Chapter 18

Classical field theory

Classical field theory deals with the general idea of a quantity that is a function of time and space, which can be used to describe wave-like physical phenomena such as sound and light, or other continuous phenomena such as fluid flow. The word 'classical' is here used in the sense 'not quantum mechanical'. We shall define a field to be *classical* if it satisfies the following criteria:

1. The state of the field at a given time is represented by furnishing, for each point in space, a finite set of numbers (e.g. a single real number or a tensor or a spinor, depending on the type of field).
2. The field can in principle be observed without disturbing it.

By contrast, a quantum field would be described by furnishing at each point in space a set of operators not numbers, and it could not in general be observed without disturbing it. It is important to maintain a tight grip on terminology here, because in many textbooks the equations described in this chapter are first introduced in the context of quantum mechanics. However I believe it is better to become acquainted with these fields in their classical guise first, and then quantize them afterwards.

In the whole of this book up to and including this chapter, all fields mentioned are classical fields. The electromagnetic field has been shown to be a tensor field. We have sometimes invoked scalar fields for illustrative purposes. In this chapter the idea of a classical spinor field will be introduced. We shall allow that high relative speeds may be involved; to make sure our results satisfy the Postulates of Relativity we shall only write Lorentz covariant field equations. The language of tensors and spinors makes this easy to accomplish.

A general discussion of classical field theory would require a text book in its own right. In this chapter the aim is to introduce some field equations that are close cousins of the wave equation.

This will allow the basic principles of quantum field theory and particle physics to be opened up in the next chapter.

Some material on Lagrangian mechanics for fields is also included.

18.1 Wave equation and Klein-Gordan equation

Notation. In this section the symbol ϕ is used to indicate a classical Lorentz-scalar field. That is, $\phi(t, x, y, z)$ is a function of time and space, whose value is unchanged under a change of reference frame. For example, in the case of a fluid, ϕ could be the proper density; in the case of an electromagnetic field, ϕ could be one of the invariants $E^2 - c^2 B^2$ or $\mathbf{E} \cdot \mathbf{B}$. We shall not take an interest here in any specific physical field, however, but rather in what are the simplest covariant differential equations that can be written down for scalar fields.

18.1.1 The wave equation

The most familiar non-trivial Lorentz covariant field equation is the scalar wave equation

$$\Box^2 \phi = 0 \qquad \text{or} \qquad -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = 0. \tag{18.1}$$

This has a complete set of plane wave solutions of the form

$$\phi(t, x, y, z) = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{18.2}$$

Substituting this solution into the wave equation yields the *dispersion relation*

$$\omega^2 - k^2 c^2 = 0. \tag{18.3}$$

This is reminiscent of the energy-momentum relationship for massless particles, $E^2 - p^2 c^2 = 0$.

18.1.2 Klein-Gordan equation

We can modify the wave equation while preserving Lorentz covariance by introducing a Lorentz scalar s. Possibilities include, for example

$$\begin{aligned}\Box^2\phi &= s,\\ \Box^2\phi &= s\phi.\end{aligned}$$

The first of these is the wave equation with a source term; we studied it in section 6.5.2. If s is non-zero it does not in general have plane wave solutions, but the second equation does. If s is independent of time and space then the solution (18.2) is valid as long as the dispersion relation

$$\omega^2 - k^2c^2 = sc^2 \tag{18.4}$$

is satisfied. This is reminiscent of the energy-momentum relationship for massive particles, $E^2 - p^2c^2 = m^2c^4$ if we set $s \propto m^2$. Therefore let us name the constant μ^2c^2 to remind us of mass, and we have

Klein-Gordon equation

$$(\Box^2 - \mu^2c^2)\phi = 0. \tag{18.5}$$

The dispersion relation now reads

$$\omega^2 - k^2c^2 = \mu^2c^4. \tag{18.6}$$

The Klein-Gordon equation was first applied to physics in the context of quantum theory, and this is the way it is often introduced in text books. However, here we are dealing with a classical field. Don't forget that in this chapter $\phi(t, x, y, z)$ is a scalar quantity that in principle could be measured without disturbing it; it is not a wavefunction. (In quantum theory the constant μ^2c^2 may be written $m^2c^2/\hbar^2$).

Yukawa potential

Let us examine the time-independent solution to the Klein-Gordan equation (the d.c. limit of the plane waves). If ϕ is independent of time then we have

$$\nabla^2\phi = \mu^2c^2\phi. \tag{18.7}$$

Now let's obtain a spherically symmetric solution. When there is no angular dependence, we have

$$\nabla^2 \phi = \frac{1}{r} \frac{d^2}{dr^2} (r\phi)$$

and therefore

$$\frac{d^2}{dr^2} (r\phi) = \mu^2 c^2 (r\phi). \tag{18.8}$$

Thinking of $(r\phi)$ as the function, this is readily solved: one finds the general solution

$$r\phi = A e^{-\mu c r} + B e^{\mu c r} \tag{18.9}$$

where A, B are constants of integration. We take it that ϕ cannot become infinite at large r, so $B = 0$ and the solution is

$$\phi = A \frac{e^{-\mu c r}}{r}. \tag{18.10}$$

This solution is called the *Yukawa potential.* It is reminiscent of the Coulomb potential, except for the exponential term which makes ϕ fall to zero much more quickly as a function of r than does the Coulomb potential. Indeed, the exponential term has a *natural length scale* given by $(\mu c)^{-1}$. We say that such a form has a 'finite range'; this does not mean it falls strictly to zero beyond a given distance, but its exponential fall is so rapid that to all intents and purposes it is zero once the distance is large compared to $(\mu c)^{-1}$. This is in contrast to the Coulomb potential which has no such restricted range.

The solution may also be thought of as a 'spherical wave', but one which decays instead of propagates—this is called an *evanescent* wave.

In the mid 1930s Hideki Yukawa was trying to understand the nature of the strong force that holds together nucleons in a nucleus. The wave-particle duality for light was established, though the full quantum field theory for electromagnetism was not. Yukawa supposed that the strong force was mediated by a particle, somewhat analogous to the photon. The essence of his insight was that if one assumed that the mediating particle had a finite rest mass, and proceeded by analogy with the case of photons, then one needed a mass term in the wave equation and then the associated force would have a finite range, as exhibited in equation (18.10). He thus predicted the existence of a previously unknown type of particle, and from the known data on the range of nuclear forces, he was able to predict (approximately) the mass of his 'heavy photon-like particle'. It was subsequently discovered in cosmic ray data and cyclotron experiments in the predicted mass range; it is now called the *meson.* It is found there are several types of meson, and it turns out that it is a composite particle (composed of one quark and one antiquark). Yukawa's theory is not correct in detail (it has to replaced by quantum chromodynamics), but it gives a sound qualitative picture and helped guide the way to the correct theory.

Klein-Gordan current

Now return to the general Klein-Gordan equation (18.5) and introduce a real quantity $\mathbf{j}$ defined by

$$\mathbf{j} \equiv i\left(\phi \boldsymbol{\nabla} \phi^* - \phi^* \boldsymbol{\nabla} \phi\right). \tag{18.11}$$

We take an interest in whether this could be a current of a conserved quantity, i.e. a quantity satisfying the continuity equation $\boldsymbol{\nabla} \cdot \mathbf{j} = -\partial \rho / \partial t$ where ρ is a density. You are invited to confirm, by using the Klein-Gordan equation, that

$$\boldsymbol{\nabla} \cdot \mathbf{j} = \frac{i}{c^2}\left(\phi \frac{\partial^2 \phi^*}{\partial t^2} - \phi^* \frac{\partial^2 \phi}{\partial t^2}\right)$$

and therefore

$$\boldsymbol{\nabla} \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t} \qquad \text{with} \qquad \rho \equiv \frac{i}{c^2}\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right). \tag{18.12}$$

Therefore we have a conservation law. The density ρ can be either positive or negative. It could represent, for example, a density of electric charge (with an appropriate constant premultiplying factor to get the physical dimensions right). Eqs. (18.11) and (18.12) combine to give the 4-current

$$\mathsf{J} = (\rho c, \mathbf{j}) = i\left(\phi \Box \phi^* - \phi^* \Box \phi\right). \tag{18.13}$$

18.2 The Dirac equation

The wave equation and the Klein-Gordan equation are both second order in time. This means that, to obtain a unique solution, it is not sufficient to specify the field ϕ at some initial time; one must also specify its first derivative with respect to time. This implies that ϕ alone cannot serve as a complete specification of the physical situation or 'state'. In many applications, such as to particle physics where we want an equation of motion, this is a drawback. We would prefer an equation that only involves the first derivative with respect to time. We shall approach this in stages.

Massless Dirac equation in 2 dimensions

The two dimensional case (i.e. one spatial dimension plus time) is straightforward when the mass is zero. Then eq. (18.5) can be written

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\phi(x,t) = 0. \tag{18.14}$$

A solution is ensured if we keep either one of the factors. Thus we can obtain a first-order equation

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi(x,t) = 0.$$

This corresponds to the relation $E - cp = 0$ which makes sense for massless particles.

A plane wave solution has the form $\phi = \exp(i(kx - \omega t))$ with $k = \omega c$. Notice that in this case the form $\exp(i(-kx - \omega t))$, which is a solution of the second-order equation, does not satisfy our first-order equation. The first-order equation only has plane wave solutions whose wavefronts propagate towards positive x. A wave described by such a solution in one spatial dimension is sometimes called a "right-mover". This is a Lorentz-invariant property because we are dealing with waves having phase velocity c. ω and k can be either both positive or both negative.

By retaining the other factor in (18.14) we obtain a first-order equation for "left-movers".

Massive Dirac equation in 2 dimensions

The assumption $\mu = 0$ lead to the simplicity of the preceding discussion. When $\mu \neq 0$ we can make some progress by recasting the Klein-Gordan equation into a pair of first-order equations:

$$(\hat{\omega} + \hat{k}_x c)\phi_1 = \mu c^2 \phi_2, \tag{18.15}$$
$$(\hat{\omega} - \hat{k}_x c)\phi_2 = \mu c^2 \phi_1, \tag{18.16}$$

where to reduce clutter we have used

$$\hat{\omega} \equiv i\frac{\partial}{\partial t}, \qquad \hat{\mathbf{k}} \equiv -i\boldsymbol{\nabla}.$$

The factor i is introduced in these definitions so that when acting on a plane wave solution, we will find

$$\hat{\omega}\phi = \omega\phi, \qquad \hat{\mathbf{k}}\phi = \mathbf{k}\phi. \tag{18.17}$$

I know it looks like we are doing quantum mechanics, but I promise you we are not! (yet).

You should verify that both ϕ_1 and ϕ_2 satisfy the original equation. Indeed, a common method of solution of such a pair of first-order equations is to obtain the 2nd order equation and then try to solve it. We have proceeded in the opposite direction.

In the limit $\mu \to 0$, ϕ_1 describes right-movers and ϕ_2 describes left-movers. However, when $\mu \neq 0$ there is no clear distinction between them because a wavefront moving to the right in one reference frame will move to the left in another when the speed is less than c. Therefore both equations are required, and the number of degrees of freedom is roughly doubled. Notice that ϕ_2 is acting like a source in the equation for ϕ_1, and ϕ_1 is acting like a source in the equation for ϕ_2. The two parts are coupled through the 'mass' μ.

We can write the pair of equations for ϕ_1 and ϕ_2 as a single matrix equation for a two-component spinor:

$$\left(\hat{\omega} + \sigma_z \hat{k}_x c\right) \psi = \mu c^2 \sigma_x \psi \tag{18.18}$$

where $\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. This is the two-dimensional Dirac equation.

The spinor is a function of position and time. We started with an equation (Klein-Gordan) for a scalar field, now we have an equation for a spinor field (in the same sense that the electric field in classical electromagnetism is a vector field). The Klein-Gordan equation has not completely gone away, because each component of the spinor field satisfies it.

Massless Dirac equation in 4 dimensions

In four dimensions, the idea is to attempt once again a factorisation of the Klein-Gordan equation $(\hat{\omega}^2 - \hat{k}^2 c^2)\phi = \mu^2 c^4 \phi$. However we now have a scalar operator $\hat{\omega}$ and a vector operator $\hat{\mathbf{k}}$ so we can't write $(\hat{\omega} + \hat{\mathbf{k}})(\hat{\omega} - \hat{\mathbf{k}})$ (well, we can write it, but it doesn't make any mathematical sense). The key to performing the factorization is to introduce a set of quantities σ_i (these will be the Pauli matrices, but pretend for a moment you don't know that) and try

$$\left(\hat{\omega} + \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} c\right) \left(\hat{\omega} - \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} c\right) \phi = 0 \tag{18.19}$$

Since the operators $\hat{\omega}$ and $\hat{\mathbf{k}}$ commute, this evaluates to

$$\left(\hat{\omega}^2 - (\boldsymbol{\sigma} \cdot \hat{\mathbf{k}})^2\right) \phi = 0$$

so we have the Klein-Gordan equation as long as $(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}})^2 = \hat{k}^2$. Now, assuming the three entities σ_i commute with $\hat{k}_j$ we have

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\hat{\mathbf{k}})^2 &= (\sigma_x\hat{k}_x + \sigma_y\hat{k}_y + \sigma_z\hat{k}_z)(\sigma_x\hat{k}_x + \sigma_y\hat{k}_y + \sigma_z\hat{k}_z)\\ &= \sigma_x^2\hat{k}_x^2 + \sigma_y^2\hat{k}_y^2 + \sigma_z^2\hat{k}_z^2\\ &+(\sigma_x\sigma_y + \sigma_y\sigma_x)\hat{k}_x\hat{k}_y + (\sigma_y\sigma_z + \sigma_z\sigma_y)\hat{k}_y\hat{k}_z + (\sigma_z\sigma_x + \sigma_x\sigma_z)\hat{k}_z\hat{k}_x\end{aligned}$$

therefore the result is $\hat{k}^2$ as long as the σ_i all square to 1 and anticommute among themselves. Recalling section 17.7 we recognise that the Pauli matrices have this property. Therefore we can represent $\boldsymbol{\sigma}$ in (18.19) by a set of three 2×2 Hermitian matrices, and the wave ϕ becomes a two-component spinor. Each of its components satisfies the Klein-Gordan equation.

Now we throw away one of the factors, to obtain either of the first-order equations

$$\begin{aligned}\left(\hat{\omega} - \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}c\right)\phi_1 &= 0,\\ \left(\hat{\omega} + \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}c\right)\phi_2 &= 0.\end{aligned} \quad (18.20)$$

These are the massless Dirac equations in 4 dimensions, also called the *Weyl equations*. We already met them in the previous chapter, eqs (17.39) and (17.45). In the previous chapter we had E and $\mathbf{p}$ as parts of a 4-vector, now we have operators $\hat{\omega}, \hat{\mathbf{k}}$ in a classical wave equation. The solutions ϕ_1 and ϕ_2 are Weyl (i.e. two-component) spinor fields.

For the plane wave solutions we can replace $\hat{\omega}$ and $\hat{k}$ by $\omega, \mathbf{k}$. Since these are components of a 4-vector (the wave 4-vector) we then obtain *precisely* the previous (non-operator) versions of the Weyl equations (17.39), (17.45), except now we must interpret the quantities as frequency and wave-vector of a wave, not energy and momentum of a particle. (The connection will be completed when we come to quantum field theory in the next chapter.)

The discussion of helicity and chirality provided in section 17.4.1 can now be applied to the plane wave solutions, where now by helicity we mean the projection of $\boldsymbol{\sigma}$ on the wave-vector. The main point is that there are two separate equations, and solutions of one are not solutions of the other.

To relate the solutions, consider the complex conjugate of the first equation of (18.20):

$$\left(\hat{\omega}/c - \sigma_x\hat{k}_x + \sigma_y\hat{k}_y - \sigma_z\hat{k}_z\right)\phi_1^* = 0.$$

Note the sign change of the σ_y term because this matrix is imaginary. Using the anticommutation relations we can write this

$$\sigma_y\left(\hat{\omega}/c + \boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\right)\sigma_y\phi_1^* = 0.$$

The left hand factor of σ_y can be dropped since it is invertible. Therefore the spinor field $\sigma_y \phi_1^*$ satisfies the opposite-helicity equation.

18.2.1 Massive Dirac equation in 4 dimensions

In turning to the massive case, we must retain the μ^2 term in the Klein-Gordan equation. We proceed as we did in the two-dimensional case, by writing a pair of coupled equations (c.f. (18.16)), taking advantage of the spin matrix 'trick' for factoring $\hat{k}^2$:

$$\left(\hat{\omega} - \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} c\right) \phi_R = \mu c^2 \chi_L, \qquad \left(\hat{\omega} + \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} c\right) \chi_L = \mu c^2 \phi_R.$$

Here, ϕ_R and χ_L are both two-component spinors. The equation guarantees that each of the four components satisfies the Klein-Gordan equation. Gathering everything together into a matrix, we have

Dirac equation

$$\begin{pmatrix} -\mu c^2 & \hat{\omega} + \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} c \\ \hat{\omega} - \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} c & -\mu c^2 \end{pmatrix} \begin{pmatrix} \phi_R \\ \chi_L \end{pmatrix} = 0. \tag{18.21}$$

This is the Dirac equation in empty space (i.e. without interactions) in four dimensions. It is exactly the same as (17.63), except now $\hat{\omega}$ and $\hat{\mathbf{k}}$ are differential operators and $\psi = (\phi_R, \chi_L)$ is a spinor field.

We shall consider the introduction of further terms describing interactions in section 18.5.

By using the γ^μ matrices (17.53) the Dirac equation can be written

Dirac equation, manifestly covariant form

$$(i\gamma^\alpha \partial_\alpha \psi - \mu c)\, \psi = 0. \tag{18.22}$$

The Einstein summation *is* implied here; the combination $\gamma^\mu \partial_\mu$ is a sum of four terms that results in a 4×4 matrix of differential operators. This form of the equation is suggestive, because it appears to be a manifestly covariant form. To prove that this is a true appearance, it is sufficient to show that $\gamma^\mu \partial_\mu$ is a 'Lorentz scalar' matrix, i.e. a 4×4 matrix which is invariant under Lorentz transformations. This means we need to show that the matrices γ^μ transform in the same way as components of a contravariant 4-vector, as the notation implies. The logic is as follows. Suppose we define the Dirac matrices in the first instance by the anti-commutation relations (17.67). Then to prove that the Dirac equation is consistent with the Postulates of Relativity, eq. (18.22) shows that we need to prove that the Dirac matrices transform as the components of a contravariant 4-vector. We accomplish this by asserting *by definition* that the Dirac matrices transform the required way. All that is needed is to show that this assertion does not contradict the anti-commutation relations. It is found that it does not.

In writing down the Dirac equation, (18.21), we used the notation ϕ_R, χ_L for the Dirac spinor, implying (without proof) that the two components have well-defined chirality, and that their chirality is right- and left-handed respectively. To prove this, note the similarity between the Dirac equation (18.21) and its classical particle version (17.63). For plane wave solutions of the differential equation, we regain the equation (17.63) (see below) but now it describes the 4-wave-vector. It follows that the spinor fields ϕ_R and χ_L must transform in the same way as their counterparts in the classical particle equation, and therefore they have right and left chirality "as advertised".

With this in hand, we can now invoke the discussion of section 17.6 to deduce that

$$\mathsf{J}^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi \tag{18.23}$$

is a 4-vector (the 4-velocity in the previous case, now the 4-current density) called the *Dirac current* and

$$\mathsf{W}^\mu = \psi^\dagger \gamma^0 \gamma^\mu \gamma^5 \psi$$

is a 4-pseudovector (previously the Pauli-Lubanski 4-vector, now the 4-spin density). We can easily show that the Dirac current satisfies the continuity equation:

$$\Box \cdot \mathsf{J} = (\partial_\alpha \bar{\psi})\gamma^\alpha \psi + \bar{\psi}\gamma^\alpha \partial_\alpha \psi = i\mu c \bar{\psi}\psi - i\mu c\bar{\psi}\psi = 0$$

where we adopted the notation $\bar{\psi} \equiv \psi^\dagger \gamma^0$ to reduce clutter, and to evaluate the first term we used the 'Dirac adjoint' of the Dirac equation, which reads

$$-i\gamma^\alpha \partial_\alpha \bar{\psi} = \mu c \bar{\psi}. \tag{18.24}$$

(see exercise **??**).

Connection between the two forms of the Dirac equation

In the above we appealed several times to the similarity between (18.21) and (17.63). For clarity we shall here expound the similarity and difference between the two equations.

The Dirac equation (18.21) is a differential equation describing a classical field. It describes a Dirac spinor field and it has a complete set of plane wave solutions. Equation (17.63), by contrast, applies to a single Dirac spinor (not a field) describing energy-momentum and spin of a classical particle.

Let us examine a plane wave solution to the Dirac equation:

$$\psi(t, x, y, z) = u \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t).$$

where u (the 'amplitude' of the wave) is a constant Dirac spinor. We have

$$\begin{aligned}\hat{\omega}\psi &= i\frac{\partial}{\partial t}\left(u e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) = \omega u e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \\ \hat{\mathbf{k}}\psi &= -i\boldsymbol{\nabla}\left(u e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) = \mathbf{k} u e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.\end{aligned}$$

Upon substituting into the Dirac equation (18.21), the exponential term factors out and the result is

$$\begin{pmatrix} -\mu c^2 & \omega + \boldsymbol{\sigma}\cdot\mathbf{k}c \\ \omega - \boldsymbol{\sigma}\cdot\mathbf{k}c & -\mu c^2 \end{pmatrix}\begin{pmatrix} u_R \\ u_L \end{pmatrix} = 0. \tag{18.25}$$

This is now an equation involving numbers not differential operations, and it describes a single Dirac spinor (the amplitude of the wave). Eq. (18.25) has the same mathematical form as equation (17.63), but a different physical interpretation. The terms appearing in the matrix are now the frequency and 3-wave vector of a wave, rather than the energy and momentum of a particle. We shall complete the connection between these two ideas in the next chapter.

Since the plane wave solutions form a complete set of solutions, one may say that (18.25) *is* the Dirac equation, in its complete generality, but displayed in Fourier space (frequency/wave-vector space) rather than time-position space. It can be written

$$(-\gamma^\alpha \mathsf{K}_\alpha - \mu c)\, u = 0 \tag{18.26}$$

where K is the 4-wave-vector.

18.3 Lagrangian mechanics for fields*

[*Section omitted in lecture-note version.*]

18.4 Conserved quantities and Noether's theorem

[*Section omitted in lecture-note version.*]

18.5 Interactions

[*Section omitted in lecture-note version.*]