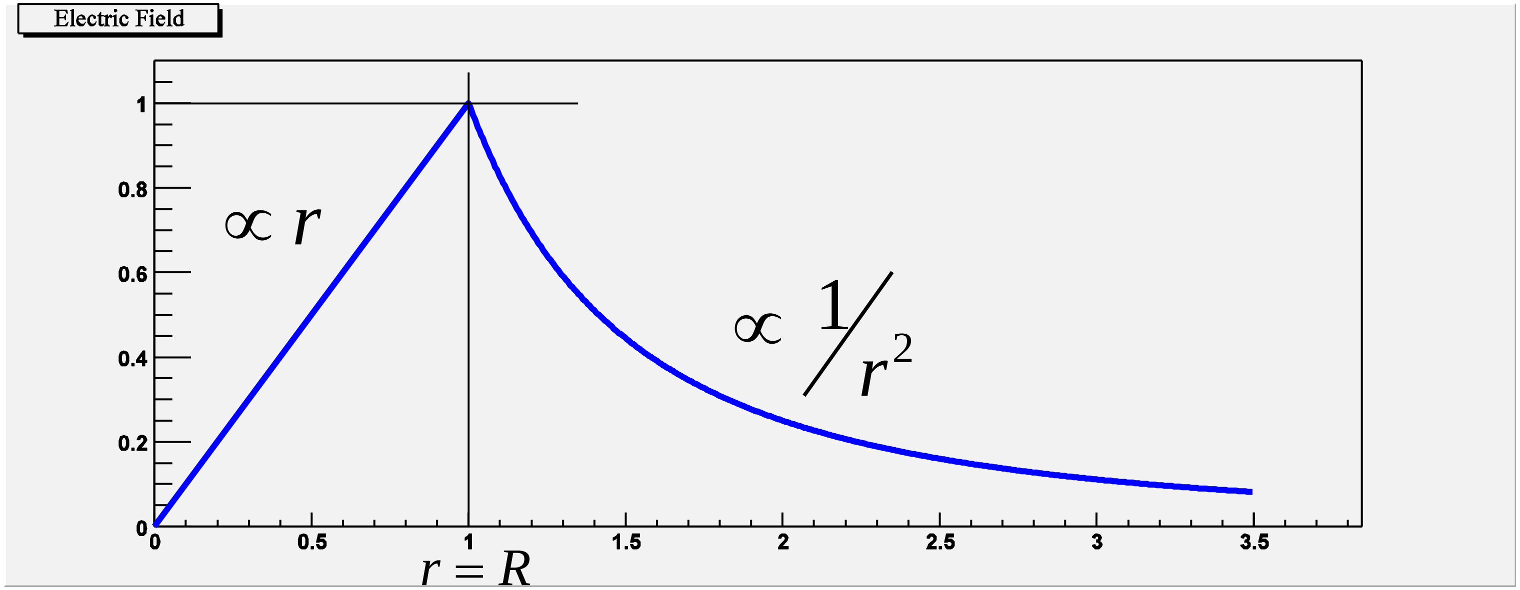
$$V_{out} = -\int_\infty^r E_r dr' = -\frac{\rho_0}{3\varepsilon_0} R^3 \cdot \left[-\frac{1}{r'}\right]_\infty^r = \underline{\underline{\frac{\rho_0}{3\varepsilon_0}\cdot\frac{R^3}{r}}}$$

on sphere $(r = R)$: $$V_S = \frac{\rho_0}{3\varepsilon_0} R^2$$

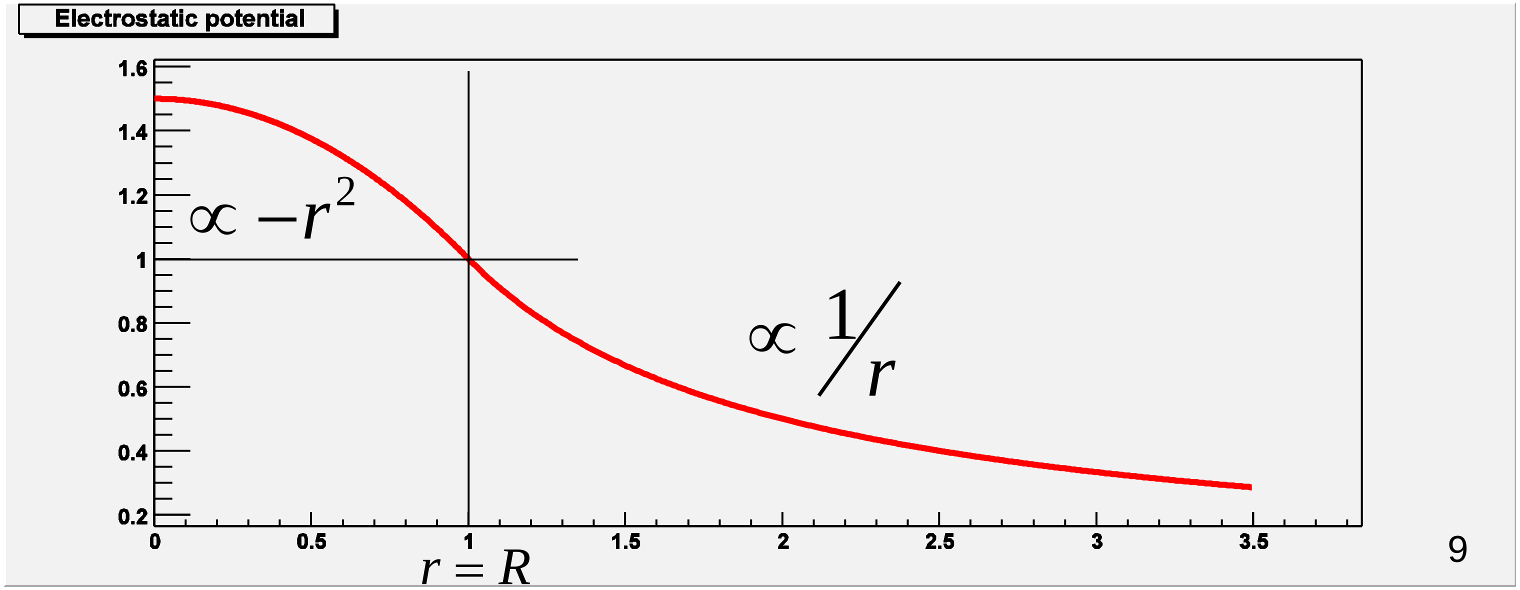
$$V_{ins} = V_S - \int_R^r E_r dr' = \frac{\rho_0}{3\varepsilon_0}\left[R^2 - \tfrac{1}{2} r^2 + \tfrac{1}{2} R^2\right] = \underline{\underline{\frac{\rho_0}{3\varepsilon_0}\left[\tfrac{3}{2} R^2 - \tfrac{1}{2} r^2\right]}}$$

E-field and potential V as function of r

$$E_{max} = \frac{\rho_0}{3\varepsilon_0} R$$



$$V_S = \frac{\rho_0}{3\varepsilon_0} R^2$$



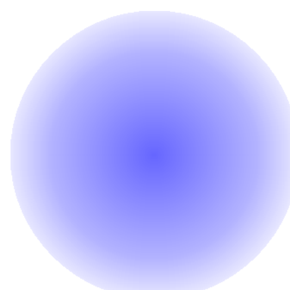
1.6. The electron charge density of a hydrogen atom in its ground state is given by

$$\rho(r) = -\frac{e}{\pi a_0^3} \exp[-2r/a_0],$$

where a_0 is the Bohr radius (5.3×10^{-11} m). Show that the electric field due to the electron cloud is given by

$$E(r) = \frac{e}{4\pi\epsilon_0} \left\{ \frac{(e^{-2r/a_0} - 1)}{r^2} + \frac{2e^{-2r/a_0}}{a_0 r} + \frac{2e^{-2r/a_0}}{a_0^2} \right\}. \quad [9]$$

$$\left[\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx \right]$$

Electron cloud: 

$$\rho(r) = -\frac{e}{\pi a_0^3} \cdot \exp\left(-\frac{2r}{a_0} \right)$$

Gauss' Law: $\oiint \mathbf{E}\cdot\mathbf{dS} = \frac{1}{\varepsilon_0}\cdot\iiint \rho dV$ ⇦ (charge enclosed)

$$E_r = \frac{1}{4\pi\varepsilon_0 r^2}\cdot\left[-\frac{e}{\pi a_0^3}\cdot\iiint \exp\left(-\frac{2r'}{a_0}\right) r'^2 \sin\theta d\theta d\varphi dr'\right]$$

$$\int_0^r x^2\exp(ax)dx = \tfrac{1}{a}x^2 e^{ax}\Big|_0^r - \tfrac{2}{a^2}x e^{ax}\Big|_0^r + \tfrac{2}{a^3}e^{ax}\Big|_0^r$$

here: $a = -\frac{2}{a_0}$ and $\iint \sin\theta d\theta d\varphi = 4\pi$

$$E_r = \frac{e}{4\pi\varepsilon_0}\left\{\frac{\exp(-2r/a_0)-1}{r^2} + \frac{2\exp(-2r/a_0)}{a_0 r} + \frac{2\exp(-2r/a_0)}{a_0^2}\right\}$$

Explain qualitatively what happens when such an atom is placed in a steady, uniform electric field, of strength E_0. [4]

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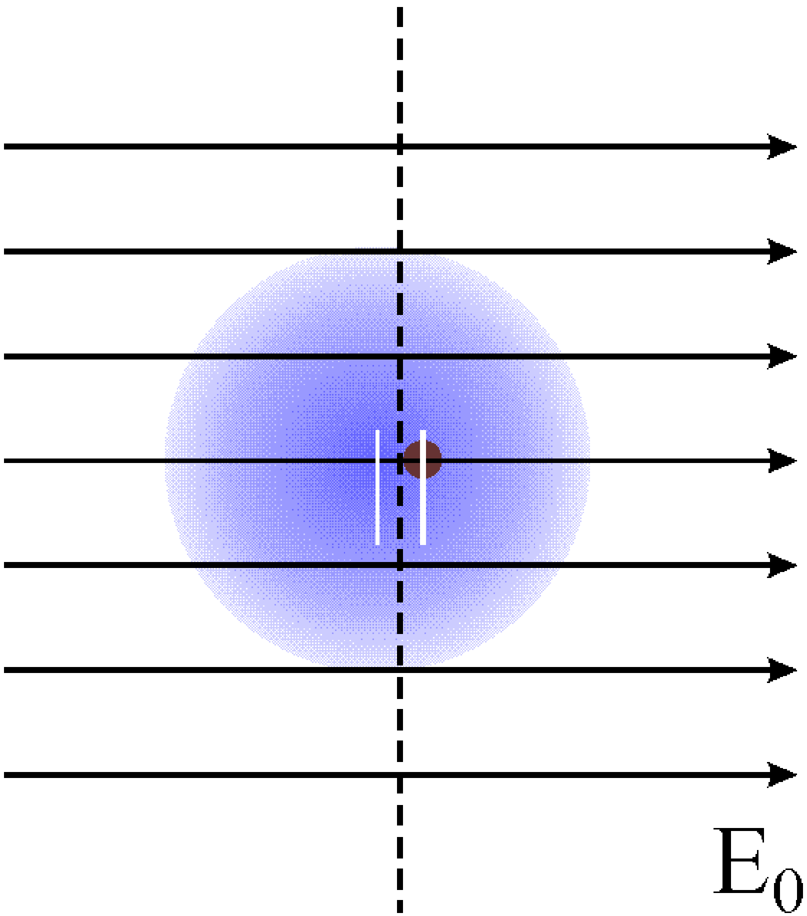


Centres of gravity of the positive nucleus and the negative electron charge distribution shift.

Force on charges due to E_0 balances the internal force of the dipole charges.

The atom exhibits an electric dipole moment.

1.7. The space between two concentric spheres, of radii a and b ($b > a$), is filled with air. Show that the capacitance C of the combination is given by

$$C = 4\pi\epsilon_0 \left(\frac{ab}{b-a} \right).$$

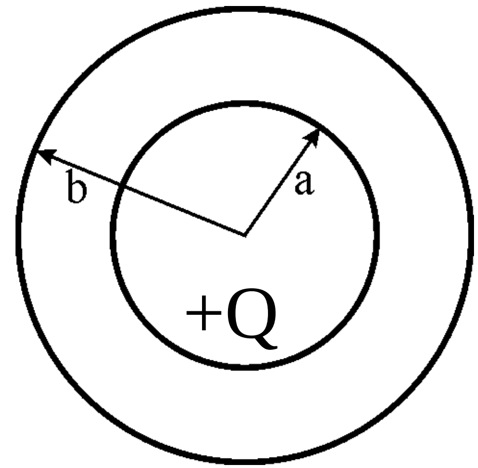
[7]

Gauss's Theorem in vacuo:

$$\oiint \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\varepsilon_0} \cdot \iiint \rho dV$$

Calculate the capacitance for a spherical capacitor:

$$C = \frac{Q}{V}$$



Between a and b : $E_r \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} \cdot Q$

$$V = -\int_b^a E_r dr = \frac{Q}{4\pi\varepsilon_0} \cdot \left[\frac{1}{r} \right]_b^a = \frac{Q \cdot (b-a)}{4\pi\varepsilon_0 ab}$$

$$C = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a}$$

The inner sphere is raised to a potential V and then isolated, the outer sphere being earthed. The outer sphere is then removed. Show that the resulting potential V' of the remaining sphere is given by

$$V' = \frac{bV}{b-a}. \qquad [5]$$

The inner sphere is raised to a potential V and then isolated, the outer sphere being earthed. The outer sphere is then removed. Find the resulting potential of the remaining sphere.

Before (and after) removal, charge stored on inner sphere:

$$Q = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a} \cdot V \qquad [1]$$

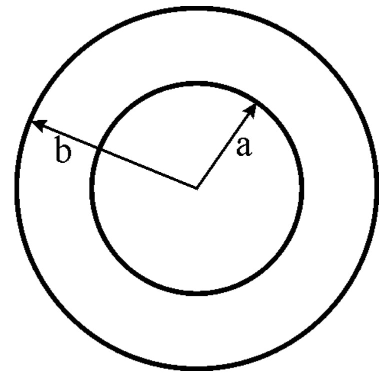
After removal, field of remaining sphere:

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\underline{\underline{V'}} = -\int_{\infty}^{a} E_r dr = -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{a}\frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0 a} = \underline{\underline{\frac{b}{b-a}\cdot V}}$$

If the values of a and b are 0.9 m and 1.0 m respectively, and given that air cannot sustain an electric field greater than 3000 V mm^{-1}, calculate the maximum potential to which the inner sphere can be initially charged. [8]

Now back to the original configuration:



$$E_{max} = 3000\text{V/mm} \qquad a = 0.9\text{m} \qquad b = 1.0\text{m}$$

E is at a maximum when r is at its smallest ⟶ consider E(a)

From [1] ($Q = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a} \cdot V$) and Gauss' Law ($E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$)

$$E_r(a) = \frac{ab}{b-a} \cdot V \cdot \frac{1}{a^2} = V \cdot \frac{b}{a} \cdot \frac{1}{b-a}$$

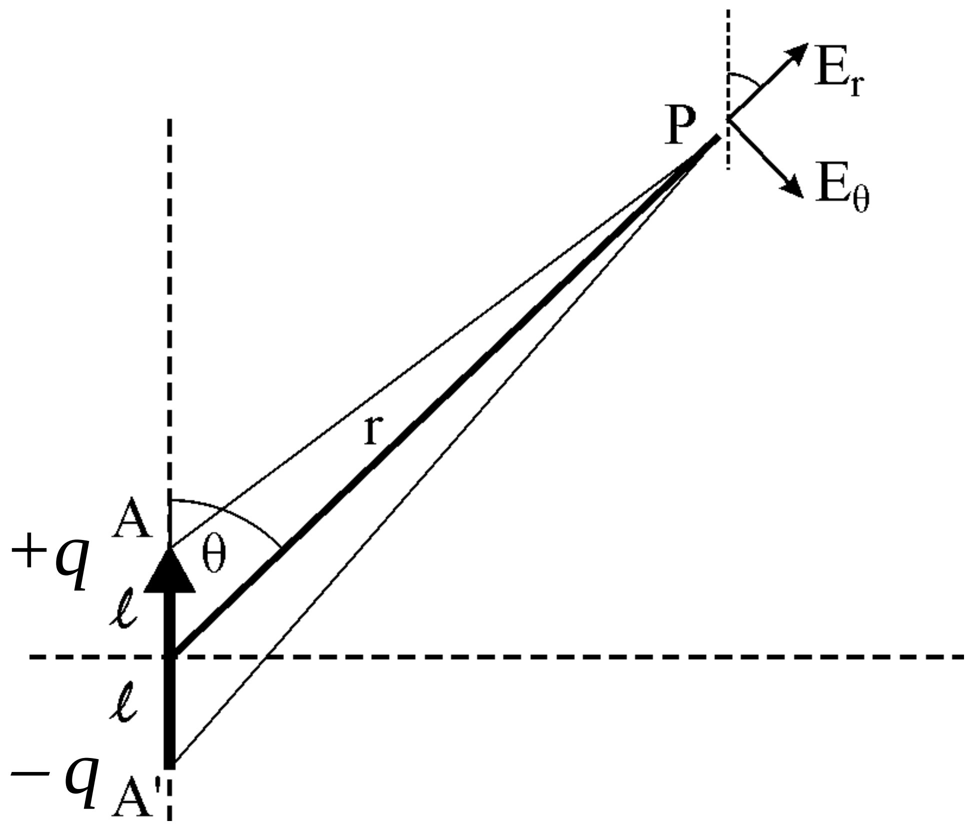
$$\underline{\underline{V_{max}}} = E_{max} \cdot \frac{a(b-a)}{b} = 3 \cdot 10^6 \frac{\text{V}}{\text{m}} \cdot \frac{0.9\text{m} \cdot 0.1\text{m}}{1\text{m}} = \underline{\underline{2.7 \cdot 10^5 \text{V}}}$$

1.8. An *electric dipole* consists of charges $-q$ and $+q$ separated by a distance $2l$, the resulting dipole moment $\mathbf{p}$ being of magnitude $2ql$ and with direction from $-q$ to $+q$. At a point (r, θ) relative to the centre and the direction of the dipole axis, derive from first principles, in the case where $r \gg l$,

(a) the electrostatic potential, [4]

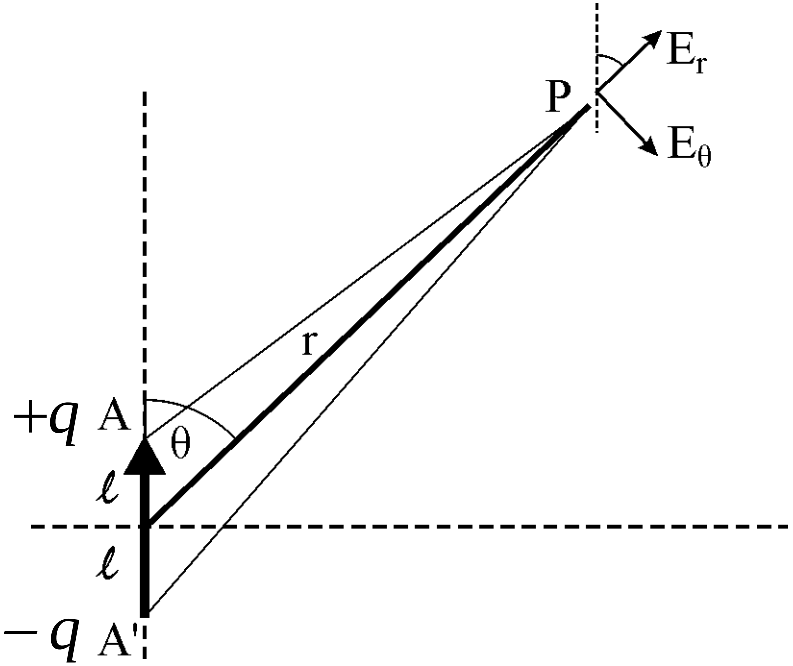
(b) the radial and tangential components of the electric field, [4]

(c) the torque exerted on such a dipole by a uniform electric field $\mathbf{E}$. [6]



The electrostatic potential of a dipole:

Charges +q at A and –q at A’



$$\overline{\mathrm{AP}}^2 = r^2 + \ell^2 - 2r\ell\cos\theta$$

$$\overline{\mathrm{A'P}}^2 = r^2 + \ell^2 + 2r\ell\cos\theta$$

$$V_P = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\overline{\mathrm{AP}}} - \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\overline{\mathrm{A'P}}}$$

Binomial expansion

$$\frac{1}{\overline{\mathrm{AP}}} = \frac{1}{r}\cdot\left[1+\left(\ell/r\right)^2 - 2\,\ell/r\cos\theta\right]^{-1/2} \approx \frac{1}{r}\cdot\left[1+\ell/r\cos\theta+\ldots\right]$$

$$\frac{1}{\overline{\mathrm{A'P}}} \approx \frac{1}{r}\cdot\left[1-\ell/r\cos\theta+\ldots\right]$$

$$\text{so: } V_P = \frac{q}{4\pi\varepsilon_0 r}\cdot\left[1+\ell/r\cos\theta - 1 + \ell/r\cos\theta\right] = \frac{2q\ell}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\cos\theta$$

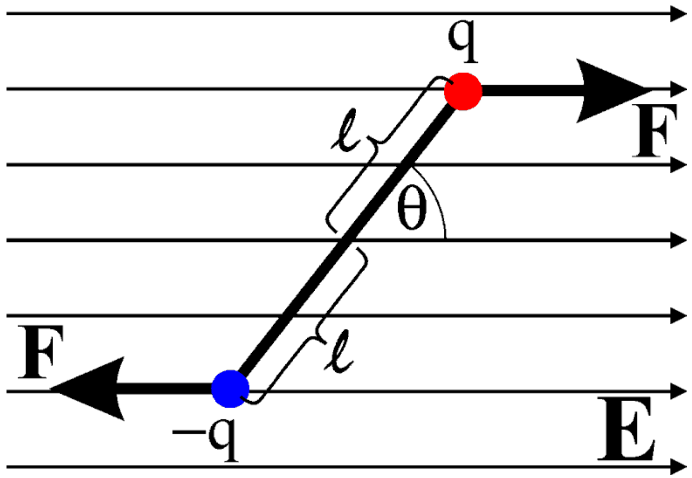
$$\underline{\underline{V_P = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}}$$

The radial and tangential components of the E-field:

$$\mathbf{E} = -grad\left(V_P\right); \qquad E_r = -\frac{\partial V_P}{\partial r} \quad \text{and} \quad E_\theta = -\frac{1}{r}\cdot\frac{\partial V_P}{\partial\theta}$$

$$\underline{\underline{E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}}} \qquad \text{and} \qquad \underline{\underline{E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}}}$$

Show that the torque exerted on a dipole by a uniform electric field **E** is **p** x **E**.



$$T = r \cdot F = 2\ell \sin\theta \cdot F =$$

$$= 2q\ell E \sin\theta$$

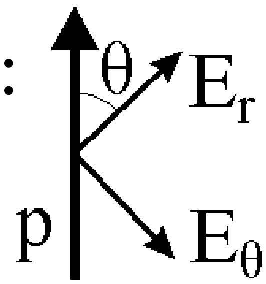
with $p = 2q\ell$:

$$\mathbf{T} = \mathbf{p} \times \mathbf{E}$$

Using these results find the angle θ for which the resultant electric field $\mathbf{E}$ at the point (r, θ) is in a direction *normal* to the axis of the dipole. [11]

Find the angle θ for which **E**(r, θ) at point P is in a direction normal to the axis of the dipole.

Take the dipole moment p to be along the *z*-axis :



Find angle for which $\mathbf{p} \cdot \mathbf{E} = p_z \cdot E_z = 0$

$$E_z = E_r \cdot \cos\theta - E_\theta \cdot \sin\theta = 0 \text{ thus } \frac{2p\cos^2\theta}{4\pi\varepsilon_0 r^3} - \frac{p\sin^2\theta}{4\pi\varepsilon_0 r^3} = 0$$

$$2\cos^2\theta = \sin^2\theta \quad \text{and} \quad \tan\theta = \pm\sqrt{2} \quad \text{or} \quad \theta = \pm 54.73^o$$

1.9. Show that the work W done in bringing a dipole of equal magnitude, $p = 2ql$, from infinity along its axis to a point at distance r from the first, is given by

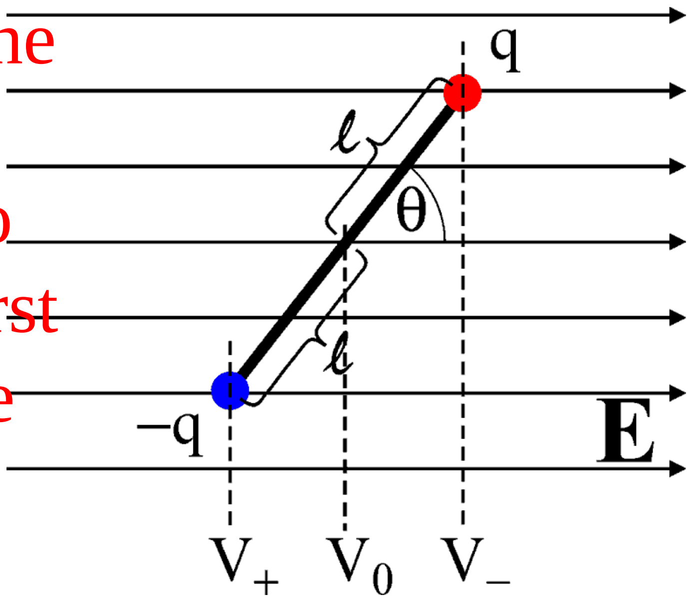
$$W = \frac{8q^2l^2}{4\pi\epsilon_0 r^3}\cos\theta,$$

where θ is the angle between the axes of the dipoles. [12]

[This is not the best worded question !]

Calculate the work done in bringing a dipole of equal magnitude from infinity to a distance r from the first along the normal to its axis.

E is the field due to the first dipole



$$U_E = (-q)\cdot V_+ + q\cdot V_-$$
$$= -q\cdot(V_+ - V_-)$$

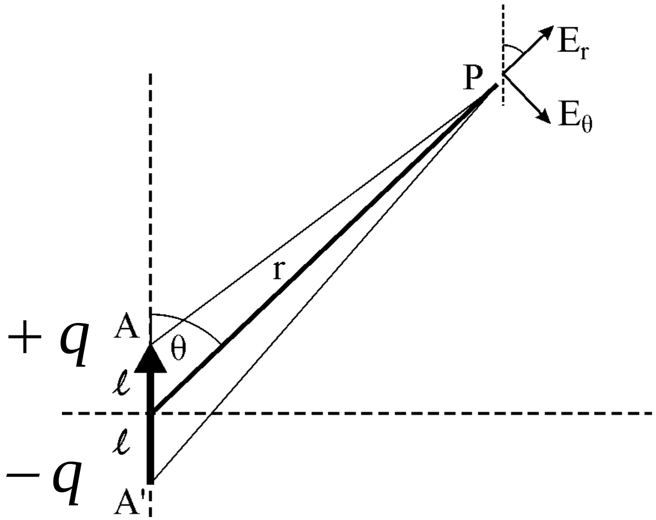
Take the origin at dipole centre

$$V_+ - V_- = -\mathbf{E}\cdot(\mathbf{r}_{q-} - \mathbf{r}_{q+})$$
$$U_E = q\mathbf{E}\cdot(-2\mathbf{l}) = -\mathbf{p}\cdot\mathbf{E}$$

When 1st dipole is at ∞, **E**=0 and U_E=0, and is then brought up

$$\underline{\underline{U_E}} = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta = 2q\ell\cdot\frac{2(2q\ell)}{4\pi\varepsilon_0 r^3}\cdot\cos\theta = \underline{\underline{\frac{8q^2\ell^2}{4\pi\varepsilon_0 r^3}\cos\theta}}$$

1.10. If a second dipole, free to rotate, is placed firstly along the line $\theta = 0$, and secondly in the plane $\theta = \pi/2$, in what direction will it point relative to the first? [4]



Second dipole placed at θ = 0 and then at θ = π/2, free to rotate :

$\theta = 0$	$E_r = \dfrac{2p}{4\pi\varepsilon_0 r^3}$	$E_\theta = 0$	p_2 Parallel
$\theta = \dfrac{\pi}{2}$	$E_r = 0$	$E_\theta = \dfrac{p}{4\pi\varepsilon_0 r^3}$	p_2 Anti-parallel