Waves reflecting from a thin layer or potential well

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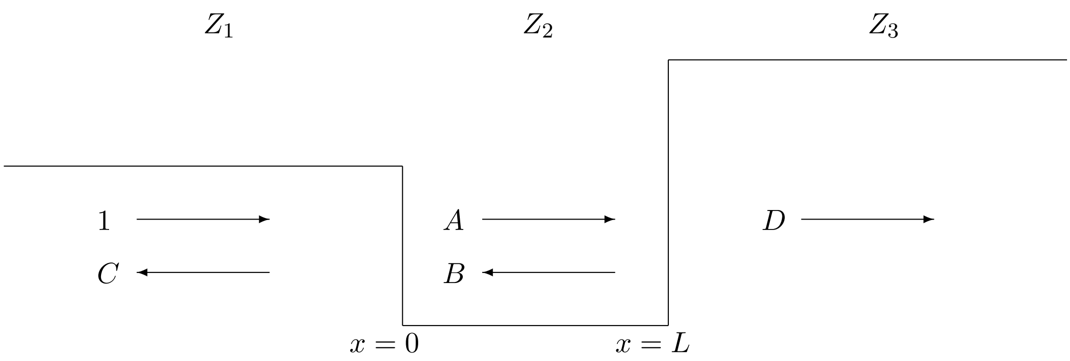


1 Preliminaries

The problem of waves reflecting from a thin layer or a potential well (or a hill) comes up in several areas. Examples include

1. Optics: electromagnetic waves incident on a piece of glass with a dielectric coating.
2. Quantum physics: de Broglie waves incident on a potential well or barrier.
3. Transmission lines: electrical signals passing between transmission lines of different impedance.

The solution of this problem can seem quite difficult the first time you encounter it, but it is not too bad if you adopt a good notation and learn how it is done. My aim here is to guide the reader through this calculation in a way that I believe to be reasonably intuitive, but ultimately to learn this you have to practice it yourself.

In the three examples listed above, in each case there are two properties of the waves that are continuous at any given boundary:

1. Electric and magnetic field amplitudes E, H.
2. Wavefunction and its gradient ψ, $(\partial\psi/\partial x)$.
3. Voltage and currrent V, I.

Also, in every case the continuous properties are related to one another in a simple way in the case of a single travelling wave going in the positive x direction:

1. If $E = E_0 e^{ikx}$ then $E = ZH$ where Z is the impedance. The impedance is a property of the medium through which the waves are propagating. In the case of non-absorbing dielectric media, the impedance is a positive real number (an expression is provided in eqn (27)).

2. If $\psi = Ae^{ikx}$ then $(\partial\psi/\partial x) = ik\psi$.

3. If $V = V_0 e^{ikx}$ then $V = ZI$ where Z is the characteristic impedance of the transmission line (see eqn (37)).

Also, for a wave travelling in the opposite direction (towards negative x), the relationship is similar but with a sign change:

1. If $E = E_0 e^{-ikx}$ then $E = -ZH$.

2. If $\psi = Ae^{-ikx}$ then $(\partial\psi/\partial x) = -ik\psi$.

3. If $V = V_0 e^{-ikx}$ then $V = -ZI$.

It is useful to treat the de Broglie wave case by writing the relationship between $(\partial\psi/\partial x)$ and ψ for a positive-going wave as

$$-i\hbar\left(\frac{\partial\psi}{\partial x}\right) = \frac{1}{Z}\psi \tag{1}$$

where $Z = (\hbar k)^{-1}$. Then we can treat all three cases using the same equations, and Z is real and positive in all cases when dealing with travelling waves (as opposed to evanescent or expontially decaying waves).

2 Main calculation

The situation we want to analyse is summarised in the figure. The potential well or dielectric layer extends from $x = 0$ to L and waves are incident from the left. We assign an amplitude 1 to the incident waves, and use A, B, C, D for the amplitudes of waves in the various regions moving in each direction. In other words, we are considering a solution for the waving quantity (electric field or wavefunction or voltage) which takes the form

$$\begin{aligned} e^{ik_1x} + Ce^{-ik_1x} \qquad & \text{for } x < 0 \\ Ae^{ik_2x} + Be^{-ik_2x} \qquad & \text{for } 0 < x < L \\ De^{ik_3x} \qquad & \text{for } x > L. \end{aligned} \tag{2}$$

Note that we are allowing that the three regions may all have different properties, so the wave vectors k_1, k_2, k_3 can be different in all three regions.

Applying the continuity conditions at the boundary at $x = 0$ we find

$$1 + C = A + B \tag{3}$$

$$\frac{1}{Z_1}(1 - C) = \frac{1}{Z_2}(A - B) \tag{4}$$

The second equation uses the fact that the second continuous physical quantity (H or $-i\hbar(\partial\psi/\partial x)$ or I) is $(1/Z)$ times the first for a right-going wave, and $(-1/Z)$ times the first for a left-going wave, and each region has its own impedance.

Applying the continuity conditions at the boundary at $x = L$ we find

$$\begin{aligned} Ae^{ik_2L} + Be^{-ik_2L} &= De^{ik_3L} && (5) \\ \frac{1}{Z_2}\left(Ae^{ik_2L} - Be^{-ik_2L}\right) &= \frac{1}{Z_3}De^{ik_3L}. && (6) \end{aligned}$$

The problem of studying reflection and transmission now reduces to the problem of solving equations (3)–(6) for C, A, B, D.

Since the reflection probability is given simply by $|C|^2$, it is best to make it our aim to find C. The job is not too hard as long as we make a good choice of how to proceed. The easiest thing to do to begin with is immediately elliminate D by taking the ratio of eqn (5) to eqn (6):

$$\frac{Z_2(A\alpha + B/\alpha)}{A\alpha - B/\alpha} = Z_3 \tag{7}$$

where

$$\alpha = e^{ik_2L}. \tag{8}$$

Introducing α here is useful to reduce clutter and thus get a clearer picture of what we have got. Both α and Z_2, Z_3 are known constants. We can now solve eqn (7) for B/A. After some easy algebra one finds

$$\frac{B}{A} = \alpha^2\frac{Z_3 - Z_2}{Z_3 + Z_2}. \tag{9}$$

Now is a good point to pause and notice that this equation makes sense. It is in fact the equation for reflection at a single interface, if the interface is located at $x = L$ (which leads to the phase factor α^2). When $Z_3 = Z_2$ we find $B = 0$: this is right, because when $Z_3 = Z_2$ there is no interface and thus no reflection at $x = L$.

The next step is to use eqns (3) and (4) to relate C to B/A. By taking the ratio of eqn (3) with (4) we find

$$\begin{aligned} \frac{Z_1(1 + C)}{1 - C} &= \frac{Z_2(A + B)}{A - B} && (10) \\ &= Z_2\frac{1 + r}{1 - r} && (11) \end{aligned}$$

where $r = B/A$. This can be solved for C. After a little algebra one finds

$$C = \frac{Z_2(1 + r) - Z_1(1 - r)}{Z_1(1 - r) + Z_2(1 + r)}. \tag{12}$$

This can also be written

$$C = \frac{Z_2 - Z_1 + r(Z_2 + Z_1)}{Z_2 + Z_1 + r(Z_2 - Z_1)} \tag{13}$$

(it is a matter of taste which form one prefers). Of course the formula only looks simple because we have bundled various factors into r, which is given by eqns (9) and (8):

$$r = e^{2ik_2L}\frac{Z_3 - Z_2}{Z_3 + Z_2}. \tag{14}$$

We have now finished, in the sense that we have an expression for the reflection amplitude C, in terms of the given quantities Z_1, Z_2, Z_3, k_2, L. At interesting case is the case of no reflection, $C = 0$. This happens when

$$Z_2 - Z_1 + r(Z_2 + Z_1) \quad = \quad 0 \tag{15}$$

therefore

$$e^{2ik_2L}\frac{Z_3 - Z_2}{Z_3 + Z_2} \quad = \quad \frac{Z_1 - Z_2}{Z_1 + Z_2}. \tag{16}$$

In order for this equation to be satisfied, the phase factor e^{2ik_2L} has to be real. There are two ways that this can come about.

1. If $k_2L = n\pi$ for integer n then $e^{2ik_2L} = 1$ and (16) is satisfied for

$$Z_3 = Z_1. \tag{17}$$

2. If $k_2L = (n + 1/2)\pi$ for integer n then $e^{2ik_2L} = -1$ and then by multiplying (16) by $(Z_3 + Z_2)(Z_1 + Z_2)$ one finds

$$(Z_2 - Z_3)(Z_1 + Z_2) = (Z_1 - Z_2)(Z_3 + Z_2). \tag{18}$$

After multiplying out the brackets four terms cancel and one obtains

$$Z_2^2 = Z_1Z_3. \tag{19}$$

This is the condition for zero reflection in this case. It says that the middle layer should have an impedance equal to the geometric mean of the other two impedances.

The first of the above two scenarios describes a half-wave layer between two media of the same impedance, or, more generally two such media separated by a layer whose thickness L is an integer multiple of $\lambda/2$, where (note) λ is the wavelength in the second region $\lambda = 2\pi/k_2$. This case is a standard example in quantum theory, with a beam of particles incident on a square potential well.

The second of the above two scenarios describes a quarter-wave layer, or, more generally, with L an odd multiple of $\lambda/4$ and $Z_3 \neq Z_1$. This case arises commonly in optics. Lenses in cameras and spectacles are often coated with such a layer to provide an anti-reflection coating. For a single layer it only works perfectly at a single wavelength in the visible spectrum, but it also works to reduce reflection somewhat over a range of wavelengths.

We will finish now with a few general comments. First, we studied the case of zero reflection since this is an interesting and easily calculated case. For this case r is real. Another moderately simple case is when r is pure imaginary. One then finds that the reflectivity is large when $|r|$ is large. This happens when the impedances differ by a large amount (as intuitively one might expect).

It is also worthy of note that we can write the expression for C (eqn (12)) in the form

$$C = \frac{Z_{\rm eff} - Z_1}{Z_{\rm eff} + Z_1} \tag{20}$$

where

$$\begin{aligned} Z_{\rm eff} &= Z_2 \frac{1+r}{1-r} && (21)\\ &= Z_2 \frac{Z_3 + Z_2 + \alpha^2 (Z_3 - Z_2)}{Z_3 + Z_2 - \alpha^2 (Z_3 - Z_2)}. && (22)\end{aligned}$$

In this way of looking at things, we regard the whole structure to the right of $x = 0$ as a single entity which presents an effective impedance to waves incident from the left. When $\alpha^2 = 1$ one gets $Z_{\rm eff} = Z_3$ and when $\alpha^2 = -1$ one gets $Z_{\rm eff} = Z_2^2/Z_3$. There is no reflection when $Z_{\rm eff} = Z_1$. For this reason the use of a suitably designed intermediate region to prevent reflection is called 'impedance matching'.

2.1 Impedance in the various scenarios

The expressions for impedance of the various types of waves we have mentioned go as follows.

We already noted that for de Broglie waves the impedance is

$$Z = \frac{1}{\hbar k} \tag{23}$$

(c.f. eqn 1). For a wave/particle of energy E, the wavevector k is related to E and the local potential energy $V(x)$ by

$$\frac{\hbar^2 k^2}{2m} = E - V. \tag{24}$$

For electromagnetic waves in a dielectric, one has, from the 3rd and 4th Maxwell's equations,

$$\begin{aligned} kE &= \omega \mu H && (25)\\ kH &= \omega \epsilon E && (26)\end{aligned}$$

where $\epsilon = \epsilon_0 \epsilon_r$ is the permittivity of the medium, and $\mu = \mu_0 \mu_r$ is the permeability. Hence one finds the phase velocity $\omega/k = (\epsilon\mu)^{-1/2}$ and

$$Z = \frac{\omega \mu}{k} = \frac{\mu}{\sqrt{\epsilon \mu}} = \sqrt{\frac{\mu}{\epsilon}}. \tag{27}$$

In terms of refractive index n, we write the phase velocity

$$\frac{\omega}{k} = \frac{c}{n} \tag{28}$$

where c is the speed of light in vacuum. Thus we find

$$n = \sqrt{\epsilon_r \mu_r} \tag{29}$$

and therefore

$$Z = \frac{Z_0 \mu_r}{n} \tag{30}$$

where $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space (value approximately 377 ohms).

For electric signals in a transmission line, one finds equations much like those for electromagnetic waves. The equations relate the voltages and currents to the capacitance and inductance per unit length of the transmission line. If one starts from '$V = Q/C$' for a capacitor, then it should not surprise us to find that voltage and current in the line are related by

$$\omega V = \frac{kI}{C} \tag{31}$$

for a wave of frequency ω and wavevector k, where (note) C is not the capacitance but the capacitance per unit length. The inductor equation meanwhile gives

$$kV = L\omega I \tag{32}$$

where L is the inductance per unit length (not to be confused with the L we used previously). Solving these two equations, one finds

$$\frac{\omega}{k} = \frac{1}{\sqrt{LC}} \tag{33}$$

and

$$Z = \sqrt{\frac{L}{C}} \tag{34}$$

In the case of coaxial cable, for example, one finds

$$C = \frac{2\pi\epsilon}{\ln(b/a)} \tag{35}$$

$$L = \frac{\mu \ln(b/a)}{2\pi} \tag{36}$$

which gives $LC = \epsilon\mu$ and

$$Z = \frac{1}{2\pi} \left(\frac{\mu}{\epsilon} \right)^{1/2} \ln(b/a) \tag{37}$$

where a, b are the inner and outer radii. This is called the characteristic impedance of the line. Typically coaxial cables are desgined to give either $Z = 50$ ohms or $Z = 70$ ohms. If one wants to deliver electrical signals to a device of some other impedance Z_3, then one might insert, between the input coaxial cable and the device, a quarter-wave segment with characteristic impedance $Z = \sqrt{Z_1 Z_3}$ (eqn (19)).

3 Transmission resonance and Fabry-Perot etalon

The case of zero reflectivity may be referred to as a 'transmission resonance', since then the transmission coefficient as a function of wavelength has a peak. To investigate this, we obtain $T = 1 - |C|^2$. Our starting point is eqn (13), which we write

$$C = \frac{\beta + r}{1 + \beta r} \tag{38}$$

where

$$\beta = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \tag{39}$$

For the cases we have considered so far, β is real, but the derivation does not require the impedances to be real, so we shall allow the possibility that one or more may be complex. We find

$$|C|^2 = \left(\frac{\beta + r}{1 + \beta r}\right)\left(\frac{\beta^* + r^*}{1 + \beta^* r^*}\right) = \frac{|\beta|^2 + |r|^2 + 2\Re[\beta r^*]}{1 + |\beta|^2|r|^2 + 2\Re[\beta r]} \tag{40}$$

so

$$T = 1 - |C|^2 = \frac{1 + |\beta|^2|r|^2 - |\beta|^2 - |r|^2 + 2\Re[\beta(r - r^*)]}{1 + |\beta|^2|r|^2 + 2\Re[\beta r]} \tag{41}$$

This is true in general (we have not restricted to any particular choice of impedance or of $k_2 L$).

Now let us treat the case $Z_3 = Z_1$. Then we have

$$r = -e^{2i\phi}\beta \tag{42}$$

where $\phi = k_2 L$ and therefore, as long as k_2 is real,

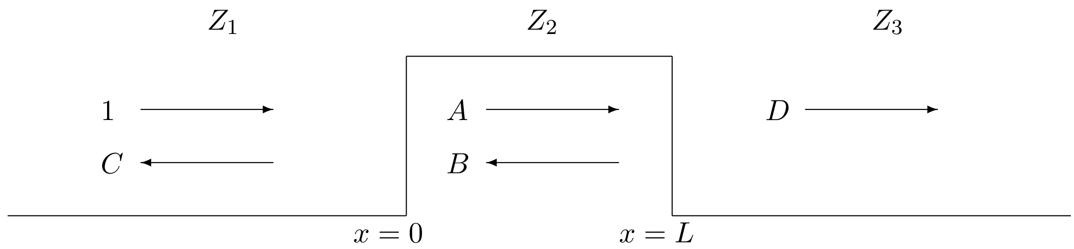
$$T = \frac{1 + |\beta|^4 - 2|\beta|^2 + 2\Re[\beta(r - r^*)]}{1 + |\beta|^4 + 2\Re[\beta r]}. \tag{43}$$

Now let us consider further the case where β is real. In this case, $\Re[\beta(r - r^*)] = 0$ and $\Re[\beta r] = -\beta^2 \cos 2\phi$ so we obtain

$$T = \frac{(1 - \beta^2)^2}{1 + \beta^4 - 2\beta^2 \cos 2\phi} = \frac{1}{1 + \frac{2\beta^2}{(1-\beta^2)^2}(1 - \cos 2\phi)}. \tag{44}$$

In optics this situation is called the *Fabry-Perot etalon* (and there exists a neat alternative method of derivation based on summing a geometric series). In quantum physics it is called a transmission resonance; examples occur for electrons passing through a gas (where the atoms provide the scattering potential) and neutrons passing through an atomic nucleus. $T = 1$ whenever $\cos 2\phi = 1$. Also, if β^2 is close to 1 then for $\cos 2\phi \neq 1$ the transmission is small. Hence the transmission can swing between 1 and a small value as a function of ϕ.

4 Quantum tunnelling and frustrated total internal reflection



Another interesting case occurs when the waves do not propagate in the normal way in the central region, because k_2 is imaginary. Then in the region $0 < x < L$ the wave takes the form

$$Ae^{\kappa L} + Be^{-\kappa L} \tag{45}$$

where $\kappa = ik_2$ (with k_2 imaginary so κ is real). In the case of electromagnetic waves, this is the form of the solution for the electric field outside a glass surface when the waves undergo total internal reflection. In the case of de Broglie waves, this happens when we have a potential barrier whose height exceeds the energy of the incident wave/particles: $V_2 > E$. In this case $\hbar^2\kappa^2/2m = V_2 - E$. Note that this also implies that Z_2 is pure imaginary.

To find the behaviour in this situation, one can proceed from equation (38), keeping in mind which quantities are complex, or start afresh. I will show both methods.

First, if we start the calculation afresh from the start, then the continuity conditions at $x = 0$ and $x = L$ now take the form

$$\begin{aligned} 1 + C &= A + B & (46)\\ \frac{1}{Z_1}(1 - C) &= \frac{1}{Z_2}(A - B) & (47)\\ Ae^{\kappa L} + Be^{-\kappa L} &= De^{ik_3 L} & (48)\\ \frac{1}{Z_2}\left(Ae^{\kappa L} - Be^{-\kappa L}\right) &= \frac{1}{Z_3}De^{ik_3 L}. & (49) \end{aligned}$$

We will make it our aim to find D. First we use the last two equations to get A and B in terms of D. By adding and subtracting one finds:

$$\begin{aligned} 2Ae^{\kappa L} &= De^{ik_3 L}(1 + Z_2/Z_3), & (50)\\ 2Be^{-\kappa L} &= De^{ik_3 L}(1 - Z_2/Z_3). & (51) \end{aligned}$$

Next we eliminate C from (46) and (47):

$$2 = A(1 + Z_1/Z_2) + B(1 - Z_1/Z_2). \tag{52}$$

Then by substituting (50) and (51) into this we find

$$2 = \frac{D}{2} e^{ik_3 L} \left[e^{-\kappa L} \left(1 + \frac{Z_2}{Z_3}\right) \left(1 + \frac{Z_1}{Z_2}\right) + e^{\kappa L} \left(1 - \frac{Z_2}{Z_3}\right) \left(1 - \frac{Z_1}{Z_2}\right) \right]. \tag{53}$$

We now have an equation for the transmission amplitude D in terms of the various impedances and κL. It remains to simplify it.

A good case to consider is $Z_1 = Z_3$. This happens when we have a potential barrier between two regions both at the same potential. In this case the brackets in the above expression evaluate to

$$\left(1 \pm \frac{Z_2}{Z_1}\right) \left(1 \pm \frac{Z_1}{Z_2}\right) = 2 \pm \frac{Z_2^2 + Z_1^2}{Z_1 Z_2} \tag{54}$$

so now (53) gives

$$2 = D e^{ikL} \left[2 \cosh \kappa L + \frac{Z_2^2 + Z_1^2}{Z_1 Z_2} \sinh \kappa L \right] \tag{55}$$

where $k = k_3 = k_1$. Therefore

$$D = e^{-ikL} \left[\cosh \kappa L + \frac{Z_2^2 + Z_1^2}{2 Z_1 Z_2} \sinh \kappa L \right]^{-1} \tag{56}$$

Now let's replace the impedances by their expressions in terms of wavevectors for the case of de Broglie waves:

$$Z_1 = 1/\hbar k, \tag{57}$$
$$Z_2 = i/\hbar \kappa \tag{58}$$

so

$$D = e^{-ikL} \left[\cosh \kappa L + i \frac{\kappa^2 - k^2}{2\kappa k} \sinh \kappa L \right]^{-1} \tag{59}$$

and therefore

$$|D|^2 = \left[\cosh^2 \kappa L + \left(\frac{\kappa^2 - k^2}{2\kappa k} \right)^2 \sinh^2 \kappa L \right]^{-1} \tag{60}$$

Now using $\cosh^2 \kappa L = 1 + \sinh^2 \kappa L$, the expression can be written in the convenient form

$$|D|^2 = \frac{1}{1 + \gamma^2 \sinh^2 \kappa L} \tag{61}$$

where

$$\gamma = \frac{\kappa^2 + k^2}{2\kappa k} \tag{62}$$
$$= \frac{V_2}{2\sqrt{(V_2 - E)E}}. \tag{63}$$

Equation (61) is an example of *quantum tunnelling.* The behaviour is called 'tunnelling' because we may picture the potential barrier as a kind of 'hill'. On a particle model, we might say that the particles do not have enough energy to climb up and over the hill, because $E < V_2$, so we should expect them to be fully reflected. They are not: $|D|^2 \neq 0$ so there is a non-zero flux on the right of the barrier, and we might say therefore that the particles have 'tunnelled through' the hill. This is an interesting phenomenon, but one should not exaggerate its difference from classical physics, because the difference depends on whether you adopt a particle picture or a wave picture for the physical entities in question. In fact things like electrons are quantum things, so one may equally well say they are waves as say they are particles. As soon as one admits they are waves, the tunnelling phenomenon is no longer so surprising, because it is exactly what one should expect waves to do. The very same behaviour is indeed seen in the propagation of classical electromagnetic waves in the phenomenon of *frustrated total internal reflection.*

Quantum tunnelling happens with non-negligible probability as long as $\gamma \sinh \kappa L$ is not too large. This requires that the barrier is neither too high nor too wide. For $\kappa L \gg 1$ one can approximate $\sinh \kappa L \simeq e^{\kappa L}/2$ and then

$$|D|^2 \simeq \frac{4}{\gamma^2} e^{-2\kappa L}. \tag{64}$$

Hence the tunnelling probability depends exponentially on the width of the barrier in the limit of a wide barrier. This exponential dependence makes the tunnelling a sensitive indicator of L. The *scanning tunnelling microscope* makes use of this to determine the distance between a tip and a surface through measurement of the electron tunnelling current.

In the scanning tunnelling microscope the region inside the potential barrier is a region of space that is empty apart from an electric field which provides the potential energy 'hill' for the electrons. Another example occurs in radioactive beta decay, where the strong nuclear force acts to provide a potential energy barrier to the escaping alpha particle.

A cautionary note. You will find on the internet, and sometimes in textbooks, dubious statements along the lines that ordinary objects could in principle pass through walls by quantum tunnelling. In fact the situation with a wall (i.e. a solid object made of atoms) is complicated by the fact that it is not empty space but full of electrons. This makes it hard for other electrons to pass through, owing to the Pauli Exclusion Principle which says (roughly) that two electrons can't occupy the same region of space if they have the same spin state and the same momentum. This Exclusion Principle is not owing to any potential energy consideration, but to an interferometric multi-electron effect which is not accounted for by the simple theory we have presented here. When thinking about quantum tunnelling it is better to restrict your thinking to simpler situations where you can be confident that you have taken everything into account. Better examples occur in the motion of electrons inside crystalline solids, and in the physics of atomic collisions, where it can happen that two atoms have to overcome a potential barrier in order to approach closely.

4.1 Alternative derivation

When the situation in region 2 is as given in eqn (45), we find

$$C = \frac{\beta + r}{1 + \beta r} \tag{65}$$

as in eqn (38), where now (using (39), (57))

$$\beta = \frac{ik - \kappa}{ik + \kappa}, \tag{66}$$

$$r = e^{2\kappa L} \frac{Z_3 - Z_2}{Z_3 + Z_2}. \tag{67}$$

Taking the case $Z_1 = Z_3$ gives $r = -e^{2\kappa L}\beta$ and therefore

$$C = \frac{\beta(1 - e^{2\kappa L})}{1 - \beta^2 e^{2\kappa L}} = \frac{\beta(e^{-\kappa L} - e^{\kappa L})}{e^{-\kappa L} - \beta^2 e^{\kappa L}}. \tag{68}$$

Now using

$$e^{\kappa L} = \cosh \kappa L + \sinh \kappa L \tag{69}$$

$$e^{-\kappa L} = \cosh \kappa L - \sinh \kappa L \tag{70}$$

we have

$$C = \frac{2\beta \sinh \kappa L}{(\beta^2 - 1)\cosh \kappa L + (\beta^2 + 1)\sinh \kappa L} \tag{71}$$

and (66) gives

$$\beta + \frac{1}{\beta} = \frac{2(\kappa^2 - k^2)}{\kappa^2 + k^2}, \tag{72}$$

$$\beta - \frac{1}{\beta} = \frac{4i\kappa k}{\kappa^2 + k^2}. \tag{73}$$

Substituting these results into (71) yields

$$C = \frac{\sinh \kappa L}{\frac{2i\kappa k}{\kappa^2 + k^2}\cosh \kappa L + \frac{\kappa^2 - k^2}{\kappa^2 + k^2}\sinh \kappa L} \tag{74}$$

which we shall write

$$C = \frac{-i\gamma \sinh \kappa L}{\cosh \kappa L + \frac{\kappa^2 - k^2}{2i\kappa k}\sinh \kappa L} \tag{75}$$

where γ is as given by eqn (62). Therefore

$$|C|^2 = \frac{\gamma^2 \sinh^2 \kappa L}{\cosh^2 \kappa L + \frac{(\kappa^2 - k^2)^2}{4\kappa^2 k^2}\sinh^2 \kappa L} = \frac{\gamma^2 \sinh^2 \kappa L}{1 + \gamma^2 \sinh^2 \kappa L} \tag{76}$$

where the last step used $\cosh^2 \kappa L = 1 + \sinh^2 \kappa L$. Eqn (61) follows immediately.

Exercise
Show that (44) and (61) amount to the same result, under the substitution $ik_2 = \kappa$, if one 'reads' the factor in the denominator of (44) as $2|\beta/(1 - \beta^2)|^2$.