Lecture 1. Basic concepts

Reference body. 'Observe' = deduce. Postulates. Standard configuration. Lorentz transformation:

$$\left.\begin{array}{rcc} t' &=& \gamma(t - vx/c^2) \\ x' &=& \gamma(-vt + x) \\ y' &=& y \\ z' &=& z \end{array}\right\}, \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Minkowski metric and definition of $\mathcal{L}$:

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{array}{rrcll} & \mathcal{L}^T g \mathcal{L} &=& g & \\ \Rightarrow & \mathsf{A}'^T g \mathsf{B}' &=& \mathsf{A}^T g \mathsf{B} & \equiv \mathsf{A}\cdot\mathsf{B} \quad \textbf{Invariant} \end{array}$$

Proper time

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$$

Familiarity with γ :

$$\beta = \sqrt{1 - 1/\gamma^2}, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}v} = \gamma^3 v/c^2, \qquad \frac{\mathrm{d}}{\mathrm{d}v}(\gamma v) = \gamma^3$$

symbol	definition	components	name(s)	invariant
X	X	$(ct, \mathbf{r})$	4-displacement, interval	$-c^2\tau^2$
U	$d\mathsf{X}/d\tau$	$(\gamma c, \gamma\mathbf{u})$	4-velocity	$-c^2$
P	$m_0\mathsf{U}$	$(E/c, \mathbf{p})$	energy-momentum, 4-momentum	$-m_0^2c^2$
A	$d\mathsf{U}/d\tau$	$\gamma(\dot\gamma c, \dot\gamma\mathbf{u} + \gamma\mathbf{a})$	4-acceleration	a_0^2

4-acceleration is orthogonal to 4-velocity: $\mathsf{U}\cdot\mathsf{A} = 0$.

Postulate 1, "Principle of Relativity":

> The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line.

Postulate 2, "Light speed postulate":

> There is a finite maximum speed for signals.

Alternative statements:
Postulate 1:

> The laws of physics take the same mathematical form in all inertial frames of reference.

Postulate 2:

> There is an inertial reference frame in which the speed of light in vacuum is independent of the motion of the source.

Lecture 2. Spacetime diagrams; velocity; method of invariants; Doppler effect

Spacetime diagrams.

Timelike	$\mathsf{X}\cdot\mathsf{X}<0$	e.g. 4-velocity
spacelike	$\mathsf{A}\cdot\mathsf{A}>0$	e.g. 4-acceleration
null	$\mathsf{K}\cdot\mathsf{K}=0$	e.g. 4-wave-vector of light wave

Transformation of velocity:

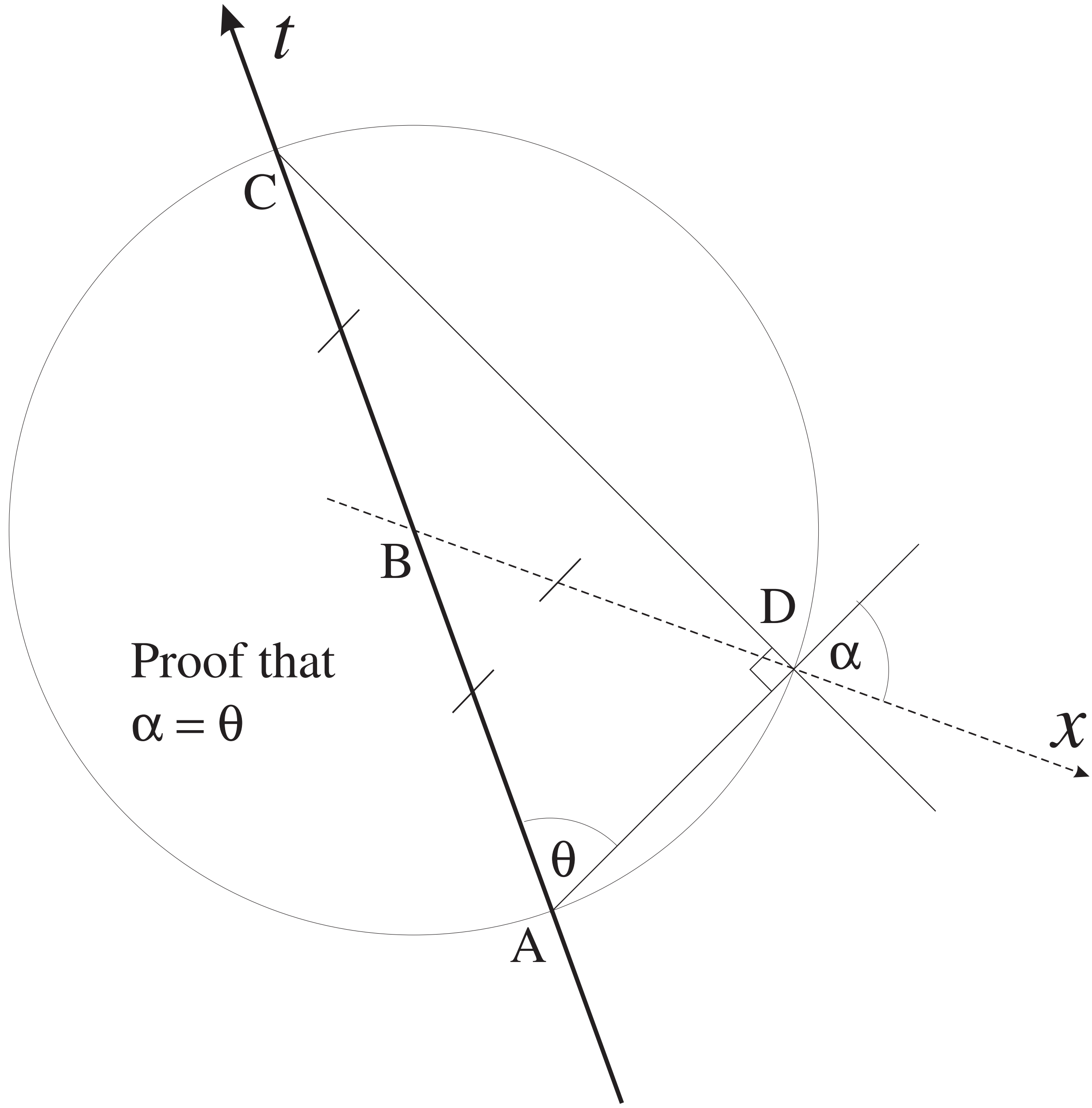
$$\mathbf{u}'_{\parallel} = \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}, \qquad \mathbf{u}'_{\perp} = \frac{\mathbf{u}_{\perp}}{\gamma_v\,(1 - \mathbf{u}\cdot\mathbf{v}/c^2)}.$$

$\longrightarrow$ particle jets

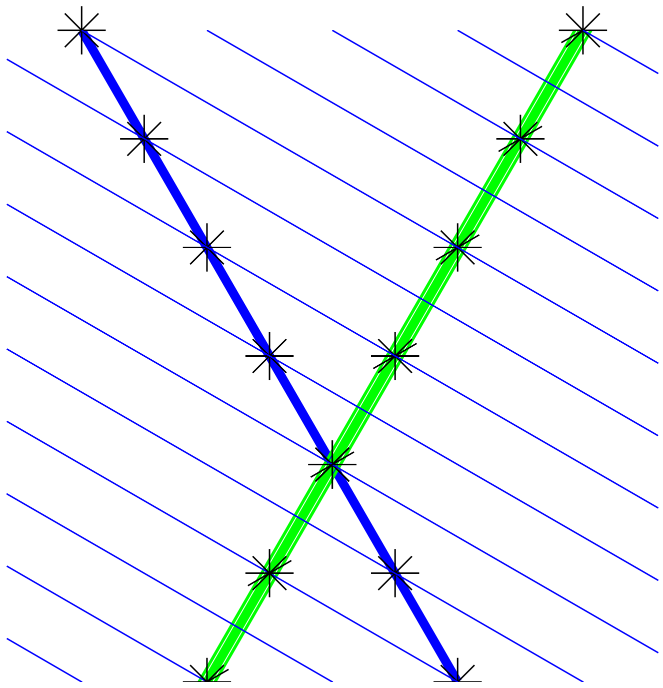
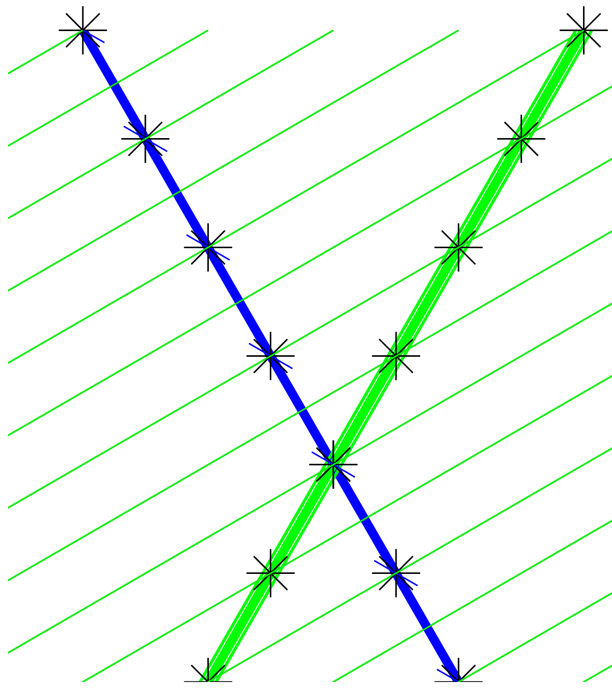
Method of invariants = "Try using an invariant if you can, and pick an easy reference frame."

4-wave vector $\mathsf{K} \equiv (\omega/c,\ \mathbf{k})$, $\qquad e^{i(\mathbf{k}\cdot r - \omega t)} = e^{i\mathsf{K}\cdot\mathsf{X}}$

Doppler effect: $\qquad \mathsf{K}\cdot\mathsf{U} \Rightarrow \qquad \dfrac{\omega}{\omega_0} = \dfrac{1}{\gamma(1-(v/v_p)\cos\theta)}.$



t
C
B
D
α
x
θ
A
Proof that
α = θ

Lecture 3. Headlight effect; rapidity; Introducing force

1. Headlight effect or 'aberration:'

Doppler effect: $\quad \mathsf{K}\cdot\mathsf{U} \Rightarrow \quad \dfrac{\omega}{\omega_0} = \dfrac{1}{\gamma(1-(v/v_p)\cos\theta)}.$

Headlight effect: $\mathsf{K} = \mathcal{L}^{-1}\mathsf{K}_0 \Rightarrow \cos\theta = \dfrac{\cos\theta_0 + v/c}{1+(v/c)\cos\theta_0} \quad \Rightarrow \dfrac{\mathrm{d}N}{\mathrm{d}\Omega} = \dfrac{N}{4\pi}\left(\dfrac{\omega}{\omega_0}\right)^2.$

$\longrightarrow$ 'brightness' (power per unit solid angle) transforms as $(\omega/\omega_0)^4$

2. Rapidity: $\tanh(\rho) \equiv \dfrac{v}{c}, \quad \Rightarrow \quad \cosh(\rho) = \gamma, \quad \sinh(\rho) = \beta\gamma, \quad \exp(\rho) = \Big(\dfrac{1}{1}$

$$\mathcal{L} = \begin{pmatrix} \cosh\rho & -\sinh\rho & 0 & 0 \\ -\sinh\rho & \cosh\rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \textit{Colinear} \text{ rapidities add.}$$

3. Force

$$\mathsf{F} \equiv \frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\tau}, \qquad \mathsf{U}\cdot\mathsf{F} = \gamma^2\left(-\frac{\mathrm{d}E}{\mathrm{d}t} + \mathbf{u}\cdot\mathbf{f}\right) = -c^2\frac{\mathrm{d}m_0}{\mathrm{d}\tau}.$$

'Pure' force:

$$\mathsf{U}\cdot\mathsf{F} = 0 \quad \Rightarrow \quad m_0 = \text{const}, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{f}\cdot\mathbf{u}$$

Transformation of force: use $\mathcal{L}\mathsf{F}$ and γ, or $(\mathrm{d}/\mathrm{d}t')(\mathcal{L}\mathsf{P})$:

$$\mathbf{f}'_\parallel = \frac{\mathbf{f}_\parallel - (\mathbf{v}/c^2)dE/dt}{1-\mathbf{u}\cdot\mathbf{v}/c^2}, \qquad \mathbf{f}'_\perp = \frac{\mathbf{f}_\perp}{\gamma_v(1-\mathbf{u}\cdot\mathbf{v}/c^2)} \qquad \Rightarrow \begin{cases} \mathbf{f}'_\parallel = \mathbf{f}_\parallel \\ \mathbf{f}'_\perp = \mathbf{f}_\perp/\gamma \end{cases} \text{ for } u = 0$$

Lecture 4. Simple dynamical problems; acceleration and rigidity.

1. Equation of motion in any given reference frame:

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \mathbf{v}) = \gamma m \mathbf{a} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v}$$

$$\Rightarrow \left\{ \begin{array}{ll} \text{(i)} & \text{acceleration is not necessarily parallel to force!} \\ \text{(ii)} & \mathbf{a} = \mathbf{f}/m \text{ at } \mathbf{v} = 0 \\ \text{(iii)} & \text{for 'pure' force } (m\text{=constant}): \qquad f_\parallel = \gamma^3 m a_\parallel, \qquad f_\perp = \gamma m a_\perp. \end{array} \right.$$

2. Uniform B field: just like classical result, but with $m \to \gamma m_0$. so $\omega = qB/\gamma m$ and $p = qBr$.

3. Motion parallel to E field: hyperbolic motion, $x^2 - c^2t^2 = (c^2/a_0)^2$ and $\ddot{\mathsf{U}} \propto \mathsf{U}$. Constant proper acceleration.

4. Rigidity; the Great Train Disaster

Lecture 5. The conservation of energy and momentum

1. Lewis and Tolman argument: $\mathbf{p} = m\alpha(v)\mathbf{v} \quad \Rightarrow \mathbf{p} = \gamma m \mathbf{v}$.

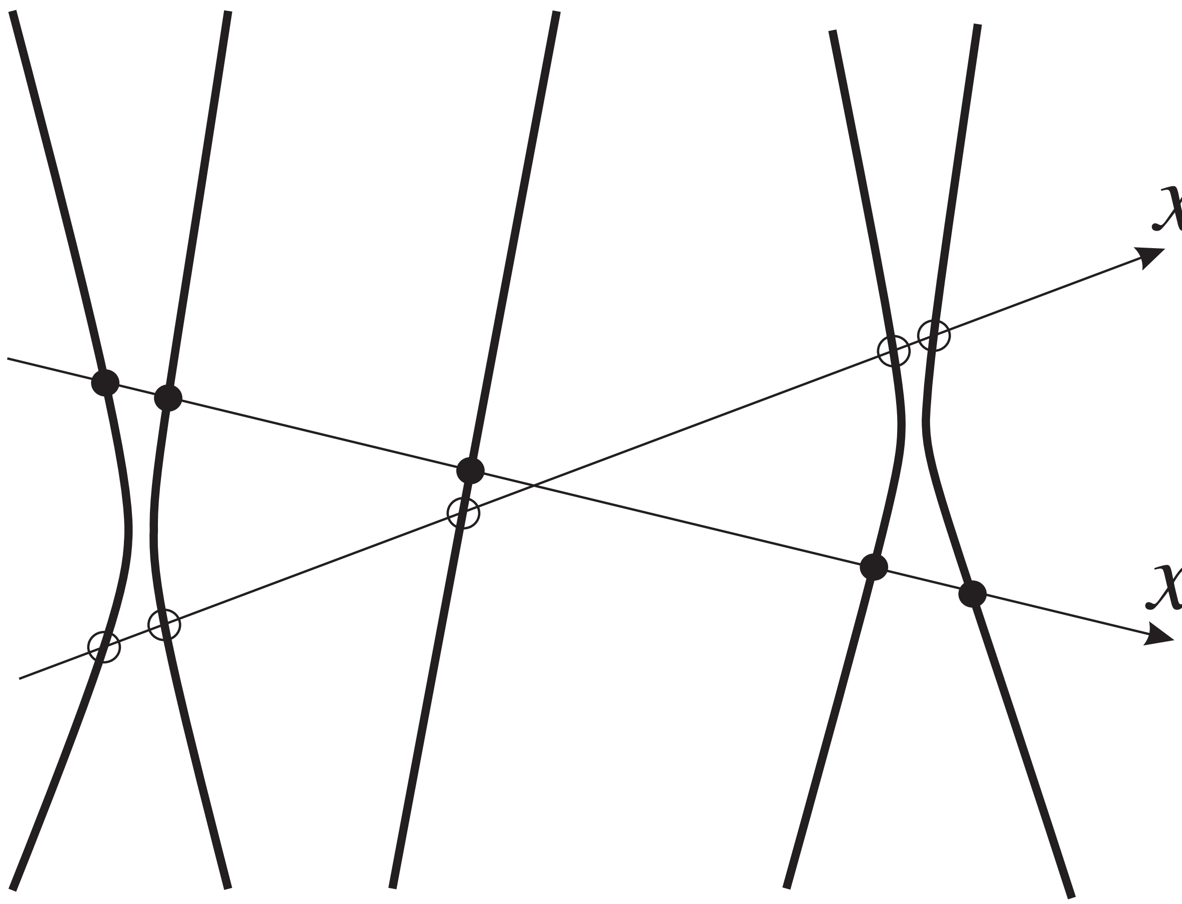
2. Impact of simultaneity on "$\mathsf{P}_{\rm tot} = \mathsf{P}_1 + \mathsf{P}_2 + \mathsf{P}_3 + \ldots \mathsf{P}_n$".

3. "Zero component lemma": if one component of a 4-vector is zero in all reference frames, then the whole 4-vector is zero.

Hence momentum conservation $\Leftrightarrow$ energy conservation.

$$\left.\begin{array}{l}\text{Main postulates,}\\ \text{momentum conservation}\end{array}\right\} \Rightarrow E_0 = mc^2, \;\; \text{equivalence of rest mass and rest energy.}$$

4. Energy flux = momentum density: $\mathbf{S} = \mathbf{g}c^2$


x'
x

Lecture 6. Collisions.

Method: *Isolate and square.*

1. Decay at rest. e.g. find the energy of one of the products:

$$E_1 = \frac{M^{*2} + m_1^2 - m_2^2}{2M^*}c^2, \qquad E_{\text{photon}} = \Delta E_{\text{rest}} - E_{\text{recoil}}.$$

2. In-flight decay. e.g. Find the rest mass of the original particle:

$$M^2 = m_1^2 + m_2^2 + \frac{2}{c^4}(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 c^2)$$

$\Rightarrow$ it suffices to measure $m_1, m_2, p_1, p_2, \theta$.

3. Particle formation. e.g. threshold energy (stationary target):

$$E_{\text{th}} = \frac{(\sum_i m_i)^2 - m^2 - M^2}{2M}c^2.$$

4. Elastic collision. e.g. find lab frame opening angle in terms of CM properties:

$$\tan(\theta_1 + \theta_2) = \left(\frac{\gamma}{\gamma^2 - 1}\right)\frac{2}{\sin\theta_0}$$

5. Compton effect (A. H. Compton, Physical Review **21**, 483 (1923)):

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta).$$

6. Composite body: $\mathbf{p} = \sum_i \mathbf{p}_i, \quad E = \sum_i E_i, \quad \mathsf{P}^2 = -m^2c^2, \quad \mathbf{v} = \mathbf{p}c^2/E.$

[7. other: 3-body decay, inverse Compton effect, etc.]

Lecture 7. 4-gradient; flow; (Thomas precession)

1. 4-gradient operator:

$$\Box \equiv \left(-\frac{1}{c}\frac{\partial}{\partial t}, \; \frac{\partial}{\partial x}, \; \frac{\partial}{\partial y}, \; \frac{\partial}{\partial z} \right)$$

If ϕ is a Lorentz-invariant scalar quantity, then

$$\Box' \phi = \mathcal{L} \, \Box \phi$$

i.e. $\Box\phi$ is a 4-vector.

2. 4-divergence

$$\Box \cdot \mathsf{F} = \frac{1}{c}\frac{\partial F^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{f}, \qquad\qquad \Box^2 = \Box \cdot \Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2.$$

3. Wave phase ϕ is Lorentz-invariant.

$$\begin{aligned} \Rightarrow \qquad \mathsf{K} &\equiv \Box\phi \quad \text{is a 4-vector} \\ \Box^2\phi &= 0 \quad \text{is the wave equation.} \end{aligned}$$

4. Flow and conservation:

4-current density	$\mathsf{J} \equiv \rho_0 \mathsf{U}$	$= (\rho c, \; \mathbf{j})$
continuity equation	$\Box \cdot \mathsf{J} = 0.$	

(5. Thomas precession:) net excess angle for a circle $\Delta\theta = 2\pi(\gamma - 1)$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = (\gamma - 1)\frac{a}{v}, \qquad \text{More generally:} \quad \boldsymbol{\omega}_T = \frac{\mathbf{a} \wedge \mathbf{v}}{c^2}\frac{\gamma^2}{1+\gamma}$$