Symmetry and Relativity: example spinor questions
3rd year physics

1. What type of 4-vector may be represented by a 2-component spinor? Write down an expression relating the 4-vector components to the spinor. 

Prove that if \( w \to e^{\alpha x/2} w \) then the 4-vector represented by the spinor \( w \) undergoes a Lorentz transformation.

The Pauli-Lubanski spin 4-vector has components \( W^a = (s \cdot p, (E/c)s) \) for a particle of spin \( s \), energy \( E \) and momentum \( p \).

Define chirality and helicity in the context of 2-spinors, velocity and spin. A single spinor \( w \) may be used to represent both the spin and the 4-momentum of a given particle. Write down two possible relationships between \( w \) and the relevant 4-vectors, hence showing that particles described this way may exist as two non-equivalent types. What experimental observation could distinguish particles of one type from the other?

2. Show that \( |AB| = |A||B| \) for arbitrary \( 2 \times 2 \) matrices \( A \) and \( B \), where \( |M| \) signifies the determinant of \( M \).

A 4-vector \( X = (t, x, y, z) \) is related to a complex \( 2 \times 2 \) matrix \( S \) by

\[
S = \begin{pmatrix}
  t + z & x - iy \\
  x + iy & t - z
\end{pmatrix}.
\]

Let \( S' = \Lambda S \Lambda^\dagger \) be similarly related to \( X' = (t', x', y', z') \), where \( \Lambda \) is a matrix of determinant 1. Find \( |S'| \), and show that \( -t'^2 + x'^2 + y'^2 + z'^2 = -t^2 + x^2 + y^2 + z^2 \).

The matrix exponential \( \exp(M) \) is defined by

\[
\exp(M) \equiv I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \ldots.
\]

Show that \( \exp(i\theta \sigma_x) = \cos(\theta) I + i \sin(\theta) \sigma_x \). By relating \( S \) to the Pauli spin matrices, or otherwise, show that, when \( S \) is transformed to \( S' = USU^\dagger \) by \( U = \exp(i\theta \sigma_x) \), the associated vector 4-vector \( X' \) is related to \( X \) by a rotation through \( 2\theta \) about the \( x \) axis.

Show that \( S \) may be expressed as an outer product of two-component spinors of rank one, as long as the elements of \( S \) satisfy a certain condition, and obtain the condition. If \( S \to \Lambda S \Lambda^\dagger \), how does the associated rank one spinor transform?