Symmetry and Relativity: example spinor questions

3rd year physics

1. What type of 4-vector may be represented by a 2-component spinor? Write down an expression relating the 4-vector components to the spinor. [4]

Prove that if $w \to e^{\rho\sigma_z/2} w$ then the 4-vector represented by the spinor w undergoes a Lorentz transformation. [8]

The Pauli-Lubanski spin 4-vector has components $W^a = (\mathbf{s}\cdot\mathbf{p}, (E/c)\mathbf{s})$ for a particle of spin $\mathbf{s}$, energy E and momentum $\mathbf{p}$.

Define chirality and helicity in the context of 2-spinors, velocity and spin. A single spinor w may be used to represent both the spin and the 4-momentum of a given particle. Write down two possible relationships between w and the relevant 4-vectors, hence showing that particles described this way may exist as two non-equivalent types. What experimental observation could distinguish particles of one type from the other? [8]

2. Show that $|AB| = |A||B|$ for arbitrary 2×2 matrices A and B, where $|M|$ signifies the determinant of M.

A 4-vector $X = (t, x, y, z)$ is related to a complex 2×2 matrix S by

$$S = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}.$$

Let $S' = \Lambda S \Lambda^\dagger$ be similarly related to $X' = (t', x', y', z')$, where Λ is a matrix of determinant 1. Find $|S|$, and show that $-t'^2 + x'^2 + y'^2 + z'^2 = -t^2 + x^2 + y^2 + z^2$. [6]

The matrix exponential $\exp(M)$ is defined by

$$\exp(M) \equiv I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots.$$

Show that $\exp(i\theta\sigma_x) = \cos(\theta)I + i\sin(\theta)\sigma_x$. By relating S to the Pauli spin matrices, or otherwise, show that, when S is transformed to $S' = USU^\dagger$ by $U = \exp(i\theta\sigma_x)$, the associated vector 4-vector X' is related to X by a rotation through 2θ about the x axis. [10]

Show that S may be expressed as an outer product of two-component spinors of rank one, as long as the elements of S satisfy a certain condition, and obtain the condition. If $S \to \Lambda S \Lambda^\dagger$, how does the associated rank one spinor transform? [4]