Lecture 1. Basic concepts (book: p.1-19; 28-34)

Reference body. 'Observe' = deduce. Postulates. Standard configuration.

Spacetime diagrams; axes and simultaneity

Lorentz transformation:

$$\left.\begin{array}{rcl} t' &=& \gamma(t - vx/c^2) \\ x' &=& \gamma(-vt + x) \\ y' &=& y \\ z' &=& z \end{array}\right\}, \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Minkowski metric and definition of Λ:

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{array}{rrcll} & \Lambda^T g \Lambda &=& g & \\ \Rightarrow & \mathsf{A}'^T g \mathsf{B}' &=& \mathsf{A}^T g \mathsf{B} & \equiv \mathsf{A} \cdot \mathsf{B} \quad \textbf{Invariant} \end{array}$$

Postulate 1, "Principle of Relativity":

> The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line.

Postulate 2, "Light speed postulate":

> There is a finite maximum speed for signals.

Alternative statements:
Postulate 1:

> The laws of physics take the same mathematical form in all inertial frames of reference.

Postulate 2:

> There is an inertial reference frame in which the speed of light in vacuum is independent of the motion of the source.

Lecture 2. 4-vectors; Proper time; Method of invariants; transformation of velocity (book p.14; 399-400; 22-25; 39-44)

Proper time

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$$

Familiarity with γ :

$$\gamma = (1-\beta^2)^{-1/2}, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}v} = \gamma^3 v/c^2, \qquad \frac{\mathrm{d}}{\mathrm{d}v}(\gamma v) = \gamma^3$$

symbol	definition	components	name(s)	invariant
X	X	$(ct, \mathbf{r})$	4-displacement, interval	$-c^2\tau^2$
U	$d\mathsf{X}/d\tau$	$(\gamma c, \gamma \mathbf{v})$	4-velocity	$-c^2$
P	$m_0\mathsf{U}$	$(E/c, \mathbf{p})$	energy-momentum, 4-momentum	$-m_0^2c^2$
A	$d\mathsf{U}/d\tau$	$\gamma(\dot\gamma c, \dot\gamma \mathbf{v} + \gamma \mathbf{a})$	4-acceleration	a_0^2

Timelike	$\mathsf{U}\cdot\mathsf{U} < 0$	e.g. 4-velocity
spacelike	$\mathsf{A}\cdot\mathsf{A} > 0$	e.g. 4-acceleration
null	$\mathsf{P}\cdot\mathsf{P} = 0$	e.g. energy-momentum of light pulse

Method of invariants = "Try using an invariant if you can, and pick an easy reference frame."

4-acceleration is orthogonal to 4-velocity: $\mathsf{U}\cdot\mathsf{A} = 0$.

Transformation of velocity:

$$\mathbf{u}'_\parallel = \frac{\mathbf{u}_\parallel - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}, \qquad \mathbf{u}'_\perp = \frac{\mathbf{u}_\perp}{\gamma_v\,(1 - \mathbf{u}\cdot\mathbf{v}/c^2)}.$$

$\longrightarrow$ particle jets

Lecture 3. Rapidity; Doppler effect; Headlight effect

1. Rapidity: $\tanh(\rho) \equiv \frac{v}{c}, \quad \Rightarrow \quad \cosh(\rho) = \gamma, \quad \sinh(\rho) = \beta\gamma, \quad e^{\rho} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}.$

$$\Lambda = \begin{pmatrix} \cosh\rho & -\sinh\rho & 0 & 0 \\ -\sinh\rho & \cosh\rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$\boxed{1} \qquad \boxed{2} \longrightarrow v_{21} \qquad \boxed{3} \longrightarrow v_{32}$

Colinear velocities: $v_{31} = \frac{v_{32} + v_{21}}{1 + v_{32}v_{21}/c^2}$

Colinear rapidities: $\rho_{31} = \rho_{21} + \rho_{32}$

2. Doppler effect

4-wave vector $\mathsf{K} \equiv (\omega/c,\ \mathbf{k}), \qquad e^{i(\mathbf{k}\cdot r - \omega t)} = e^{i\mathsf{K}\cdot\mathsf{X}}$

Doppler effect: $\mathsf{K}\cdot\mathsf{U} \Rightarrow \qquad \frac{\omega}{\omega_0} = \frac{1}{\gamma(1 - (v/v_p)\cos\theta)}.$

3. Headlight effect or 'aberration':

Headlight effect: $\mathsf{K} = \Lambda^{-1}\mathsf{K}_0 \Rightarrow \quad \cos\theta = \frac{\cos\theta_0 + v/c}{1 + (v/c)\cos\theta_0} \quad \Rightarrow \frac{d\Omega}{d\Omega_0} = \left(\frac{\omega_0}{\omega}\right)^2.$

$\longrightarrow$ **brightness** (power per unit solid angle) transforms as $(\omega/\omega_0)^4$ for isotropic source

Lecture 4. Force; simple dynamical problems

1. Force

$$\mathsf{F} \equiv \frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\tau}, \qquad \mathsf{U}\cdot\mathsf{F} = \gamma^2\left(-\frac{\mathrm{d}E}{\mathrm{d}t} + \mathbf{u}\cdot\mathbf{f}\right) = -c^2\frac{\mathrm{d}m_0}{\mathrm{d}\tau}.$$

'Pure' force:

$$\mathsf{U}\cdot\mathsf{F} = 0 \;\;\Rightarrow\;\; m_0 = \text{const}, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{f}\cdot\mathbf{u}$$

2. Transformation of force: use $\Lambda\mathsf{F}$ and γ, or $(\mathrm{d}/\mathrm{d}t')(\Lambda\mathsf{P})$:

$$\mathbf{f}'_{\parallel} = \frac{\mathbf{f}_{\parallel} - (\mathbf{v}/c^2)dE/dt}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}, \qquad \mathbf{f}'_{\perp} = \frac{\mathbf{f}_{\perp}}{\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)} \qquad \Rightarrow \begin{cases} \mathbf{f}'_{\parallel} = \mathbf{f}_{\parallel} \\ \mathbf{f}'_{\perp} = \mathbf{f}_{\perp}/\gamma \end{cases} \text{for } u = 0$$

3. Equation of motion in any given reference frame:

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = \gamma m\mathbf{a} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v}$$

$\Rightarrow$

- (i) acceleration is not necessarily parallel to force!
- (ii) $\mathbf{f} = m\mathbf{a}$ only valid at $\mathbf{v} = 0$
- (iii) for 'pure' force (m=constant): $f_{\parallel} = \gamma^3 m a_{\parallel}, \quad f_{\perp} = \gamma m a_{\perp}.$

4. Uniform B field: just like Newtonian result, but with m replaced by γm. Hence $\omega = qB/\gamma m$ and $p = qBr$.

5. Motion parallel to E field: hyperbolic motion, $x^2 - c^2t^2 = (c^2/a_0)^2$ and $\ddot{\mathsf{U}} \propto \mathsf{U}$. Constant proper acceleration.

Lecture 5. Some more kinematics; The conservation of energy and momentum

1. Constant proper acceleration and hyperbolic motion: $\dfrac{\mathrm{d}\rho}{\mathrm{d}\tau} = a_0$.

2. Rigidity; the Great Train Disaster

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3. Lewis and Tolman argument: $\mathbf{p} = m\alpha(v)\mathbf{v} \quad \Rightarrow \mathbf{p} = \gamma m \mathbf{v}$.

4. Impact of simultaneity on "$\mathsf{P}_{\text{tot}} = \mathsf{P}_1 + \mathsf{P}_2 + \mathsf{P}_3 + \ldots \mathsf{P}_n$".

5. "Zero component lemma": if one component of a 4-vector is zero in all reference frames, then the whole 4-vector is zero.

Hence momentum conservation $\Leftrightarrow$ energy conservation.

$$\left.\begin{array}{l}\text{Main postulates,}\\ \text{momentum conservation}\end{array}\right\} \Rightarrow E_0 = mc^2, \;\; \text{equivalence of rest mass and rest energy.}$$

Lecture 6. Collisions.
Methods:

$$\mathsf{P}\cdot\mathsf{P} = -m^2c^2 \;\;\Rightarrow\;\; E^2 - p^2c^2 = m^2c^4 \tag{1}$$

$$\mathbf{p} = \gamma m\mathbf{v}, \;\; E = \gamma mc^2 \quad\;\; \Rightarrow\;\; \mathbf{v} = \frac{\mathbf{p}c^2}{E} \tag{2}$$

"Isolate and square." (3)

1. Decay at rest. e.g. find the energy of one of the products:

$$E_1 \;=\; \frac{M^2 + m_1^2 - m_2^2}{2M}c^2, \qquad E_{\rm photon} = \Delta E_{\rm rest} - E_{\rm recoil}.$$

2. In-flight decay. e.g. Find the rest mass of the original particle:

$$M^2 = m_1^2 + m_2^2 + \frac{2}{c^4}(E_1E_2 - \mathbf{p}_1\cdot\mathbf{p}_2c^2)$$

$\Rightarrow$ it suffices to measure $m_1, m_2, p_1, p_2, \theta$.

3. Particle formation. e.g. threshold energy (stationary target):

$$E_{\rm th} = \frac{(\sum_i m_i)^2 - m^2 - M^2}{2M}c^2.$$

4. Elastic collision. e.g. find angles in the lab frame

5. Compton effect (A. H. Compton, Physical Review **21**, 483 (1923)):

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta).$$

[other: 3-body decay, inverse Compton effect, etc.]

Lecture 7. Composite body; 4-gradient; flow

1. The idea of a composite body with a net energy and momentum, hence rest mass and velocity. $\mathbf{p} = \sum_i \mathbf{p}_i$, $E = \sum_i E_i$, $\mathsf{P} \equiv (E/c, \mathbf{p})$, $m \equiv \sqrt{-\mathsf{P}^2/c^2}$, $\mathbf{v} = \mathbf{p}c^2/E$.

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2. Concept of a Lorentz-invariant scalar field.
e.g. $\mathbf{B} \cdot \mathbf{E}$, $E^2 - c^2 B^2$, but NOT charge density or potential energy.

3. 4-gradient operator:

$$\Box \equiv \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

If ϕ is a Lorentz-invariant scalar quantity, then: $\Box' \phi = \Lambda \Box \phi$
i.e. $\Box\phi$ is a 4-vector.

4. 4-divergence

$$\Box \cdot \mathsf{F} = \frac{1}{c}\frac{\partial F^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{f}, \qquad \Box^2 = \Box \cdot \Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2.$$

5. Wave phase ϕ is Lorentz-invariant.

$$\Rightarrow \quad \mathsf{K} \equiv \Box\phi \quad \text{is a 4-vector}$$
$$\Box^2\phi = 0 \quad \text{is the wave equation.}$$

6. Flow and conservation:

4-current density $\mathsf{J} \equiv \rho_0 \mathsf{U} = (\rho c, \mathbf{j})$

continuity equation $\Box \cdot \mathsf{J} = 0.$