Lecture 19. Spinors

1. Def: a 'vector+flag' that can be mapped onto a pair of complex numbers: $\boldsymbol{s} = \begin{pmatrix} a \\ b \end{pmatrix}$

2. Extracting a null 4-vector from a spinor:
$$\boxed{\mathsf{X}^\mu = \langle s|\,\sigma^\mu\,|s\rangle}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. Rotating the coordinate system: spinor $\boldsymbol{s} \to U\boldsymbol{s}$

$$\begin{aligned}
U_x = e^{i(\theta/2)\sigma_x} &= \cos(\theta/2)I + i\sin(\theta/2)\sigma_x = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \\
U_y = e^{i(\theta/2)\sigma_y} &= \cos(\theta/2)I + i\sin(\theta/2)\sigma_y = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \\
U_z = e^{i(\theta/2)\sigma_z} &= \cos(\theta/2)I + i\sin(\theta/2)\sigma_z = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.
\end{aligned}$$

4. Any 2×2 matrix Λ with unit determinant Lorentz-transforms a spinor.
Such matrices can be written
$$\Lambda = \exp\left(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 - \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2\right)$$
where ρ is rapidity and θ is rotation angle. If Λ is unitary the transformation is a rotation in space; if Λ is Hermitian it is a boost.

5. Main idea: $\begin{cases} \text{1st rank spinor} & \leftrightarrow \quad \text{null 4-vector} \\ \text{one type of 2nd rank spinor} & \leftrightarrow \quad \text{any 4-vector } (\Rightarrow \text{transformation rule}) \end{cases}$

1. *Spinor basics*

i. A "spinor" is essentially a mathematical tool.
ii. A rank 1 spinor is very much like a 4-vector; (a rank 2 spinor is like a tensor).
iii. Spinors are used in quantum as well as classical physics; we shall only do classical physics.
iv. Everything you can do with vectors and tensors you can also do with spinors!
v. But don't worry, we will focus on describing just two basic physical quantities: energy-momentum and angular-momentum.

Two ways of thinking about a spinor.

1. It is a two-component vector having complex coefficients:

$$\boldsymbol{s} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad\qquad (a, b \in \text{complex numbers})$$

2. It is a null 4-vector, with real (not complex) components, with a 'flag' attached.

If we write the spinor as

$$\boldsymbol{s} = s e^{i\alpha/2} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix}$$

then the associated 4-vector (sometimes called 'flagpole') is

$$\begin{pmatrix} r \\ r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix}$$

where $r = s^2$. The spatial (3-vector) part can also be called a Bloch vector.

Lecture 20. Chirality, Weyl equations, Dirac spinor

1.
$$\begin{array}{llll} \text{contraspinor} & \boldsymbol{s}'_R = \Lambda \boldsymbol{s}_R, & \mathsf{X}^\mu & = \langle s_R | \, \sigma^\mu \, | s_R \rangle \\ \text{cospinor} & \boldsymbol{s}'_L = (\Lambda^\dagger)^{-1} \boldsymbol{s}'_L, & \mathsf{X}_\mu & = \langle s_L | \, \sigma^\mu \, | s_L \rangle \end{array}$$

contraspinor is called "right handed" or "+ve chirality"
cospinor is called "left handed" or "−ve chirality" *but this terminology invites confusion* (e.g. we don't call contravariant 4-vectors "right handed")

2. Weyl equations and **parity violation**

$$\begin{array}{lrcl} \mathsf{P}_\lambda \sigma^\lambda \boldsymbol{w}_R = 0 & \Rightarrow (E/c - \mathbf{p} \cdot \boldsymbol{\sigma}) \boldsymbol{w}_R & = & 0, \\ \mathsf{P}_\lambda \sigma^\lambda \boldsymbol{w}_L = 0 & \Rightarrow (E/c + \mathbf{p} \cdot \boldsymbol{\sigma}) \boldsymbol{w}_L & = & 0. \end{array}$$

→ eigenvalue equations, helicity is positive/negative for contraspinors/cospinors.

3. Treat massive particles using a pair of spinors, ϕ_R, χ_L.
Form $\mathsf{U} = \mathsf{A} - \mathsf{B}$, $\mathsf{W} = (\mathsf{A} + \mathsf{B}) mcS$: pair of orthogonal non-null 4-vectors.
Hence

$$\mathsf{U}^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi, \qquad \mathsf{W}^\mu = mcS \Psi^\dagger \Sigma^\mu \Psi,$$

where $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$ "Dirac matrices"

4. Lorentz transform of Dirac spinor: $\Psi \rightarrow \begin{pmatrix} \Lambda(v) & 0 \\ 0 & \Lambda(-v) \end{pmatrix} \Psi$
leads to "particle Dirac equation"

$$\begin{pmatrix} -m & E + \boldsymbol{\sigma} \cdot \mathbf{p} \\ E - \boldsymbol{\sigma} \cdot \mathbf{p} & -m \end{pmatrix} \begin{pmatrix} \phi_R(\mathbf{p}) \\ \chi_L(\mathbf{p}) \end{pmatrix} = 0.$$

Lecture 21. Variations on the wave equation

1. **Wave equation**: $\Box^2\phi = 0$ or $-\dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = 0.$

Plane wave solutions

$$\phi(t,x,y,z) = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \Rightarrow \omega^2 - k^2c^2 = 0. \qquad \text{(c.f. } E^2 - p^2c^2 = 0.)$$

2. **Klein Gordan equation**: $(\Box^2 - \mu^2c^2)\phi = 0.$

Plane wave solutions: $\omega^2 - k^2c^2 = \mu^2c^4$

Associated 4-current $\mathsf{J} = i\left(\phi\Box\phi^* - \phi^*\Box\phi\right),\qquad \Box\cdot\mathsf{J} = 0$

3. **Dirac equation in 1D** (i.e. 1 spatial dimension plus time):

zero mass:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi_1 = 0, \qquad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\phi_2 = 0.$$

non-zero mass:

$$\left.\begin{array}{rcl} i\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\phi_1 &=& \mu c^2\phi_2 \\ i\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi_2 &=& \mu c^2\phi_1 \end{array}\right\} \qquad \Rightarrow \qquad \left(\hat{\omega} + \sigma_z\hat{k}_x c\right)\psi = \mu c^2\sigma_x\psi$$

4. **Dirac equation in 3D** (i.e. 3 spatial dimensions plus time):

zero mass:

$$\begin{array}{rcl} (\hat{\omega} - \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}c)\phi_1 &=& 0, \\ (\hat{\omega} + \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}c)\phi_2 &=& 0. \end{array}$$

non-zero mass:

$$\left.\begin{array}{rcl} (\hat{\omega} - \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}c)\,\phi_R &=& \mu c^2\chi_L, \\ (\hat{\omega} + \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}c)\,\chi_L &=& \mu c^2\phi_R \end{array}\right\} \qquad \text{or} \qquad i\gamma^\lambda\partial_\lambda\Psi = \mu c\Psi$$

Dirac current $\mathsf{J}^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi \quad = (\psi^\dagger\psi,\ c\psi^\dagger\boldsymbol{\alpha}\psi)$

and 4-spin $\quad \mathsf{W}^\mu = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi$