Lecture 8. Relativity and electromagnetism

1. Force per unit charge $\longrightarrow \mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$

Change frame: $\mathbf{f} \to \mathbf{f}'$, hence:

Transformation of electromagnetic field

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma\left(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}\right), \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2\right), \end{aligned}$$

where S$'$ has velocity $\mathbf{v}$ in S.

e.g. capacitor, particle beam

2. Correct understanding of $\mathsf{J} = (\rho c,\ \mathbf{j})$ and charge conservation.

3. Fields due to a moving point charge:

$$\begin{aligned} \mathbf{E}' &= \frac{\gamma Q \mathbf{r}'}{4\pi\epsilon_0(\gamma^2(x')^2 + (y')^2 + (z')^2)^{3/2}}, \\ \mathbf{B}' &= -\frac{\mathbf{v} \wedge \mathbf{E}'}{c^2}. \end{aligned}$$

Lecture 9. 4-vector potential; Maxwell's equations; introducing tensors

1. Lorentz covariance of Maxwell's equations
2. Scalar and vector potential

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \wedge \mathbf{A}, \\ \mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \end{aligned}$$

$\longrightarrow$ automatically satisfy M2, M3.

3. Gauge transformation: $\left\{ \begin{array}{lll} \mathbf{A} & \to & \mathbf{A} + \boldsymbol{\nabla}\chi, \\ \phi & \to & \phi - \frac{\partial \chi}{\partial t} \end{array} \right.$
4. 4-vector potential

$$\mathsf{A} \equiv \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix}, \qquad \text{Gauge transformation:} \;\; \mathsf{A} \to \mathsf{A} + \Box\chi$$

5. Lorenz gauge, Maxwell's equations in a manifestly covariant form:

Maxwell's equations (!)

$$\Box^2 \mathsf{A} = \frac{-1}{c^2 \epsilon_0} \mathsf{J}, \qquad \text{with} \;\; \Box \cdot \mathsf{A} = 0.$$

6. General idea of 3-dimensional tensors such as conductivity, susceptibility, ...
7. Outer product $\mathbb{F} = \mathsf{A}\mathsf{B}^T \Rightarrow \quad \mathbb{F}' = \Lambda \mathbb{F} \Lambda^T.$ Also $\mathbb{F} \cdot \mathsf{B} \equiv \mathbb{F} g \mathsf{B}$

Maxwell equations and Lorentz force equation:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{M1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{M2}$$

$$\nabla \wedge \mathbf{E} = -\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} \tag{M3}$$

$$c^2 \nabla \wedge \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t}, \tag{M4}$$

$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

Change frame:

$$\begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix}, \qquad \begin{array}{rcl} \mathbf{E}_\parallel &=& \mathbf{E}'_\parallel \\ \mathbf{E}_\perp &=& \gamma\left(\mathbf{E}'_\perp - \mathbf{v} \wedge \mathbf{B}'\right), \\ \mathbf{B}_\parallel &=& \mathbf{B}'_\parallel \\ \mathbf{B}_\perp &=& \gamma\left(\mathbf{B}'_\perp + \mathbf{v} \wedge \mathbf{E}'/c^2\right). \end{array}$$

$$\begin{aligned} \frac{\partial(\cdots)}{\partial x} &= \frac{\partial(\cdots)}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial(\cdots)}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial(\cdots)}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial(\cdots)}{\partial z'}\frac{\partial z'}{\partial x} \\ \frac{\partial(\cdots)}{\partial y} &= \text{etc.} \end{aligned}$$

Lecture 10. Tensor analysis and index notation

1. Basic idea: ϕ, A^a, $\mathbb{F}^{ab}$, g_{ab}, $\Lambda^{a'}{}_b$

2. Summation convention: $A^{ab}X_b$ means $\sum_{b=0}^{3} A^{ab}X_b$ … 'dummy' index

3. Contravariant/covariant. $A^T g B = A'^T g' B' \quad \Rightarrow \quad g' = (\Lambda^{-1})^T g \Lambda^{-1}$
   Contravariant: $\mathsf{X} \to \Lambda \mathsf{X}$
   Covariant: $(g\mathsf{X}) \to (\Lambda^{-1})^T (gX)$

4. Index lowering: $\mathsf{F}_a \equiv g_{a\mu}\mathsf{F}^\mu$ so $\mathsf{U}\cdot\mathsf{F} = \mathsf{U}^\lambda \mathsf{F}_\lambda$

5. Legal tensor operations: sum, outer product, contract.

6. Caution when comparing with matrix notation

$$\mathbb{A}^{a\lambda}\mathsf{B}_\lambda \;\leftrightarrow\; \mathbb{A}\cdot\mathsf{B}$$
$$\text{but} \quad \mathbb{A}^{\lambda a}\mathsf{B}_\lambda = \mathsf{B}_\lambda \mathbb{A}^{\lambda a} \;\leftrightarrow\; \mathsf{B}\cdot\mathbb{A}$$

7. Invariants: contract down to a scalar. e.g. $\mathsf{A}^\lambda \mathsf{B}_\lambda$, T^λ_λ, $T^{\mu\nu}T_{\mu\nu}$

8. Differentiation. $\partial_a \equiv \frac{\partial}{\partial x^a} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Thus $\partial_a = \Box_a = g_{a\lambda}\Box^\lambda$ and $\partial^a = \Box^a$.

   $\Box \leftrightarrow \partial^a$. e.g. continuity equation $\partial^\lambda \mathsf{J}_\lambda = 0$

   Product rule: $\partial^a(U^b V^c) = (\partial^a U^b)V^c + U^b(\partial^a V^c)$

Lecture 11. Electromagnetic field theory via field tensor $\mathbb{F}$

1. Lorentz covariance, simplicity $\Big\}$ $\Rightarrow$ $\begin{cases} \text{4-force} = \text{charge} \times \text{field} \times \text{4-velocity} \\ \mathsf{F} = q\mathbb{F}\cdot\mathsf{U} \end{cases}$ **Field tensor**
2. Pure force ($\mathsf{F}\cdot\mathsf{U} = 0$) $\Leftrightarrow$ $\mathbb{F} = -\mathbb{F}^T$

$$\Rightarrow \mathbb{F} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

3. propose field equation $\qquad \Box\cdot\mathbb{F} = -\mu_0\rho_0\mathsf{U}$, i.e. $\partial_\lambda \mathbb{F}^{\lambda b} = -\mu_0\rho_0\mathsf{U}^b$
4. need a further equation; try $\qquad \mathbb{F} = \Box\wedge\mathsf{A}$, i.e. $\mathbb{F}^{ab} = \partial^a\mathsf{A}^b - \partial^b\mathsf{A}^a$

$\Rightarrow \quad \partial^c\mathbb{F}^{ab} + \partial^a\mathbb{F}^{bc} + \partial^b\mathbb{F}^{ca} = 0.$

$\Rightarrow$ The physical world ?

Implications

5. Antisymmetric $\mathbb{F}$ $\Rightarrow$ charge conservation: $\partial_\mu\partial_\nu\mathbb{F}^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\lambda \mathsf{J}^\lambda = 0$

6. Invariants $\quad D \equiv \frac{1}{2}\mathbb{F}^{\mu\nu}\mathbb{F}_{\mu\nu} = B^2 - E^2/c, \qquad \alpha \equiv \frac{1}{4}\tilde{\mathbb{F}}^{\mu\nu}\mathbb{F}_{\mu\nu} = \mathbf{B}\cdot\mathbf{E}/c.$

e.g. orthogonal in one frame $\Rightarrow$ orthogonal in all ($\alpha = 0$)
purely magnetic in one frame $\Rightarrow$ not purely electric in another ($D > 0$).

7. Finding the frame (if there is one) in which B or E vanishes.

Lecture 12. Introducing angular momentum, and some general tensor manipulations

1. Vector product, e.g. $\mathbb{L} = \mathsf{X}\mathsf{P}^T - \mathsf{P}\mathsf{X}^T$ or $\mathbb{L}^{ab} = \mathsf{X}^a\mathsf{P}^b - \mathsf{X}^b\mathsf{P}^a$.

2. Conservation of angular momentum and the motion of the centroid.

$$\mathbf{x}_{\rm c} \equiv \frac{\sum_i \mathbf{x}_i E_i}{E_{\rm tot}}$$

3. Transformation of an antisymmetric tensor:

$$\mathbb{F} = \begin{pmatrix} 0 & a_x & a_y & a_z \\ -a_x & 0 & b_z & -b_y \\ -a_y & -b_z & 0 & b_x \\ -a_z & b_y & -b_x & 0 \end{pmatrix}, \qquad \mathbb{F}' = \Lambda \mathbb{F} \Lambda^T \Rightarrow \begin{array}{rcl} \mathbf{a}'_{\parallel} &=& \mathbf{a}_{\parallel}, \\ \mathbf{a}'_{\perp} &=& \gamma(\mathbf{a}_{\perp} + \mathbf{v} \wedge \mathbf{b}/c), \\ \mathbf{b}'_{\parallel} &=& \mathbf{b}_{\parallel}, \\ \mathbf{b}'_{\perp} &=& \gamma(\mathbf{b}_{\perp} - \mathbf{v} \wedge \mathbf{a}/c). \end{array}$$

4. Dual: $\tilde{\mathbb{F}}_{ab} = \dfrac{1}{2}\epsilon_{ab\mu\nu}\mathbb{F}^{\mu\nu}$; hence $\mathbf{a} \to -\mathbf{b}$; $\mathbf{b} \to \mathbf{a}$.

5. Differentiation examples

Tips

1. Name your indices sensibly; make repeated indices easy to spot.
2. Look for scalars. e.g. $\mathbb{F}_{\lambda\mu}\mathbb{A}^a_b\mathbb{F}^{\lambda\mu}$ is $s\mathbb{A}^a_b$ where $s = \mathbb{F}_{\lambda\mu}\mathbb{F}^{\lambda\mu}$.
3. You can always change the names of dummy (summed over) indices; if there are two or more, you can swap names.
4. The 'see-saw rule'

$$A_\lambda B^\lambda = A^\lambda B_\lambda \quad \text{(works for any rank)}$$

5. In the absence of differential operators, everything commutes.

Lecture 13. Wave equation and general solution of Maxwell's equations

1. Poisson equation and its solution (reminder) ('Green's method'):

$$\nabla^2 \phi = \frac{-\rho}{\epsilon_0}, \qquad \phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}_{\rm s})}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_{\rm s}|} {\rm d}V_{\rm s}.$$

2. How to calculate $\nabla^2(1/r)$

3. Wave equation and its ('retarded') solution

$$\frac{-1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = \frac{-\rho(\mathbf{r}, t)}{\epsilon_0}, \qquad \phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}_{\rm s},\ t - |\mathbf{r} - \mathbf{r}_{\rm s}|/c)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_{\rm s}|} {\rm d}V_{\rm s}.$$

N.B. 'source event', 'field event'.

Hence

$$\mathsf{A} = \frac{1}{4\pi\epsilon_0 c} \int \frac{\mathsf{J}(\mathbf{r}_{\rm s}, t - r_{\rm sf}/c)}{r_{\rm sf}} {\rm d}V_{\rm s}.$$

4. Potentials of an arbitrarily moving charged particle

$$\mathsf{A} = \frac{q}{4\pi\epsilon_0} \frac{\mathsf{U}/c}{(-\mathsf{R} \cdot \mathsf{U})}. \qquad \text{('Liénard-Wiechart' potentials)}$$

Lecture 14. Electromagnetic radiation

Fields of an accelerated charge:

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0\kappa^3}\left(\frac{\mathbf{n}-\mathbf{v}/c}{\gamma^2 r^2} + \frac{\mathbf{n}\wedge[(\mathbf{n}-\mathbf{v}/c)\wedge\mathbf{a}]}{c^2 r}\right) \\
\mathbf{B} &= \mathbf{n}\wedge\mathbf{E}/c \\
\text{where}\quad \mathbf{n} &= \mathbf{r}/r, \quad \kappa = 1 - v_r/c = 1 - \mathbf{n}\cdot\mathbf{v}/c
\end{aligned}$$

In terms of the displacement $\mathbf{r}_0 = \mathbf{r} - \mathbf{v}r/c$ from the projected position,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r_0^3(\gamma^2\cos^2\theta + \sin^2\theta)^{3/2}}\left(\gamma\mathbf{r}_0 + \frac{\gamma^3}{c^2}\mathbf{r}\wedge[\mathbf{r}_0\wedge\mathbf{a}]\right)$$

1. General features: bound field, radiative field
2. Case of linear motion coming to rest
3. Radiation from slowly moving dipole oscillator

$$\begin{aligned}
\mathbf{A} &= \frac{q}{4\pi\epsilon_0 c^2}\frac{\mathbf{v}}{(r_{\rm sf} - \mathbf{r}_{\rm sf}\cdot\mathbf{v}/c)}, \\
\Rightarrow \text{far field:}\quad \mathbf{B} &= \frac{\omega^2 q x_0}{4\pi\epsilon_0 c^3}\frac{\sin\theta}{r}\sin(kr-\omega t)\,\hat{\boldsymbol{\phi}}, \\
E &= cB
\end{aligned}$$

4. The half-wave dipole antenna. Emitted power $= I_{\rm rms}^2 \times (73\text{ ohm})$.

Lecture 15. Radiated power

1. Radiated power (Larmor)

$$\mathrm{d}\mathcal{P} = N r^2 \mathrm{d}\Omega = \frac{q^2}{4\pi\epsilon_0}\frac{a^2\sin^2\theta}{4\pi c^3}\mathrm{d}\Omega \qquad \Rightarrow \quad \mathcal{P}_L = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{a_0^2}{c^3}.$$

2. Linear particle accelerator: $a_0 = f_0/m = f/m, \longrightarrow$ loss $\simeq 0$

3. Dipole oscillator

$$\mathcal{P}_L = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c^3}(\gamma^3\omega^2 x_0\cos\omega t)^2, \qquad \Rightarrow \bar{\mathcal{P}}_L \simeq \frac{1}{3}\frac{q^2}{4\pi\epsilon_0 c^3}\omega^4 x_0^2$$

4. Headlight effect:
received energy per unit time at the detector, per unit solid angle

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\Omega} = \frac{q^2 a^2}{4\pi\epsilon_0 c^3}\frac{\sin^2\theta}{(1-(v/c)\cos\theta)^6} \qquad \text{for linear motion}$$

5. Circular motion: $a_0 = f_0/m = \gamma f/m = \gamma^2 a, \quad \Delta E = \frac{q^2}{3\epsilon_0 r}\gamma^4 (v/c)^3.$

(6. Synchrotron radiation: 'lighthouse' pulses with frequency spread $\Delta\omega \simeq \gamma^3\omega_0$.)

(7. Self-force and radiation reaction)

Lecture 16. Spin; parity inversion symmetry

1. $L^{ab}(0) = L^{ab}(R) + \left(R^a P^b_{\rm tot} - R^b P^a_{\rm tot}\right)$

Total angular momentum $J^{ab} = S^{ab} + L^{ab}$

2. **Pauli-Lubanski vector** $\mathsf{W}_a \equiv \tilde{J}_{a\lambda}\mathsf{P}^\lambda = \frac{1}{2}\epsilon_{a\lambda\mu\nu}J^{\mu\nu}\mathsf{P}^\lambda \quad \Rightarrow \mathsf{W}^a = (\mathbf{s}\cdot\mathbf{p},\ \ (E/c)\mathbf{s})$

In the rest frame: $(0,\ mc\mathbf{s}_0)$ so $\mathsf{W}\cdot\mathsf{U} = 0$ and

$$\mathbf{s}_\parallel = \mathbf{s}_{0\parallel}, \quad \mathbf{s}_\perp = \frac{\mathbf{s}_{0\perp}}{\gamma}$$

Hence for a photon, W is null and points along P.

---

3. Mirror reflection; polar and axial vectors

4. Parity inversion: $x \to -x,\ y \to -y,\ z \to -z$

$$\mathbf{x} \to -\mathbf{x},\ \ \mathbf{p} \to -\mathbf{p} \qquad \Rightarrow \mathbf{x}\wedge\mathbf{p} \to \mathbf{x}\wedge\mathbf{p}$$

5. Classical physics covariant under parity inversion

6. Parity non-conserving process

Lecture 17. Lagrangian mechanics (symmetry again!)

1. Reminder of Least Action and Euler-Lagrange equations

$$\begin{aligned} \text{Lagrangian} \quad \mathcal{L} &= \mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t) \equiv T - V \\ \text{Action} \quad S[q(t)] &= \int_{q_1,t_1}^{q_2,t_2} \mathcal{L}(q, \dot{q}, t) \mathrm{d}t \end{aligned}$$

stationary for path satisfying **Euler-Lagrange equations**: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \dfrac{\partial \mathcal{L}}{\partial q_i}$.

$$\frac{\partial \mathcal{L}}{\partial q_i} = \text{“generalized force”}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{“canonical momentum”}$$

Hamiltonian: $\mathcal{H}(q, \tilde{p}, t) \equiv \sum_i^n \tilde{p}_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t)$

2. Special Relativity (version 1).

Freely moving particle: $\mathcal{L} = -mc^2/\gamma$, $\quad \tilde{\mathbf{p}} = \gamma m \mathbf{v}$

Particle in an e-m field: $\mathcal{L} = -mc^2/\gamma + q(-\phi + \mathbf{v} \cdot \mathbf{A})$, $\quad \tilde{\mathbf{p}} = \gamma m \mathbf{v} + q\mathbf{A}$

Taking a derivative along the worldline: $\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \dfrac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}$

Hamiltonian $\mathcal{H} = \gamma mc^2 + q\phi = \left((\tilde{\mathbf{p}} - q\mathbf{A})^2 c^2 + m^2 c^4\right)^{1/2} + q\phi$.

3. Special Relativity (version 2), using τ instead of t in the action:

$$\mathcal{L}(\mathsf{X}, \mathsf{U}) = -mc(-\mathsf{U} \cdot \mathsf{U})^{1/2} + q\mathsf{U} \cdot \mathsf{A}, \qquad S[\mathsf{X}(\tau)] = \int_{(\mathsf{X}_1)}^{(\mathsf{X}_2)} \mathcal{L}(\mathsf{X}, \mathsf{U}, \tau) d\tau,$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial \mathcal{L}}{\partial \mathsf{U}^a} = \frac{\partial \mathcal{L}}{\partial \mathsf{X}^a}, \qquad \tilde{\mathsf{P}}_a = m\mathsf{U}_a + q\mathsf{A}_a, \qquad m\frac{\mathrm{d}\mathsf{U}}{\mathrm{d}\tau} = q(\Box \wedge \mathsf{A}) \cdot \mathsf{U}$$

Use of a parameter to minimize the action.

Integrating with respect to proper time means the value of τ at the end event is different for each path.

Problem!: calculus of variations needs fixed start and end values.

Introduce a parameter λ:

$$\int \mathcal{L}(\mathsf{X}, \dot{\mathsf{X}}, \tau)\mathrm{d}\tau = \int_{\lambda_1}^{\lambda_2} \mathcal{L}\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\mathrm{d}\lambda.$$

Now the Lagrangian is

$$\tilde{\mathcal{L}} = \mathcal{L}\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \mathcal{L}\frac{1}{c}\left(-g_{\mu\nu}\frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}\mathsf{X}^\nu}{\mathrm{d}\lambda}\right)^{1/2}.$$

(using $\mathrm{d}\tau^2 = \mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)/c^2$)

giving Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\mathsf{X}}^a} = \frac{\partial\tilde{\mathcal{L}}}{\partial\mathsf{X}^a}$$

in which $\dot{\mathsf{X}} = \mathrm{d}\mathsf{X}/\mathrm{d}\lambda$.

Now pick $\lambda = \tau$ *along the solution worldline.*

For that worldline, and for that worldline only (but it is the only one we are interested in from now on), we must then find $\mathrm{d}\tau/\mathrm{d}\lambda = 1$ and $\tilde{\mathcal{L}} = \mathcal{L}$ and $\dot{\mathsf{X}}^a = \mathsf{U}^a$.

Now

$$\frac{d\mathsf{A}_a}{d\tau} = U^\lambda \partial_\lambda \mathsf{A}_a$$

so

$$\begin{aligned} m\frac{d\mathsf{U}_a}{d\tau} &= q\left((\partial_a \mathsf{A}_\lambda) - (\partial_\lambda \mathsf{A}_a)\right)\mathsf{U}^\lambda \\ \text{or} \qquad \frac{d\mathsf{P}}{d\tau} &= q(\Box \wedge \mathsf{A}) \cdot \mathsf{U} \end{aligned}$$

Lecture 18. Energy-momentum flow; stress-energy tensor

1. Conservation of energy: $$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{N} + \mathbf{E} \cdot \mathbf{j}.$$

$$\left.\begin{array}{lrcl} \text{energy density} & u & = & \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 c^2 B^2 \\ \text{Poynting vector} & \mathbf{N} & = & \epsilon_0 c^2 \mathbf{E} \wedge \mathbf{B}. \end{array}\right\} \quad \text{follow from Maxwell's equations}$$

2. Transfer of 4-momentum per unit volume from fields to matter:

Let $\mathsf{W} = (\mathbf{E} \cdot \mathbf{j}/c,\ \rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B})$ then

$$\mathsf{W}^b = -\,\partial_\lambda \mathbb{T}^{\lambda b} \qquad [\,\mathsf{W} = -\Box \cdot \mathbb{T}\,,$$

and one finds (by using Maxwell's equations):

$$\mathbb{T}^{ab} = \left(\begin{array}{c|c} u & \mathbf{N}/c \\ \hline \mathbf{N}/c & \sigma_{ij} \end{array}\right) \quad \text{where} \quad \sigma_{ij} = u\delta_{ij} - \epsilon_0(E_i E_j + c^2 B_i B_j)$$

This can also be written

$$\mathbb{T}^{ab} = \epsilon_0 c^2 \left(-\mathbb{F}^{a\mu}\mathbb{F}_\mu^{\ b} - \frac{1}{2} g^{ab} D \right), \qquad \text{where} \quad D = \tfrac{1}{2}\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}.$$

$$[\ \text{i.e.}\ \ \mathbb{T} = \epsilon_0 c^2 \left(-\mathbb{F} \cdot \mathbb{F} - \frac{1}{2} g D \right).\ ]$$

Conservation of 4-momentum of matter and field together

$$(\ \mathbf{E} \cdot \mathbf{j}/c,\ \ \rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B}\ ) = -\left(\frac{1}{c}\frac{\partial}{\partial t},\ \nabla \cdot \right) \left(\begin{array}{c|c} u & \mathbf{N}/c \\ \hline \mathbf{N}/c & \boldsymbol{\sigma} \end{array}\right)$$

Poynting's argument (John Henry Poynting (1852-1914)):

We want to find expressions for u and $\mathbf{N}$, such that $\quad -\dfrac{\partial u}{\partial t} = \boldsymbol{\nabla}\cdot\mathbf{N} + \mathbf{E}\cdot\mathbf{j}$.

Using M4 to express $\mathbf{j}$ in terms of the fields:

$$\mathbf{E}\cdot\mathbf{j} = \epsilon_0 c^2 \mathbf{E}\cdot(\boldsymbol{\nabla}\wedge\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\,.$$

but for any pair of vectors $\mathbf{E},\ \mathbf{B}$,

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\wedge\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\wedge\mathbf{B}).$$

so

$$\mathbf{E}\cdot\mathbf{j} = -\epsilon_0 c^2\boldsymbol{\nabla}\cdot(\mathbf{E}\wedge\mathbf{B}) + \epsilon_0 c^2\mathbf{B}\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) - \frac{\partial}{\partial t}\left(\tfrac{1}{2}\epsilon_0\mathbf{E}\cdot\mathbf{E}\right).$$

Now use M3:

$$\mathbf{E}\cdot\mathbf{j} = -\epsilon_0 c^2\boldsymbol{\nabla}\cdot(\mathbf{E}\wedge\mathbf{B}) - \frac{\partial}{\partial t}\left(\tfrac{1}{2}\epsilon_0 c^2\mathbf{B}\cdot\mathbf{B} + \tfrac{1}{2}\epsilon_0\mathbf{E}\cdot\mathbf{E}\right)$$

Which shows that a possible assignment is

$$\begin{aligned} u &= \tfrac{1}{2}\epsilon_0 c^2 B^2 + \tfrac{1}{2}\epsilon_0 E^2\,, \\ \mathbf{N} &= \epsilon_0 c^2\mathbf{E}\wedge\mathbf{B}. \end{aligned}$$