A comment on notation. The Lorentz transformation is written either $\mathcal{L}$ or Λ. 3-vectors are indicated by bold font: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$; 4-vectors are indicated by capitol letters in this font: $\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots$; 2nd rank tensors by this font $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$. In the later sections where index notation is used, the font no longer matters but is mostly adhered to anyway. Indices run over 0,1,2,3 (t, x, y, z), with the exception of i, j which run over 1,2,3 (x, y, z). For convenience of spotting them, indices being summed over are given Greek letters such as λ or μ. The Minkowski metric is taken as $g = \text{diag}(-1, 1, 1, 1)$ (in rectangular coordinates). The scalar product is

$$\mathsf{A} \cdot \mathsf{B} \equiv \mathsf{A}^T g \mathsf{B} \equiv \mathsf{A}^\lambda \mathsf{B}_\lambda$$

In the lectures we occasionally use the notation $\Box$ for the 4-gradient operator, by analogy with the familiar $\boldsymbol{\nabla}$, but mostly we use ∂^a. In many books the symbol $\Box$ is used for the d'Alembertian operator, which in our notation is $\Box^2$. The summary is

thing	notation here	notation elsewhere
$\left(-\frac{1}{c}\left(\frac{\partial}{\partial t}\right), \boldsymbol{\nabla}\right)$	$\Box$ or ∂^a	∂^a
$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2$	$\Box^2$	$\Box$

In texts adopting the other possible metric $(+1, -1, -1, -1)$, ∂^a changes sign but ∂_a does not.

Relativity: problems

Andrew M. Steane

Some constants

c	299792458	m/s
electron	0.511	MeV
proton	938.3	MeV
π^-	139.6	MeV

1 Basic ideas, simple kinematics and dynamics

(**Lectures 1-5**)

Basic ideas

1. The Lorentz transformation Λ is defined such that $\Lambda^T g \Lambda = g$ where g is the Minkowski metric, taken as $g = \mathrm{diag}(-1, 1, 1, 1)$. Show that for any pair of 4-vectors A, B, the scalar product $\mathsf{A} \cdot \mathsf{B} \equiv \mathsf{A}^T g \mathsf{B}$ is Lorentz-invariant.

2. Using a spacetime diagram, or otherwise, prove that
(i) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval,
(ii) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval.

3. Using algebra, or otherwise, show that
(i) for any time-like vector there exists a frame in which its spatial part is zero,
(ii) any vector orthogonal to a time-like vector must be space-like,
(iii) with one exception, any vector orthogonal to a null vector is spacelike, and describe the exception.

4. Show that (i) the instantaneous 4-velocity of a particle is parallel to the worldline (i.e. demonstrate that you understand the meaning of this claim—if you do then it is obvious),
(ii) if the 4-displacement between any two events is orthogonal to an observer's worldline, then the events are simultaneous in the rest frame of that observer.

5. Define *proper time*. A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by (ct, x, y, z) in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$.

6. Two particles have velocities $\mathbf{u}$, $\mathbf{v}$ in some reference frame. The Lorentz factor for their relative velocity $\mathbf{w}$ is given by

$$\gamma(w) = \gamma(u)\gamma(v)(1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

Prove this twice, by using each of the following two methods.

(i) In the given frame, the worldline of the first particle is $\mathsf{X} = (ct, \ \mathbf{u}t)$. Transform to the rest frame of the other particle to obtain

$$t' = \gamma_v t(1 - \mathbf{u} \cdot \mathbf{v}/c^2)$$

Obtain dt'/dt and apply the result of the previous question.
(ii) Use the invariant $\mathsf{U} \cdot \mathsf{V}$, first showing that it is equal to $-c^2\gamma(w)$.

7. In a given inertial frame S, two particles are shot out from a point in orthogonal directions with equal speeds v. At what rate does the distance between the particles increase in S? What is the speed of each particle relative to the other?

Doppler effect and aberration

8. Derive a formula for the frequency ω of light waves from a moving source, in terms of the proper frequency ω_0 in the source frame and the angle in the observer's frame, θ, between the direction of observation and the velocity of the source.

   A galaxy with a negligible speed of recession from Earth has an active nucleus. It has emitted two jets of hot material with the same speed v in opposite directions, at an angle θ to the direction to Earth. A spectral line in singly-ionised Mg (proper wavelength $\lambda_0 = 448.1$ nm) is emitted from both jets. Show that the wavelengths $\lambda_\pm$ observed on Earth from the two jets are given by

   $$\lambda_\pm = \lambda_0 \gamma (1 \pm (v/c)\cos\theta)$$

   (you may assume the angle subtended at Earth by the jets is negligible). If $\lambda_+ = 420.2$ nm and $\lambda_- = 700.1$ nm, find v and θ.

   In some cases the receding source is difficult to observe. Suggest a reason for this.

9. The emission spectrum from a source in the sky is observed to have a periodic fluctuation, as shown in the data displayed in figure 1.

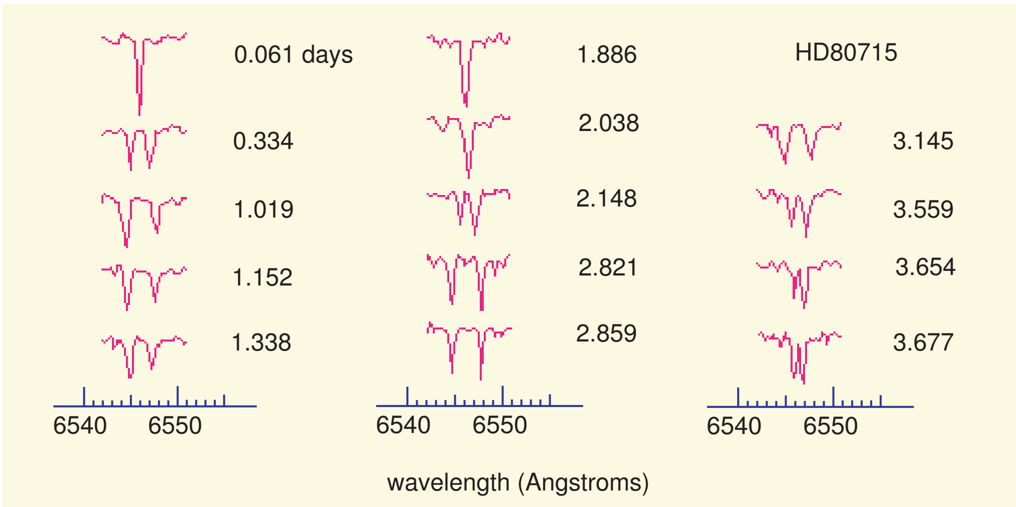


Figure 1: Spectra of light received from a astronomical object at specific times during an observation period of a few days.

It is proposed that the source is a binary star system. Explain how this could give rise to the data. Extract an estimate for the component of orbital velocity in the line of sight.

[Optional: assuming the stars have equal mass, estimate also the distance between them and their mass.]

10. Consider a wavetrain whose width and length are sufficiently large that it can be treated as monochromatic and with negligible diffraction. Define the width of a wavefront to be the distance between two points on a given wavefront at a given instant in time in some reference frame. Show that this width is the same in all frames. Hence deduce how the intensity of a plane wave transforms. [Hint: either set up equations for the movement of the ends of the wavefront, and use a Lorentz transformation, or, for a clever method, give a carefully argued treatment using 4-vectors and invariants.]

11. **Moving mirror**. A plane mirror moves uniformly with velocity $\mathbf{v}$ in the direction of its normal in a frame S. An incident light ray has angular frequency ω_i and is reflected with angular frequency ω_r.
(i)Show that

$$\omega_i \sin\theta_i = \omega_r \sin\theta_r$$

where θ_i, θ_r are the angles of incidence and reflection.

(ii) [Optional—harder]. Also show that establish

$$\frac{\tan(\theta_i/2)}{\tan(\theta_r/2)} = \frac{1+v/c}{1-v/c}.$$

[Hint: first establish by trigonometric manipulation that $\cos\theta = (1-t^2)/(1+t^2)$ where $t = \tan(\theta/2)$, then employ this in the Doppler formula relating $\cos\theta$ to $\cos\theta_0$ in order to obtain a relation between t and t_0. Then apply this relation to the two rays.]

Motion under a given force

12. Does Special Relativity place any bounds on the possible sizes of forces or accelerations?

13. **Twin paradox**. (i) Evaluate the acceleration due to gravity at the Earth's surface (9.8 m/s^2) in units of years and lightyears.
(ii) In the twin paradox, the travelling twin leaves Earth on board a spaceship undergoing motion at constant proper acceleration of 9.8 m/s^2. After 5 years of proper time for the spaceship, the direction of the rockets are reversed so that the spaceship accelerates towards Earth for 10 proper years. The rockets are then reversed again to allow the spaceship to slow and come to rest on Earth after a further 5 years of spaceship proper time. How much does the travelling twin age? How much does the stay-at-home twin age?

14. **Constant force**. Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows.
(i) Write down the solution for $\mathbf{p}$ as a function of time, taking as initial condition $\mathbf{p}(0) = \mathbf{p}_0$.
(ii) Show that the Lorentz factor as a function of time is given by $\gamma^2 = 1 + \alpha^2$ where $\boldsymbol{\alpha} = (\mathbf{p}_0 + \mathbf{f}t)/mc$.
(iii) You can now write down the solution for $\mathbf{v}$ as a function of time. Do so.
(iv) Now restrict attention to the case where $\mathbf{p}_0$ is perpendicular to $\mathbf{f}$. Taking the x-direction along $\mathbf{f}$ and the y-direction along $\mathbf{p}_0$, show that the trajectory is given by

$$\begin{aligned} x &= \frac{c}{f}\left(m^2c^2 + p_0^2 + f^2t^2\right)^{1/2} + \text{const} \\ y &= \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const} \end{aligned}$$

where you may quote that $\int (a^2 + t^2)^{-1/2}\mathrm{d}t = \log\left(t + \sqrt{a^2 + t^2}\right)$
(v) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation.
N.B. the calculation as a function of proper time is best done another way, see later problems.

15. For motion under a pure (rest mass preserving) inverse square law force $\mathbf{f} = -\alpha\mathbf{r}/r^3$, where α is a constant, derive the energy equation $\gamma mc^2 - \alpha/r = \text{constant}$.

Optional extra questions

16. Derive the equations describing the transformation of velocity:

$$\mathbf{u}'_{\parallel} = \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}, \qquad \mathbf{u}'_{\perp} = \frac{\mathbf{u}_{\perp}}{\gamma_v\left(1 - \mathbf{u}\cdot\mathbf{v}/c^2\right)}.$$

17. Excited ions in a fast beam have a uniform velocity and emit light on a given internal transition. The wavelength observed in the direction parallel to the beam is 359.5 nm, the wavelength observed in the direction perpendicular to the beam in the laboratory is 474.4 nm. Find the wavelength in the rest frame of the ions, and the speed of the ions in the laboratory.

18. Show that the motion of a particle in a uniform magnetic field is in general helical, with the period for a cycle independent of the initial direction of the velocity. [Hint: what can you learn from $\mathbf{f}\cdot\mathbf{v}$?]

19. In a frame S a guillotine blade in the (x, y) plane falls in the negative y direction towards a block level with the x axis and centred at the origin. The angle of the edge of the blade is such that the point of intersection of blade and block moves at a speed in excess of c in the positive x direction. In some frame S′ in standard

configuration with S, this point moves in the *opposite* direction along the block. Now suppose that when the centre of the blade arrives at the block, the whole blade instantaneously evaporates in frame S (for example, it could be vapourized by a very powerful laser beam incident from the z direction). A piece of paper placed on the block is therefore cut on the negative x-axis only. Explain this in S′.

20. Two photons travel along the x-axis of S, with a constant distance L between them. Find the distance between them as observed in S′. How is this result connected to the Doppler effect?

21. In an atomic beam spectroscopy experiment, the atoms are excited by a well-collimated laser beam propagating at right angles to the axis of the atomic beam. The laser frequency is scanned, and fluorescence is detected when the atoms absorb and re-emit the light on a narrow atomic transition. Suppose the spread of atomic speeds along the beam is 1000 m/s, with a mean speed of $v = 1000$ m/s, and the atomic transition wavelength is 300 nm in the rest frame of the atom. Calculate the maximum and minimum transverse Doppler effects for atoms propagating along the atomic beam axis, expressing your answers as frequency shifts in kHz.

    Give a formula for the longitudinal Doppler effect along the laser beam direction for an atom propagating at the average speed at a small angle θ to the atomic beam axis, accurate to first order in θ.

    Hence calculate the degree of collimation of the beam required in order that the transverse effect dominates the observed broadening of the transition frequency, and the relative precision of the laser frequency scan required if the transverse effect is to be observable.

22. Consider a particle moving in a straight line with velocity v, rapidity ρ and proper acceleration a_0. Prove that $d\rho/d\tau = a_0/c$. [Hint: use the fact that colinear rapidities are additive.]

23. A particle moves hyperbolically with proper acceleration a_0, starting from rest at $t = 0$. At $t = 0$ a photon is emitted towards the particle from a distance c^2/a_0 behind it. Prove that in the instantaneous rest frames of the particle, the distance to the photon is always c^2/a_0.

24. Prove that the necessary and sufficient condition for there to exist a reference frame in which three particles have parallel 3-velocities is that their 4-velocities $\mathsf{U}, \mathsf{V}, \mathsf{W}$ be linearly dependent.

25. For any two future-pointing time-like vectors V_1, V_2, prove that $\mathsf{V}_1 \cdot \mathsf{V}_2 = -V_1 V_2 \cosh\rho$ where ρ is the relative rapidity of frames in which V_1 and V_2 are purely temporal.

2 Energy-momentum conservation; collisions; 4-gradient

(**Lectures 6-9**)

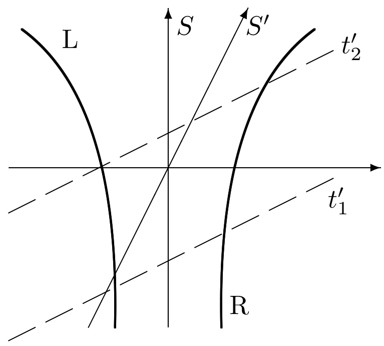
Basic ideas

1. The upper atmosphere of the Earth receives electromagnetic energy from the sun at the rate 1400 Wm^{-2}. Find the rate of loss of mass of the sun due to all its emitted radiation. [The Earth-sun distance is 499 light seconds.]

2. Calculate the mass increase of a block of copper heated from 0°C to 1000°C, assuming the specific heat capacity is constant at 420 J $\mathrm{kg}^{-1}\mathrm{K}^{-1}$ and the initial mass is 10 kg.

3. Show that if a 4-vector has a component which is zero in *all* frames, then the entire vector is zero. What insight does this offer into energy and momentum?

4. The following spacetime diagram shows the wordlines of two accelerating particles L, R and two inertial observers S, S′. The dashed lines are lines of simultaneity in frame S′ at two times t'_1, t'_2.



The two particles have the same mass.

First consider the observations in frame S. At any given time in S, the particles have equal and opposite momenta, therefore their total three-momentum $\mathbf{p}_{\rm tot} = \mathbf{p}_{\rm L} + \mathbf{p}_{\rm R} = 0$. In other words, $\mathbf{p}_{\rm tot}$ is constant (and zero). Now consider the observations in frame S′. Initially the two particles have almost the same velocity relative to S′. Then, between times t'_1 and t'_2, R comes to rest relative to S′, while L changes its velocity relative to S′ by only a small amount. Therefore the total particle momentum in frame S′ roughly halves between times t'_1 and t'_2. The total momentum of the two particles is certainly not constant in S′.

What does this situation tell us about total momentum? Is total momentum a meaningful physical concept? If so, then is it always conserved? Under what conditions does it transform as part of a four-vector?

5. Obtain the transformation equations for 3-force, by starting from the Lorentz transformation of energy-momentum, and then differentiating with respect to t'. [Hint: argue that the relative velocity $\mathbf{v}$ of the reference frames is constant, and use or derive an expression for dt/dt' .]

In the following collision questions we will use $c = 1$ throughout—and I would encourage students to do the same. Then one has $E^2 - p^2 = m^2$ in general, and $E = p$ if $m = 0$.

6. A particle of rest mass m and kinetic energy $3m$ strikes a stationary particle of rest mass $2m$ and sticks to it. Find the rest mass and speed of the composite particle.

7. A system consists of two photons, each of energy E, propagating at right angles in the laboratory frame. Find the rest mass of the system and the velocity of its CM frame relative to the laboratory frame.

Particle formation

8. Prove that the threshold energy in the laboratory frame for a particle m hitting a free stationary target M, such that collision products of total rest mass $\sum_i m_i$ can be produced, is given by

$$E_{\text{th}} = \frac{(\sum_i m_i)^2 - m^2 - M^2}{2M} c^2.$$

9. (i) Pion formation can be achieved by the process $\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \pi_0$. A proton beam strikes a target containing stationary protons. Calculate the minimum kinetic energy which must be supplied to an incident proton to allow pions to be formed, and compare this to the rest energy of a pion.
(ii) A photon is incident on a stationary proton. Find, in terms of the rest masses, the threshold energy of the photon if a neutron and a pion are to emerge.
(iii) A particle formation experiment creates reactions of the form $b + t \to b + t + n$ where b is an incident particle of mass m, t is a target of mass M at rest in the laboratory frame, and n is a new particle. Define the 'efficiency' of the experiment as the ratio of the rest energy of the new particle to the supplied kinetic energy of the incident particle. Show that, at threshold, the efficiency thus defined is equal to

$$\frac{M}{m + M + m_n/2}.$$

10. [part of B1 2004 q3, with some modifications.]
    For an isolated system of particles, let

$$s^2 = \left(\sum E_i\right)^2 - \left(\sum \mathbf{p}_i\right)^2$$

    where the sums are taken over the particles in the system at some given time. What is s for a single particle of mass m?

    In the laboratory frame a particle of mass m and momentum p_m is incident on a particle of mass M, at rest. Find an expression for the total available energy in the centre-of-mass frame. Show that the momentum of the particle of mass m in the centre-of-mass frame is given by $p'_m = Mp_m/s$.

11. Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction, in terms of E_0 and the electron rest mass m. High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4} eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

Decay

12. Particle tracks are recorded in a bubble chamber subject to a uniform magnetic field of 2 tesla. A vertex consisting of no incoming and two outgoing tracks is observed. The tracks lie in the plane perpendicular to the magnetic field, with radii of curvature 1.67 m and 0.417 m, and separation angle 21°. It is believed that they belong to a proton and a pion respectively. Assuming this, and that the process at the vertex is decay of a neutral particle into two products, find the rest mass of the neutral particle.

13. A particle with known rest mass M and energy E decays into two products with known rest masses m_1 and m_2. Find the the energies E_1, E_2 (in the lab frame) of the products, by the following steps:
    (i) Find the energies E'_1, E'_2 of the products in the CM frame.
    (ii) Show that the momentum of either decay product in the CM frame is

$$p = (c/2M)\left[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2 m_2^2\right]^{1/2}$$

    (iii) Find the Lorentz factor and the speed v of the CM frame relative to the lab.
    (iv) Write down, in terms of v, γ, p, E'_1 and E'_2, expressions for E_1, E_2 when the products are emitted (1) along the line of flight (2) at right angles to the line of flight in the CM frame.

14. [B1 2006 q8, reworded to remove an ambiguity]
    It is proposed to generate a pure beam of either electron neutrinos or electron antineutrinos by accelerating ions of unstable nuclei to relativistic speeds and then allowing them to decay in a long straight section of the accelerator.

An unstable ion of rest mass M decays after it has been accelerated to total energy E and Lorentz factor $\gamma = E/Mc^2$ and emits a neutrino of energy E_ν at an angle of θ to the beam direction. (i) Derive an expression for the neutrino's energy E_ν^* in the rest frame of the ion in terms of E_ν, θ and the velocity of the ion βc. (ii) Show that in the rest frame of the ion, the neutrino's path is inclined to the beam direction by the angle θ^* that satisfies

$$\cos\theta^* = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} .$$

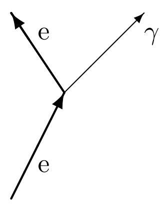
Ions are accelerated to $\gamma = 100$ and decay in the straight section of the accelerator. A cylindrical detector that is coaxial with the beam and has radius $r = 30\,\mathrm{m}$, is placed $D = 300\,\mathrm{km}$ downstream. Show that the angle between the beam direction and that of a neutrino which will hit the outer edge of the detector, measured in the rest frame of the ion, is approximately given by

$$\cos\theta^* = \frac{1 - \gamma^2\theta^2}{1 + \gamma^2\theta^2 - \theta^2/2}$$

where $\theta \simeq r/D$. Given that the emission of neutrinos is isotropic in the ion rest frame, find the fraction of the neutrinos that pass through the detector.

Show that in the ion rest frame the detector subtends an angle $2\theta_r^*$ at the ion at the emission event, where $\theta_r^* = \tan^{-1}(\gamma r/D)$. Why does θ_r^* differ from θ^*?

15. This diagram illustrates a process in which an electron emits a photon:



Prove that the process is impossible. Prove also that a photon cannot transform into an electron-positron pair in free space, and that a photon in free space cannot decay into a pair of photons with differing directions of propagation.

A comment: you will see diagrams like this in particle physics. The lesson is that since the process is impossible classically, then if it is nevertheless included in a calculation, it must be part of a larger process involving a sum over quantum amplitudes. Terms in the sum that do not conserve energy-momentum are said to involve a virtual *particle which does not propagate in the normal manner but has a wavefunction that decays like an evanescent wave.*

16. **Compton scattering**. Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $\mathsf{P} + \mathsf{P}_e = \mathsf{P}' + \mathsf{P}'_e$. [Hint: we

would like to eliminate the final electron 4-momentum P'_e, so make this the subject of the equation and square.] A collimated beam of X rays of energy 17.52 keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of 90°, including a quantitative indication of the scale.

Introducing four-gradient

17. Describe the way density and flux transform under the Lorentz transformation. Write down the continuity equation in 4-vector notation.
18. An oscillator undergoes periodic motion. State, with reasons, which of the following are Lorentz-invariant:
(i) the amplitude of the oscillation,
(ii) the phase of the oscillation,
(iii) the maximum velocity.
19. A wave motion has a phase ϕ given by $\phi(x, y, z, t) = \mathbf{k} \cdot \mathbf{r} - \omega t$ where $\mathbf{k}$ is a constant vector and ω is a constant frequency. Evaluate $\Box\phi$ and comment.

Optional extra questions

20. A 'photon rocket' propels itself by emitting photons in the rearwards direction. The rocket is initially at rest with mass m. Show that when the rest mass has fallen to αm the speed (as observed in the original rest frame) is given by
$$\frac{v}{c} = \frac{1 - \alpha^2}{1 + \alpha^2}$$
[Hint: don't bother with equations of motion, use conservation of momentum].
It is desired to reach a speed giving a Lorentz factor of 10. What value of α is required? Supposing the rocket cannot pick up fuel en route, what proportion of its initial mass must be devoted to fuel if it is to make a journey in which it first accelerates to $\gamma = 10$, then decelerates to rest at the destination (the destination being a star with negligible relative speed to the sun)?
21. A rocket propels itself by giving portions of its mass m a constant velocity $\mathbf{u}$ relative to its instantaneous rest frame. Let S′ be the frame in which the rocket is at rest at time t. Show that, if v' is the speed of the rocket in S', then to first order in dv',
$$(-dm)u = m dv'.$$
Hence prove that, when the rocket attains a speed v relative to its initial rest frame, the ratio of final to initial rest mass of the rocket is
$$\frac{m_f}{m_i} = \left(\frac{1 - v/c}{1 + v/c}\right)^{c/2u}$$

Note that the least expenditure of mass occurs when $u = c$, i.e. the 'photon rocket'.

Prove that if the rocket moves with constant proper acceleration a_0 for a proper time τ, then $m_f/m_i = \exp(-a_0\tau/u)$.

22. A decay mode of the neutral Kaon is $K^0 \to \pi^+ + \pi^-$. The Kaon has momentum 300 MeV/c in the laboratory, and one of the pions is emitted, in the laboratory, in a direction perpendicular to the velocity of the Kaon. Find the momenta of both pions.

23. **Three-body decay** A particle Y decays into three other particles, with labels indicated by $Y \to 1 + 2 + 3$. Working throughout in the CM frame:
(i) Show that the 3-momenta of the decay products are coplanar.
(ii) Show that the energy of particle 3 is given by

$$E_3 = \frac{(m_Y^2 + m_3^2 - m_1^2 - m_2^2)c^4 - 2E_1E_2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2}{2m_Y c^2}$$

(iii) Show that the maximum value of E_3 is

$$E_{3,\max} = \frac{m_Y^2 + m_3^2 - (m_1 + m_2)^2}{2m_Y} c^2$$

and explain under what circumstances this maximum is attained.
(iv) Show that, when particle 3 has its maximum possible energy, particle 1 has the energy

$$E_1 = \frac{m_1(m_Y c^2 - E_{3,\max})}{m_1 + m_2}$$

[Hint: first argue that 1 and 2 have the same speed in this situation]
(v) Let X be the system composed of particles 1 and 2. Show that its rest mass is given by

$$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2$$

(vi) Write down an expression for the energy E^* of particle 2 in the rest frame of X, in terms of m_1, m_2 and m_X.
(vii) Show that, when particle 3 has an energy of intermediate size, $m_3c^2 < E_3 < E_{3,\max}$, the energy of particle 2 in the original frame (the rest frame of Y) is in the range

$$\gamma(E^* - \beta p^* c) \le E_2 \le \gamma(E^* + \beta p^* c)$$

where p^* is the momentum of particle 2 in the X frame, and γ, β refer to the speed of that frame relative to the rest frame of Y.

24. Consider a head-on elastic collision between a moving 'bullet' of rest mass m and a stationary target of rest mass M. Show that the post-collision Lorentz factor γ of the bullet cannot exceed $(m^2 + M^2)/(2mM)$. (This means that for large energies almost all the energy of the bullet is transferred to the target, very different from the classical result). [Hint: consider $\mathsf{P}_\mathrm{t} - \mathsf{Q}_\mathrm{b}$ where P_t is the initial 4-momentum of the target and Q_b is the final 4-momentum of the bullet.]

25. Particles of mass m and kinetic energy T are incident on similar particles at rest in the laboratory. Show that, if elastic scattering takes place, then the minimum angle between the final momenta in the laboratory is given by

$$\cos\theta_{\rm min} = \frac{T}{T + 4mc^2}$$

26. Show (by considering one component at a time, or otherwise) that
(i) $\Box(\phi V) = V\Box\phi + \phi\Box V$
(ii) $\Box \cdot (\phi \mathsf{F}) = \mathsf{F} \cdot \Box\phi + \phi\Box \cdot \mathsf{F}$

3 A little more kinematics, and electromagnetism via 4-vectors

(**Lectures 10-14**)

Angle changes associated with Lorentz boosts

1. The axis of a cylinder lies along the x' axis. The cylinder has no translational motion in S$'$, but it rotates about its axis with angular speed ω'. When observed in S the cylinder travels and rotates.
   (i) Prove that in S at any instant the cylinder is twisted, with a twist per unit length $\gamma\omega' v/c^2$. [Hint: consider the rotation of the flat surfaces at the two ends of the cylinder; a line painted on either surface rotates like the hand of a clock].
   (ii) Is the cylinder in mechanical equilibrium?. Comment on whether or not you expect there to be internal sheer forces in the cylinder in frame S.

2. In S$'$ a rod parallel to the x' axis moves in the y' direction with velocity u. Show that in S the rod is inclined to the x-axis at an angle $-\tan^{-1}(\gamma u v/c^2)$.

3. The Thomas precession (off-syllabus) is a rotation of a rigid body that results from an acceleration without torque. The rotation frequency (the Thomas precession frequency) is given by

$$\boldsymbol{\omega}_T = \frac{\mathbf{a}\wedge\mathbf{v}}{c^2}\frac{\gamma^2}{1+\gamma}$$

   for a body moving with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$.
   (i) Obtain an expression for the Thomas precession frequency for an electron moving in an electric field $\mathbf{E}$.
   (ii) The spin-orbit interaction causes the intrinsic spin angular momentum of an electron to precess in the electron rest frame at the Larmor frequency

$$\boldsymbol{\omega}_L = \frac{-g_{\rm s}\mu_B}{\hbar}\frac{\mathbf{v}\wedge\mathbf{E}}{c^2}.$$

   At what frequency does the spin precess in an inertial reference frame? (You may assume $v \ll c$).

4. The Pauli-Lubanski spin vector W is a 4-vector related to angular momentum. For a particle of energy E and momentum $\mathbf{p}$ its components are given by

$$\mathsf{W} = (\mathbf{s}\cdot\mathbf{p},\ (E/c)\mathbf{s})$$

   where $\mathbf{s}$ is the 3-spin, i.e. the intrinsic angular momentum.
   (i) Show that this 4-vector is orthogonal to the 4-momentum ($\mathsf{W}\cdot\mathsf{P}=0$) and that in the limit $v\to c$, W is proportional to P [hint: start in the rest frame and apply

a boost].
(ii) (From Part B, 2015). For a massive particle, we may define a spin 4-vector $s^a = \mathsf{W}^a/mc$. In the absence of an applied torque, the spin 4-vector of an accelerating particle evolves as

$$\frac{\mathrm{d}s^a}{\mathrm{d}\tau} = \frac{s_\lambda \dot{u}^\lambda}{c^2} u^a$$

where u^a is the 4-velocity and the dot signifies $\mathrm{d}/\mathrm{d}\tau$. Show that the 3-spin evolves as

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}\tau} = \frac{\gamma^2}{c^2}\left[(\mathbf{s}\cdot\dot{\mathbf{v}})\mathbf{v} - (\mathbf{v}\cdot\dot{\mathbf{v}})\mathbf{s}\right]$$

and find $\mathbf{s}(\tau)$ for a particle accelerated along a straight line with speed $v(\tau) = c\,[1 - \exp(-2\Gamma\tau)]^{1/2}$, where Γ is constant.

Electromagnetism

5. How does a 2nd rank tensor change under a Lorentz transformation? By transforming the field tensor, and interpreting the result, prove that the electromagnetic field transforms as:

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} & \mathbf{E}'_{\perp} &= \gamma\left(\mathbf{E}_{\perp} + \mathbf{v}\wedge\mathbf{B}\right), \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} & \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B}_{\perp} - \mathbf{v}\wedge\mathbf{E}/c^2\right). \end{aligned} \tag{1}$$

[Hint: you may find the algebra easier if you treat **E** and **B** separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]

6. Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.

7. Obtain the electric field of a uniformly moving charge, as follows. Place the charge at the origin of the primed frame S′ and write down the field in that frame, then transform to S using the equations for the transformation of the fields (not the force transformation method) and the coordinates. Be sure to write your result in terms of coordinates in the appropriate frame. Sketch the field lines. Prove (from the transformation equations, or otherwise) that the magnetic field of a uniformly moving charge is related to its electric field by $\mathbf{B} = \mathbf{v}\wedge\mathbf{E}/c^2$.

8. An isolated parallel plate capacitor has charge $\pm Q$ on the plates. It is initially at rest in the laboratory frame. Assuming the capacitor's proper dimensions are fixed, what uniform motion should be given to the capacitor in order to increase the electric field between the plates? Does this result in an increased force of attraction between the plates?

9. A current-carrying wire is electrically neutral in its rest frame S. The wire has cross-sectional area A and a current I flows uniformly through this cross-section. Write down the 4-vector current density in the rest frame of the wire. Obtain the 4-vector current density in the rest frame S′ of the current carriers (you may assume they all have the same charge and drift velocity). Hence show that in this frame there is a non-zero charge density in the wire. Does this imply that charge is not Lorentz invariant? Explain. Find the electric field in S′ produced by the charge density of the wire, in the region outside the wire, and show that it is the same as the field obtained by transformation of the magnetic field in frame S.

10. (i) Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$, ϕ such that

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}, \qquad \mathbf{E} = -\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \boldsymbol{\nabla}\phi.$$

(ii) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ and ϕ.
(iii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators and Lorentz scalars (a *manifestly covariant* form).

11. A sphere of radius a in its rest frame is uniformly charged with charge density $\rho = 3q/4\pi a^3$ where q is the total charge. Find the fields due a moving charged sphere by two methods, as follows.
N.B. it will be useful to let the rest frame of the sphere be S′ (not S) and to let the frame in which we want the fields be S. This will help to avoid a proliferation of primes in the equations you will be writing down. Let S and S′ be in the standard configuration.
(i) Field method. Write down the electric field as a function of position in the rest frame of the sphere, for the two regions $r' < a$ and $r' \geq a$ where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. Use the field transformation equations to find the electric and magnetic fields in frame S (re-using results from previous questions where possible), making clear in what regions of space your formulae apply.
(ii) Potential method. In the rest frame of the sphere the 3-vector potential is zero, and the scalar potential is

$$\phi' = \frac{q}{8\pi\epsilon_0 a}\left(3 - r'^2/a^2\right) \qquad \text{for } r' < a \tag{2}$$

$$\phi' = \frac{q}{4\pi\epsilon_0 r'} \qquad \text{for } r' \geq a. \tag{3}$$

Form the 4-vector potential, transform it, and thus show that both ϕ and $\mathbf{A}$ are time-dependent in frame S. Hence derive the fields for a moving sphere. [Beware when taking gradients that you do not muddle $\partial/\partial x$ and $\partial/\partial x'$, etc.]

Retarded potentials and radiative emission

12. (i) Write down the solution to Poisson's equation for the case of a point charge q.
(ii) In electrostatics, how is the electric potential at a point in space obtained if the charge distribution is known?
(iii) Now consider the wave equation

$$\Box^2 \phi = -\frac{\rho}{\epsilon_0}$$

Show that the spherical wave form $\phi = \kappa g(t - r/c)/r$ (where $\kappa \equiv 1/4\pi\epsilon_0$) is a solution of the wave equation for $r \neq 0$ if ρ is zero everywhere except at the origin.
(iv) We would like to show that this is a solution also as $r \to 0$, if the charge density ρ is concentrated at a point at the origin. Using your knowledge of Poisson's equation, or otherwise, show that this is true as long as $g(t) = \int \rho(t) dV$.
(v) Hence write down the solution to the wave equation for a given arbitrary time-dependent distribution of charge.
(vi) Why is this called a *retarded* solution?

13. In a frame S a point charge first moves uniformly along the negative x-axis in the positive x direction, reaching the point $(-d, 0, 0)$ at $t = -\Delta t$, and then it is brought to rest at the origin at $t = 0$. Sketch the lines of electric field in S at $t = 0$, in the region $(x + d)^2 + y^2 + z^2 > (c\Delta t)^2$.

14. The electromagnetic field of a charge in an arbitrary state of motion is given by

$$\begin{aligned} \mathbf{E} &= \frac{q}{4\pi\epsilon_0\kappa^3}\left(\frac{\mathbf{n} - \mathbf{v}/c}{\gamma^2 r^2} + \frac{\mathbf{n} \wedge [(\mathbf{n} - \mathbf{v}/c) \wedge \mathbf{a}]}{c^2 r}\right) \\ \text{where} \quad \mathbf{n} &= \mathbf{r}/r, \quad \kappa = 1 - v_r/c = 1 - \mathbf{n} \cdot \mathbf{v}/c \\ \mathbf{B} &= \mathbf{n} \wedge \mathbf{E}/c \end{aligned}$$

where $\mathbf{r}$ is the vector from the source point to the field point, and $\mathbf{v}$, $\mathbf{a}$ are the velocity and acceleration of the charge at the source event. Without detailed derivation, outline briefly how this result may be obtained. How is the source event identified?

A charged particle moves along the x axis with constant proper acceleration ('hyperbolic motion'), its worldline being given by

$$x^2 - t^2 = \alpha^2$$

in units where $c = 1$. Find the electric field at $t = 0$ at points in the plane $x = \alpha$, as follows.
(i) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$. Show that the source event is at

$$x_{\rm s} = \alpha + \frac{y^2}{2\alpha}$$

(ii) Show that the velocity and acceleration at the source event are

$$v_{\rm s} = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}, \qquad a_{\rm s} = \frac{\alpha^2}{x_s^3}.$$

(iii) Consider the case $\alpha = 1$, and the field point $y = 2$. Write down the values of $x_{\rm s}$, $v_{\rm s}$, $a_{\rm s}$. Draw on a diagram the field point, the source point, and the location of the charge at $t = 0$. Mark at the field point on the diagram the directions of the vectors $\mathbf{n}$, $\mathbf{v}$, $\mathbf{a}$, $\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{a})$. Hence, by applying the formula above, establish the direction of the electric field at $(t, x, y, z) = (0, 1, 2, 0)$.
(iv) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x axis so that their separation is constant, comment on the forces they exert on one another.

15. The far field due to an elementary wire segment $\mathrm{d}z$ carrying oscillating current I is given by

$$\mathrm{d}E = \frac{I \sin\theta}{2\epsilon_0 c r} \frac{\mathrm{d}z}{\lambda} \cos(kr - \omega t)$$

Compare and contrast the case of a short antenna and the *half-wave dipole antenna.* Roughly estimate E in the the far field for each case by proposing a suitable model for the distribution of current $I(z)$ in the antenna. What happens (qualitatively) for still longer antennas?

Optional extra questions

16. Two frames are said to be 'aligned' if an observer at rest in one of the frames finds that the two sets of axes are parallel.
Frame S$'$ is aligned with frame S and moves along the x-direction of S at speed v. Frame S$''$ is aligned with frame S$'$ and moves along the y'-direction of S$'$ at speed u. Let $\mathbf{w}$ be the relative velocity of S$''$ and S. Find the angle θ between $\mathbf{w}$ and the x-axis of frame S, and the angle θ'' between $\mathbf{w}$ and the x''-axis of frame S$''$. Hence show that S and S$''$ are not aligned with one another.

17. Show that the result of 'adding' a velocity $\mathbf{u}'$ to a velocity $\mathbf{v}$ is not in general the same as 'adding' a velocity $\mathbf{v}$ to a velocity $\mathbf{u}'$. Also show that the magnitudes of the two results are the same.

18. Verify that the complex 3-vector $\mathbf{k} = \mathbf{E} + ic\mathbf{B}$ transforms under a Lorentz boost as does a 3-vector under a rotation through an imaginary angle. Deduce that $E^2 - c^2B^2$ and $\mathbf{E} \cdot \mathbf{B}$ are Lorentz invariant.

19. Give a 4-vector argument to show that the 4-vector potential of a point charge q in an arbitrary state of motion is given by

$$\mathsf{A} = \frac{q}{4\pi\epsilon_0} \frac{\mathsf{U}/c}{(-\mathsf{R} \cdot \mathsf{U})}$$

where U and R are suitably chosen 4-vectors which you should define in your answer.

20. A pair of parallel particle beams separated by a distance d have the same uniform charge per unit length λ. In the laboratory frame, a magnetic field is applied with a direction and strength just sufficient to overcome the repulsion between the beams, so that they both propagate in a straight line at constant speed v. Find the size B of this magnetic field, by both of the following methods:
(i) Do the whole calculation in the laboratory frame.
(ii) Start with a calculation of the force exerted by either beam on a particle in the other, in the rest frame of the beams. Transform this force to the laboratory frame and hence deduce the required B field in that frame.
What form does the externally applied field take in the rest frame of the beams?

4 Further methods; energy-momentum flow; spinors

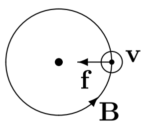
(**Lectures 15-20**)

In the following when index notation is adopted, we show, where useful, the same equation in matrix notation. In the latter notation it is understood that all quantities are contravariant and the dot signifies the presence of the metric in the matrix multiplication. For example, $\mathbb{A}^{a\lambda}\mathsf{B}_\lambda$ *would be written* $\mathbb{A}\cdot\mathsf{B}$ *which is defined* $\mathbb{A}\cdot\mathsf{B}\equiv\mathbb{A}g\mathsf{B}$.

Reflection symmetry and angular momentum

1. Define *polar* and *axial* vectors. Which of the following vectors are polar, which axial: position, velocity, electric field, magnetic field, electric current density, angular momentum, electric dipole moment, magnetic dipole moment?

2. Define the operation of *parity inversion* in three spatial dimensions. Prove that all predictions of classical electromagnetism show mirror symmetry (i.e. no preference for behaviour of one handedness over the other). Illustrate your answer by describing (by means of a diagram) the effect of a parity transformation on the following situation, which shows a positively charged particle moving parallel to a current-carrying wire in the direction of the current (out of the plane of the paper):



3. The 4-angular momentum of a single particle about the origin is defined

$$L^{ab}\equiv\mathsf{X}^a\mathsf{P}^b-\mathsf{X}^b\mathsf{P}^a$$

(i) Prove that, in the absence of forces, $\mathrm{d}L^{ab}/\mathrm{d}\tau=0$.
(ii) Exhibit the relationship between the space-space part and the 3-angular momentum vector $\mathbf{L}=\mathbf{x}\wedge\mathbf{p}$.
(iii) The total angular momentum of a collection of particles about the pivot R is defined

$$L^{ab}_{\mathrm{tot}}(\mathsf{R})=\sum_i(\mathsf{X}^a_i-\mathsf{R}^a)\mathsf{P}^b_i-(\mathsf{X}^b_i-\mathsf{R}^b)\mathsf{P}^a_i$$

where the sum runs over the particles (that is, X and P are 4-vectors not 2nd rank tensors, i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

4. *This question develops some familiarity with the differential operator $\Box$ or ∂^a. I find I prefer ∂^a when carrying out the calculation, but writing the starting point and the final result using $\Box$ can be a clean way to see what you have got. It is a matter of taste.*

   The 4-vector field F is given by $\mathsf{F} = 2\mathsf{X} + \mathsf{K}(\mathsf{X}\cdot\mathsf{X})$ where K is a constant 4-vector and $\mathsf{X} = (ct, x, y, z)$ is the 4-vector displacement in spacetime.
   Evaluate the following:
   (i) $\Box \cdot \mathsf{X}$ (or $\partial_\lambda x^\lambda$)
   (ii) $\Box(\mathsf{X}\cdot\mathsf{X})$ (or $\partial^a(x_\lambda x^\lambda)$)
   (iii) $\Box^2(\mathsf{X}\cdot\mathsf{X})$
   (iv) $\Box \cdot \mathsf{F}$ (or $\partial_\lambda \mathsf{F}^\lambda$)
   (v) $\Box(\Box\cdot\mathsf{F})$ (or $\partial^a(\partial_\lambda \mathsf{F}^\lambda)$)
   (vi) $\Box^2 \sin(\mathsf{K}\cdot\mathsf{X})$ (or $\Box^2 \sin(k_\lambda x^\lambda)$)
   (vii) $\Box\mathsf{X}$ (or $\partial^a x^b$)

Lagrangian mechanics

5. (i) How is a canonical momentum related to a Lagrangian?
   (ii) Show that the Lagrangian

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -\frac{mc^2}{\gamma} + q(-\phi + \mathbf{v}\cdot\mathbf{A})$$

   leads to the canonical momentum $(\gamma m\mathbf{v} + q\mathbf{A})$ and to the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}).$$

   (iii) In this formalism, write down the Hamiltonian function $\mathcal{H}$ for a particle of charge q moving in a magnetic field $\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}$. Make sure you express $\mathcal{H}$ in terms of the appropriate variables.

Electromagnetism

6. The electromagnetic field tensor $\mathbb{F}$ (sometimes called Faraday tensor) is defined such that the four-force on a charged particle is given by

$$\mathsf{F}^a = q\mathbb{F}^{a\lambda}\mathsf{U}_\lambda \qquad\qquad [\ \mathsf{F} = q\mathbb{F}\cdot\mathsf{U}$$

   By comparing this to the Lorentz force equation $\mathbf{f} = q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})$ which defines electric and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $d\mathsf{P}/d\tau$), show that the components of the field tensor are

$$\mathbb{F}^{ab} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

7. Assuming the relation of fields $\mathbf{E}, \mathbf{B}$ to potentials $\phi, \mathbf{A}$, show that the field tensor can be written

$$\mathbb{F}^{ab} = \partial^a \mathsf{A}^b - \partial^b \mathsf{A}^a.$$

(Note, the right hand side here is the 4-vector equivalent of a curl operation). [Hint: use cyclic permutation to avoid unnecessary repetition]. Now write down $\partial^c \mathbb{F}^{ab}$ in terms of ∂ operators and A. By keeping track of the sequence of indices, show that

$$\partial^c \mathbb{F}^{ab} + \partial^a \mathbb{F}^{bc} + \partial^b \mathbb{F}^{ca} = 0.$$

(In an axiomatic approach, one could argue in the opposite direction, asserting the above as an axiom and then deriving the relation of fields to potentials).

8. Show that the field equation

$$\partial_\lambda \mathbb{F}^{\lambda\nu} = -\mu_0 \rho_0 \mathsf{U}^\nu \qquad\qquad [\,\Box \cdot \mathbb{F} = -\mu_0 \rho_0 \mathsf{U}$$

is equivalent to

$$\Box^2 \mathsf{A} - \Box(\Box \cdot \mathsf{A}) = -\mu_0 \mathsf{J},$$

where $\mathsf{J} \equiv \rho_0 \mathsf{U}$ (here ρ_0 is the proper charge density, J is the 4-current density). Comment.

Field energy and momentum

9. (i) The electric field in a linear accelerator is 10^6 V/m. Find the power emitted by an electron traveling down the accelerator. Express your result in eV per metre assuming the electrons travel at close to the speed of light. *You may quote Larmor's formula for emitted power.*
(ii) A magnetic field of 1 tesla is used to maintain electrons in their orbits around a synchrotron of radius 10 m. Show that the electron energy is approximately 3 GeV. Find the radiative energy loss per revolution.
(iii) What is the main reason why the loss rate is so much higher in part (ii) than in part (i)?

10. *This question involves merely algebraic manipulations; the main requirement is good organisation. It shows one way to attack momentum flow in electromagnetic fields; students pressed for time could omit it.*

(i) Show (from Maxwell's equations, or otherwise) that

$$\rho \mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial \mathbf{g}}{\partial t} + \epsilon_0 \left[(\boldsymbol{\nabla} \cdot \mathbf{E})\mathbf{E} + (\boldsymbol{\nabla} \wedge \mathbf{E}) \wedge \mathbf{E} + c^2 (\boldsymbol{\nabla} \wedge \mathbf{B}) \wedge \mathbf{B} \right]. \qquad (4)$$

where $\mathbf{g} = \mathbf{N}/c^2$ and $\mathbf{N}$ is the Poynting vector. We wish to show that this can be written

$$\rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial \mathbf{g}}{\partial t} - \mathbf{s} \tag{5}$$

such that

$$s_i = \frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z}$$

with

$$\sigma_{ij} = \tfrac{1}{2}\epsilon_0(E^2 + c^2B^2)\delta_{ij} - \epsilon_0(E_iE_j + c^2B_iB_j).$$

To this end, take the following steps (or use another method if you prefer):
(ii) Show that the x component of $(\boldsymbol{\nabla}\wedge\mathbf{E})\wedge\mathbf{E}$ is $(\partial_z E_x - \partial_x E_z)E_z - (\partial_x E_y - \partial_y E_x)E_y$.
(iii) Consider the x component of $\mathbf{s}$. It is $s_x = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(x)}$ where (taking $c = 1$)

$$\boldsymbol{\sigma}^{(x)} = \frac{\epsilon_0}{2}(E^2 + B^2)\begin{pmatrix}1\\0\\0\end{pmatrix} - \epsilon_0(E_x\mathbf{E} + B_x\mathbf{B})$$

Writing $E^2 = E_x^2 + E_y^2 + E_z^2$ and using the identity $\boldsymbol{\nabla}\cdot(f\mathbf{A}) = \mathbf{A}\cdot(\boldsymbol{\nabla}f) + f\boldsymbol{\nabla}\cdot\mathbf{A}$, evaluate just the electric field part of $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(x)}$.
(iv) Confirm that you have matched the x component of the electric field part of the square bracket in (4). Explain why the magnetic part also follows.
(v) Is it necessary to calculate explicitly the other components?
(vi) Multiplying eq. (5) by a small volume δV, we have

$$\delta V\rho\mathbf{E} + \delta V\mathbf{j}\wedge\mathbf{B} = -\frac{\delta V\partial\mathbf{g}}{\partial t} - \delta V\mathbf{s}. \tag{6}$$

Explain the physical significance of the terms in this equation (and thus justify all this hard work!).

11. Show that σ_{ij} in the previous question is the space-space part of

$$\begin{aligned} \mathbb{T}^{ab} &= \epsilon_0 c^2\left(-\mathbb{F}^{a\lambda}\mathbb{F}_\lambda^{\ b} - \frac{1}{2}g^{ab}D\right), \\ [\text{ i.e. } \mathbb{T} &= \epsilon_0 c^2\left(-\mathbb{F}\cdot\mathbb{F} - \frac{1}{2}gD\right).\ ] \\ \text{where } D &= \frac{1}{2}\mathbb{F}_{\lambda\mu}\mathbb{F}^{\lambda\mu}. \end{aligned}$$

Use the stress-energy tensor $\mathbb{T}^{ab}$ to find the forces exerted by the magnetic field inside a long cylindrical solenoid of radius 3 cm and field 1 tesla. *Mu-metal* is an alloy of high magnetic permeability that can be used to provide shielding against magnetic fields. If a piece of mu-metal is placed against the end of a solenoid, it 'confines' the magnetic field to the interior of the solenoid. By interpreting the stress-energy tensor for the field on each side of the mu-metal sheet, discover whether the latter is attracted or repelled by the solenoid, and find the net force.

12. Write down the stress-energy tensor and the 4-wave vector for an electromagnetic plane wave propagating in the x direction.

    Such a wave is observed in two frames in standard configuration. Show that the values of radiation pressure P, momentum density g, energy density u and frequency ν in the two frames satisfy

    $$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2}$$

    (Optional: can you prove this for any relative motion of the frames? [Hint: write $\mathbb{T}^{ab}$ in terms of K^a]).

    A confused student proposes that these quantities should transform like ν'/ν not ν'^2/ν^2, on the grounds that energy-momentum $\mathsf{N} = (uc, \mathbf{N})$ is a 4-vector and so should transform in the same way as the wave-vector. What is wrong with this argument?

Spinors

Concerning spinors, the intention is that students acquire a basic familiarity with the main ideas. These are: broadly what spinors are, how to rotate and Lorentz transform them, how to extract 4-vectors from spinors. Parity violation in the weak interaction is not treated here though it will be mentioned in lectures, and treated more fully in the particle physics course. The Dirac spinor and Dirac equation will be briefly introduced in the lectures, but the beautiful connection to antimatter will only be briefly indicated, and is off syllabus for this part of the course.

13. Describe the spinors $(1, 0)$ and $(1, i)$, by giving for each one the direction of the flagpole and flag.

14. (i) Find the determinant of the matrix $X = tI + x\sigma_x + y\sigma_y + z\sigma_z$ and comment. If X is transformed by $X \to MXM^\dagger$, what set of transformations M preserve the determinant?

    (ii) Consider the outer product $U = \boldsymbol{u}\boldsymbol{u}^\dagger$ where $\boldsymbol{u}$ is a spinor written as a column vector. Show that U is Hermitian and that the associated 4-vector, given by the recipe presented in part (i), is

    $$\mathsf{U}^\alpha = \frac{1}{2}\boldsymbol{u}^\dagger \sigma^\alpha \boldsymbol{u}$$

    (iii) Show that this 4-vector is null.

    (iv) Show that if $\boldsymbol{u}$ is transformed by

    $$\boldsymbol{u}' = (\cosh(\rho/2)I - \sinh(\rho/2)\sigma_z)\, \boldsymbol{u}$$

    (where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) then the associated 4-vector U is transformed by a Lorentz boost along the z direction with rapidity ρ.

15. $\phi(t, x, y, z)$ is a spinor field satisfying

$$\frac{\partial \phi}{\partial t} + (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\phi = 0 \tag{7}$$

where the $\boldsymbol{\sigma}$ is the vector of Pauli matrices. ϕ can be expressed as a two-component column vector of fields u, v. Show that each of the components satisfies the wave equation with $c = 1$. [Hint: there are clever ways to do this, but if you don't know them, then just write out the 2×2 matrix of differential operators and thus get 2 coupled first-order differential equations for u and v. Then figure out how to eliminate v from the pair of equations.]

Optional extra questions

16. Show that the following two scalar quantities are Lorentz invariant:

$$\begin{aligned} D &= B^2 - E^2/c^2 \\ \alpha &= \mathbf{B} \cdot \mathbf{E}/c. \end{aligned}$$

[Hint: for the second, introduce the dual field tensor $\tilde{\mathbb{F}}_{ab} = \frac{1}{2}\epsilon_{ab\lambda\mu}\mathbb{F}^{\lambda\mu}$].
Show that if $\alpha = 0$ but $D \neq 0$ then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example such frame (by specifying its velocity relative to one in which the fields are $\mathbf{E}$, $\mathbf{B}$). Suggest a type of field for which both $\alpha = 0$ and $D = 0$.

17. Consider the general problem of motion of a particle of charge q in a uniform constant electromagnetic field. The equation of motion is

$$q(\mathbb{F}g)\mathsf{U} = m\frac{d\mathsf{U}}{d\tau}$$

(make sure you understand the reason why the metric g is included in this equation). For a uniform constant field, $\mathbb{F}$ is independent of space and time, and therefore this matrix equation is precisely the same as the one obtained in a classical normal modes problem, and can be solved by the same methods.
(i) Propose a solution $\mathsf{U} = \mathsf{U}_0 \exp(\Gamma\tau)$ and thus convert the equation into an eigenvalue equation[1] with eigenvalues $\lambda = m\Gamma/q$.
(ii) Without loss of generality, we can take the z-axis along $\mathbf{B}$ and $\mathbf{E}$ in the xz plane. Show that the eigenvalues are

$$\lambda^2 = -\frac{D}{2} \pm \sqrt{D^2/4 + \alpha^2}$$

[1] $\mathbb{F}g$ is not symmetric so the right-eigenvectors are not the same as the left-eigenvectors. We only need the right-eigenvectors here.

where D and α are the invariants defined in problem 16.
(iii) Consider the case $\alpha = 0$. What does this tells us about the fields? Interpret the solution corresponding to a zero eigenvalue [Hint: Lorentz force].
(iv) Find $\mathsf{U}(\tau)$ for a particle initially at rest in a uniform purely electric field.
(v) Show that the right eigenvectors of the matrix
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ may be written $\begin{pmatrix} 1 \\ i \end{pmatrix}$, $\begin{pmatrix} i \\ 1 \end{pmatrix}$.
Hence find $\mathsf{U}(\tau)$ for a particle moving in a plane perpendicular to a uniform purely magnetic field.

18. In a frame S there is a uniform electric field E along the y direction and a uniform magnetic field $B = 5E/3c$ along the z direction. A particle of mass m, charge q is released from rest on the x axis. What time elapses before it returns to the x-axis?

19. (i) Prove that the time rate of change of the angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ of a particle about an origin O is equal to the couple $\mathbf{r} \wedge \mathbf{f}$ of the applied force about 0.
(ii) If L^{ab} is the particle's 4-angular momentum, and we define the 4-couple $G^{ab} \equiv \mathsf{X}^a\mathsf{F}^b - \mathsf{X}^b\mathsf{F}^a$, prove that $(d/d\tau)L^{ab} = G^{ab}$, and that the space-space part of this equation corresponds to the previous 3-vector result.

20. If $(\mathbf{E}, \mathbf{B})$ and $\mathbf{E}', \mathbf{B}')$ are two different electromagnetic fields, show that $\mathbf{E} \cdot \mathbf{E}' - c^2\mathbf{B} \cdot \mathbf{B}'$ and $\mathbf{E} \cdot \mathbf{B}' + \mathbf{B} \cdot \mathbf{E}'$ are invariants.

21. In a certain frame S_0 having 4-velocity U, a 2nd rank tensor T has but one non-zero component, $T^{00} = c^2$. Find the components of T in the general frame S, in which $\mathsf{U} = \gamma_u(c, \mathbf{u})$.

The Dirac matrices in the chiral basis are

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

22. A Dirac spinor

$$\Psi = \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix}$$

transforms under a change of reference frame as

$$\Psi \to \begin{pmatrix} \Lambda(v) & 0 \\ 0 & \Lambda(-v) \end{pmatrix} \Psi$$

where $\Lambda(v) = \exp\left(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 - \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2\right)$
and $\Lambda(-v) = (\Lambda(v)^\dagger)^{-1} = \exp\left(i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}/2 + \boldsymbol{\sigma} \cdot \boldsymbol{\rho}/2\right)$.
(i) Show that the combination

$$\left(\phi_R^\dagger, \chi_L^\dagger\right) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix} = \phi_R^\dagger\chi_L + \chi_L^\dagger\phi_R$$

is Lorentz-invariant.

(ii) How may two non-null orthogonal 4-vectors be extracted from the Dirac spinor? Briefly exhibit both the relationship to a pair of null 4-vectors, and the use of Dirac matrices to express the result in a compact notation.

23. A Dirac spinor is used to represent the energy-momentum and spin 4-vectors of a single particle. By using a boost along the z direction and then extracting the spinor properties, or otherwise, show that in the limit $v \to c$, the spin is aligned with the momentum.

24. Prove that the electromagnetic stress-energy tensor satisfies the following two identities:

$$T^{\lambda}_{\lambda} = 0, \qquad T^{a}_{\ \lambda} T^{\lambda}_{\ b} = (I\epsilon_0/2)^2 \delta^a_b$$

where $I^2 = (E^2 - c^2B^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2c^2$. [Hint: start by establishing the identity in a particular frame, such as one in which $\mathbf{E}$ is parallel to $\mathbf{B}$].

25. The Klein-Gordan equation is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = -\mu^2c^2\phi$$

(i) Prove that the equation is Lorentz-covariant as long as ϕ is a Lorentz scalar field.
(ii) In the two dimensional case (i.e. one spatial dimension plus time), show that the left hand side factorizes into terms involving only first-order derivatives.
(iii) If $\mathbf{m}$ is a vector, each of whose components is a constant 2×2 matrix, and $\boldsymbol{\nabla}$ is the 3-gradient operator, show that $(\mathbf{m}\cdot\boldsymbol{\nabla})^2 = \boldsymbol{\nabla}^2$ if the component matrices m_i anticommute among themselves and square to 1 (i.e. $m_i^2 = I$). Identify a set of matrices with these properties.
(iv) Factorize the left hand side of the Klein-Gordan in the four-dimensional case, and hence obtain the Dirac equation. [To reduce clutter, you may find it helpful to introduce the notation $\hat{\boldsymbol{\omega}} \equiv i\left(\frac{\partial}{\partial t}\right)$, $\hat{\mathbf{k}} \equiv -i\boldsymbol{\nabla}$.]
(v) Briefly discuss the plane wave solutions of the Dirac equation.

26. In synchrotron radiation, in which direction is most of the energy emitted in the rest frame of the accelerating charge? Describe qualitatively the pattern of the radiation field in the rest frame of the centre of the synchrotron apparatus.

27. Why is it not feasible for mobile phones to use radio waves?

28. The Pauli-Lubanski spin vector is defined by

$$\mathsf{W}_a = \frac{1}{2}\epsilon_{a\nu\lambda\mu}S^{\lambda\mu}\mathsf{P}^{\nu}$$

where $S^{\lambda\mu}$ is the angular momentum about the centroid (or 'centre of mass'). By evaluating components one by one, or otherwise, show that the components of W are given by

$$\mathsf{W} = (\mathbf{s}\cdot\mathbf{p},\ (E/c)\mathbf{s})$$

where **s** is the 3-spin.