

Chapter 6

Relativity and electromagnetism

[Section omitted in lecture-note version.]

Transformation of electromagnetic field

$$\begin{aligned}\mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B}), \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} &= \gamma(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2).\end{aligned}\tag{6.1}$$

As usual, the subscripts \parallel and \perp refer to the components parallel and perpendicular to the relative velocity \mathbf{v} of the reference frames.

[Section omitted in lecture-note version.]

6.1 Maxwell's equations

The theory of electromagnetism discovered by Faraday, Ampère, Maxwell and others is encapsulated by the Lorentz force equation (??) and the

Maxwell equations:

$$\begin{aligned}
 \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
 \nabla \cdot \mathbf{B} &= 0 \\
 \nabla \wedge \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
 c^2 \nabla \wedge \mathbf{B} &= \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}, \tag{6.2}
 \end{aligned}$$

where ρ is the charge per unit volume in some region of space and \mathbf{j} is the current density¹. ϵ_0 is a fundamental constant called the permittivity of free space.

In relativity theory the issue immediately arises, are these equations ok? Can we go ahead and apply them in any reference frame we might choose, or do they include a hidden assumption that one reference frame is preferred above others?

The answer turns out to be that the equations are fine as they are: they do not prefer one reference frame to another. To prove this, we can consider a change of reference frame, and work out how Maxwell's equations are affected. We already know how the position and time coordinates will change, and we know how the charge density and current density will change (because together they form a 4-vector, eq. (5.17), assuming that charge is conserved), so we can work out how Maxwell's equations and the Lorentz equation will look in the new coordinate system. After a lot of algebra, the answer turns out to be

$$\begin{aligned}
 \nabla' \cdot \mathbf{E}' &= \frac{\rho'}{\epsilon_0} \\
 \nabla' \cdot \mathbf{B}' &= 0 \\
 \nabla' \wedge \mathbf{E}' &= -\frac{\partial \mathbf{B}'}{\partial t'} \\
 c^2 \nabla' \wedge \mathbf{B}' &= \frac{\mathbf{j}'}{\epsilon_0} + \frac{\partial \mathbf{E}'}{\partial t'}, \tag{6.3}
 \end{aligned}$$

$$\mathbf{f}' = q(\mathbf{E}' + \mathbf{u}' \wedge \mathbf{B}') \tag{6.4}$$

where $\nabla' \cdot$ and $\nabla' \wedge$ are the div and curl operators in the primed coordinate system (i.e. $\nabla' = (d/dx', d/dy', d/dz')$), and \mathbf{E}' , \mathbf{B}' are given by equations (6.1).

Eq. (6.4) confirms that the symbols \mathbf{E}' and \mathbf{B}' refer to vector fields which fit the definition of electric and magnetic fields in reference frame S' . Eqs. (6.3) then confirm that the Maxwell equations are the same in the second reference frame as they were in the first one. Therefore every physical phenomenon they describe will show no preference for one reference frame above

¹In SI units the last equation is often written $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + (\partial \mathbf{E} / \partial t) / (\epsilon_0 \mu_0)$ where μ_0 is another constant called the permeability of free space, defined by $\mu_0 \equiv 1 / (\epsilon_0 c^2)$. In the SI system c , ϵ_0 and μ_0 all have exactly defined values; these are given in appendix 24.

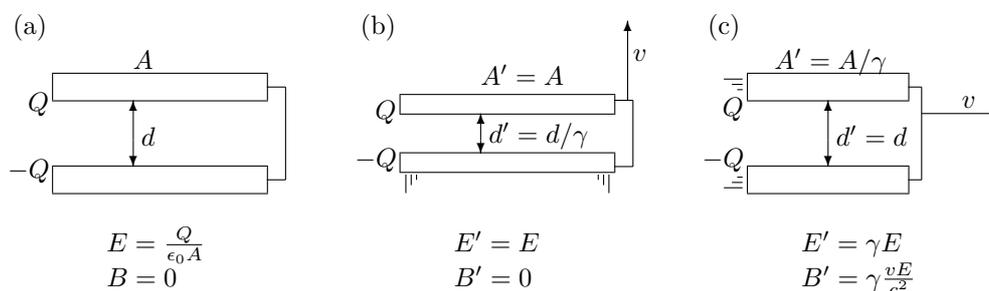


Figure 6.1: The moving capacitor. (a) The situation in frame S where the capacitor is at rest. (b) The situation in a frame moving along the field direction, normal to the plates. (c) The situation in a frame moving perpendicular to the field direction, parallel to the surface of the plates. The charges are invariant; the capacitor dimensions change as shown. (These simple cases are easily remembered and cover much of what one needs to know about field transformation.)

another, so the Principle of Relativity is upheld. The set of Maxwell equations is said to be *Lorentz covariant*. The word ‘covariant’ rather than ‘invariant’ is used for technical and historical reasons. One can think of it as expressing the idea that whereas all the bits and pieces in the equations ($\mathbf{E}, B, j, \rho, t, x, y, z$) do change from one reference frame to another, they all conspire together, or co-vary, in such a way that the form of the equations does not change.

The lengthy algebra we mentioned (but did not go into), to derive (6.3) and (6.1), can be considered a ‘brute force’ method to show that Maxwell’s equations are Lorentz covariant and to find out how the fields transform. One of the aims of this chapter is to introduce some powerful concepts and tools that will enable us to prove the former and to derive the latter a slicker way. We will re-express the equations using 4-vectors and the \square operator, so that their Lorentz covariance is obvious. This will make the result seem less like a ‘conspiracy’ and more like an elegant symmetry.

6.1.1 Moving capacitor plates

To get some insight into equations (6.1), let’s consider some simple cases. Consider for example a parallel plate capacitor, carrying charges $Q, -Q$ on two parallel plates of area A and separation d , at rest in reference frame S (see figure 6.1). The electric field between the plates of such a capacitor is uniform, directed perpendicular to the plates, and of size $E = Q/\epsilon_0 A$.

Now consider a reference frame S’ moving parallel to \mathbf{E} . The charges on the plates are invariant, the area is unchanged since it is transverse to the motion, while the plate separation is Lorentz-contracted to $d' = d/\gamma$. However, the electric field is independent of d' . One finds therefore

$E' = E$, in agreement with eq. (6.1)i.

Next suppose that instead of moving parallel to \mathbf{E} , S' moves relative to S in a direction perpendicular to \mathbf{E} (i.e. it moves parallel to the plates). Now $d' = d$ but the Lorentz contraction leads to $A' = A/\gamma$, therefore the charge per unit area on the plates is larger in S' , and we have $E' = \gamma E$, in agreement with eq. (6.1)ii.

In fact this simple argument from the capacitor plates is sufficient to prove (6.1)i and (6.1)ii in general when the relative velocity is either parallel to or perpendicular to \mathbf{E} , and there is no magnetic field in the first (unprimed) reference frame. This is because the field at a given point must transform in the same way, independent of what charges or movement of charge gave rise to it.

The case of S' moving in an arbitrary direction relative to the capacitor plates is treated in the exercises.

The capacitor example also illustrates the second term in equation (6.1)iv. A flat sheet of charge moving parallel to its own plane represents a sheet of current. It gives rise to a magnetic field above and below it, in a direction parallel to the sheet and perpendicular to the current, of size $\mu_0 I/2w$ where I is the current flowing through a width w of the sheet (this is easily proved from Ampere's Law or by integrating the field due to a wire). Applying this result to the case of a capacitor, we have two oppositely charged sheets moving at speed v in reference frame S' . For \mathbf{v} perpendicular to \mathbf{E} the magnetic fields of the two sheets add (in the region in between the capacitor plates) to give $B' = \mu_0 I'/w'$ where $I' = Qv/L'$ and L', w' are the dimensions of the plates in S' . Their product $w'L' = A' = A/\gamma$ owing to Lorentz contraction of L , so we have

$$B' = \frac{\mu_0 Qv}{A'} = \frac{\gamma QV}{c^2 \epsilon_0 A} = \gamma \frac{vE}{c^2}$$

in agreement with (6.1)iv.

Charge from nowhere?

Similar arguments can be made concerning the transformation of magnetic fields, but one needs to be more careful because there are more movements of charge to keep track of. Consider the following, which seems paradoxical at first. An ordinary current-carrying wire is electrically neutral, but has a current I in it. Therefore the 4-vector current density is $\mathbf{J} = (\rho c, \mathbf{j}) = (0, I/A)$, where A is the cross-sectional area of the wire. Now adopt some other reference frame, moving parallel to the wire, which we shall take to define the x axis. The Lorentz transformation gives the charge density in the new reference frame: it is $\rho' = \gamma(\rho - vj/c^2) = -\gamma vI/(Ac^2)$. This charge density is non-zero! So where did the charge come from? It wasn't there in the first reference frame; now it has 'magically' appeared.

Before we resolve this, consider another paradox. A stationary electron in the vicinity of the wire, say 1 metre from it, experiences no force in the first reference frame, since its velocity is

zero and a neutral wire does not produce an electric field. Therefore it does not accelerate. But now consider a reference frame moving at the drift velocity v of the electrons in the wire. This drift velocity is small. It is related to the current by $I = Anqv$ where n is the number density of electrons in the wire and q is the charge of an electron. For a typical metal such as copper, $n \simeq 8 \times 10^{28} \text{ m}^{-3}$, so for a 10 Amp current in a wire of diameter 1 mm, we find $v \simeq 1 \text{ mm/s}$. In the new reference frame the electron flow is zero, but now all the other parts of the wire (the nuclei and bound electrons) are in motion. They carry a net positive charge, so their motion constitutes a current $I' = \gamma I$, where the γ comes from the Lorentz transformation of \mathbf{J} . (We could neglect γ here because it is extremely close to 1, but let's keep it anyway). In the new frame, therefore, there is a magnetic field around the wire $B' = \mu_0 \gamma I / (2\pi r)$: this is an example of equation (6.1)iv. Now, the interesting part is that in the new reference frame, the electron situated near the wire is in motion, so it experiences a magnetic force! The force is

$$f' = qvB' = \frac{qv\mu_0\gamma I}{2\pi r}. \quad (6.5)$$

We find the B field is about 2 micro-tesla, and the force is $f' \simeq 3 \times 10^{-28}$ newton, leading to an acceleration approximately 350 ms^{-2} away from the wire. So, according to this argument, the wire will very quickly accelerate electrons in a large volume around it ... whereas in the first reference frame we found no such acceleration.

These two paradoxes are, of course, related. The non-zero charge density in the new reference frame is correct. It creates an electric field in the second frame and thus a further contribution to the force on any particle near the wire: this exactly balances the magnetic force we just calculated.

Figure 6.2 explains what is going on. An object that is overall electrically neutral but which carries a current must have two sets of charged particles in it: one positive and one negative. The overall neutrality, in a given reference frame S , means these sets have equal densities, $n_+ = n_- = n$, in S . The non-zero current means that one set of particles is moving and the other is not, or else they both move with different velocities. When we change to another reference frame, the Lorentz contraction is by a different amount for one set of particles than for the other, because of their different velocities. Indeed, in going from the frame where the copper nuclei are at rest to the frame where the conduction electrons are at rest, the nuclei get closer together while the conduction electrons *spread out* because we are transforming to their rest frame. So $n'_+ = \gamma n_+$ and $n'_- = n_- / \gamma$. The charge density in S' is then

$$\rho' = q_+ \left(\gamma n_+ - \frac{n_-}{\gamma} \right) = \gamma n q_+ \left(1 - \frac{1}{\gamma^2} \right) = \gamma n q_+ v^2 / c^2 = -\gamma j v / c^2$$

where we used $j = nqv = n(-q_+)v$. j is the current density in S , and q_+ is the charge on a proton. (Here j and v are in opposite directions so ρ' is positive.) This result agrees with the one we obtained by transforming \mathbf{J} .

To complete the analysis, let's check the electric field produced by this non-zero charge density. We have a line of charge, with charge per unit length $\lambda' = \rho' A$. The electric field at distance r

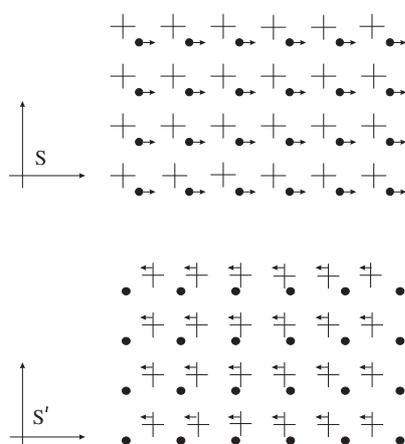


Figure 6.2: A neutral current-carrying wire consists of positive and negative charges of equal number density in frame S (upper diagram). The negative charges are shown as dots; the arrows indicate their drift velocity. In frame S' moving at the drift velocity the current is caused by the positive charges moving to the left. Compared with frame S , the lattice of positive charges suffers a Lorentz contraction, while the opposite happens to the negative charges (since in S they were moving and in S' they are not). Therefore in frame S' the wire is not neutral: it carries a net positive charge density. Charge is still conserved (count the dots and crosses!); the extra density has come at the expense of the charge distribution elsewhere, where the current flow must be in the opposite direction to complete the electrical circuit.

from such a line charge is

$$E' = \frac{\lambda'}{2\pi\epsilon_0 r} = \frac{\gamma n q_+ v^2 A}{2\pi\epsilon_0 c^2 r} = \frac{v\mu_0\gamma I}{2\pi r}.$$

Compare this with (6.5). You can see that the electric and magnetic forces in S' are everywhere balanced.

Such a perfect balance of forces that, if they were not balanced, would have substantial effects, should arouse our suspicion. It looks like a conspiracy, but we don't like conspiracies in Nature. We think they are a sign that we haven't got the right perspective on something. In this case the answer is that the two forces are not two but one: we must regard the electric and magnetic parts as two parts of one thing. If the "one force" is zero, then we have only ourselves to 'blame' for supposedly 'marvellous' effects if we start interpreting it as two forces. Of course we will find they are balanced.

The strength of materials

Let's examine another issue nicely illustrated by the parallel-plate capacitor. In section 4.1.1 we noted that a moving body loses its strength in the direction transverse to its motion. Now, most ordinary bodies are made of atoms, and the forces inside them, when they are stretched or compressed away from their natural length, are almost entirely electromagnetic in origin: a complicated combination of the electrostatic attractions between the unlike charges (nuclei and electrons), repulsions between the like charges, and the magnetic forces. It requires a quantum mechanical treatment to treat materials correctly, but to get a simple insight, suppose we argue that an attempt to break an ordinary object by pulling on it is somewhat like pulling apart a pair of capacitor plates. You shouldn't treat this simple idea as anything like a quantitative model of the structure of materials, but it does illustrate the kind of thing that happens to electromagnetic forces inside an object when it is set in motion.

For a stationary capacitor, the force on any given charge q in one of the plates is equal to q times the electric field due to the *other* plate (you can soon convince yourself that the forces from other charges within the same plate will cancel to very good approximation near the middle of a large enough plate). Therefore the force on such a charge is

$$f = qE_1 = \frac{qQ}{2\epsilon_0 A}.$$

where E_1 is the field due to the charges on one plate (this is half the total field between the plates). Now consider a reference frame in which the capacitor is moving in a direction parallel to the plates, i.e. perpendicular to \mathbf{E} . According to equation (6.1) the electric field between the plates is now *larger*, but according to equation (4.6)ii the force on the particle we picked is now *smaller*. What is going on? In the new reference frame there is a magnetic as well as an electric contribution to the force. The magnetic field due to either one of the plates on its own is

$$B_1 = \frac{\mu_0 I'}{2w'} = \frac{\gamma v E_1}{c^2}$$

Ask a silly question . . . “Who cares about the 3-force? It is just part of a 4-vector, and it is not really fundamental: it is a way of keeping track of momentum changes. If the spatial part of a 4-vector changes in some way, it is simply a hang-over from pre-spacetime thinking to agonise about this. We need to think in terms of the whole 4-vector, including the temporal part. The 4-vector \mathbf{F} is what it is, independent of reference frame.”

Answer. I agree with this position, up to a point. It is true that spacetime physics should be discussed with the right language, i.e. 4-vectors. However, in the application to physical examples we have to pick a reference frame. The fact that at high speeds the electric and magnetic contributions tend to cancel for transverse forces is memorable, and worth noticing. Also, we found that to treat the motion of particles subject to forces, the 3-force can sometimes offer the most direct route to the result.

and the charged particle now has speed v , in a direction perpendicular to \mathbf{B}'_1 . The magnetic force in this example has a direction opposite to the electric force. It follows that the total force on the particle in the new reference frame is

$$f' = q(E'_1 - vB'_1) = qE'_1(1 - v^2/c^2) \quad (6.6)$$

$$= q \frac{\gamma E_1}{\gamma^2} = \frac{qE_1}{\gamma}. \quad (6.7)$$

Thus the argument from Maxwell's equations does agree with the prediction from the Lorentz transformation of forces: physical objects get weaker in the transverse direction when they are in motion (see the box however for a comment on all this).

At speeds small compared to c , the magnetic contribution to the force is very much smaller than the electric contribution. Some people, on observing the factor v^2/c^2 in eq. (6.6), like to say that it is as if magnetic effects are a ‘relativistic correction’ to electric effects. When we put a current in a wire, and observe the magnetic field through its effect on a nearby compass needle, for example, one might say that we are observing at first hand the influence of a tiny relativistic correction! In practice magnetic effects can very often be traced to a moving electric charge². Since no magnetic monopoles have ever been discovered, and since motion is relative while charge is not, one may well feel that the electric field is the ‘senior partner’. I would prefer to say that magnetic and electric fields are two parts of a single thing, as I already mentioned, but it is good to be aware of the relative sizes of the effects. In the case of a current-carrying wire, the electrostatic effects have been cancelled extremely well by the presence of equal amounts of positive and negative charge in the wire, to a precision of order $v^2/c^2 \simeq 10^{-20}$, which allows us to see the tiny magnetic contribution.

At speeds approaching c , on the other hand, the electric and magnetic contributions have similar sizes.

² . . . but not always: it is found that magnetic dipoles are associated with the intrinsic spin angular momentum of charged particles; this spin cannot be associated with a movement of matter.

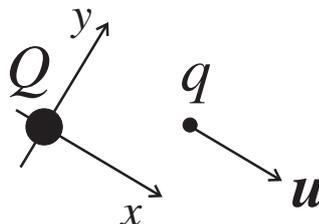


Figure 6.3: General situation for a test particle moving near a source particle. We can arrange the axes so that the test particle moves in the xy plane and parallel to the x direction.

6.2 The fields due to a moving point charge

[Section omitted in lecture-note version.]

General solution

Now let's attack the general problem. Place the test particle at an arbitrary location relative to the source, and give it an arbitrary velocity \mathbf{u} in frame S. Without loss of generality, we can place the origin of the S coordinate system at the source particle, and orient the axes so that the test particle moves in the xy plane with the x -axis parallel to \mathbf{u} , see figure 6.3. Let the coordinate system of S' be in the standard configuration with S, with relative velocity $\mathbf{v} = \mathbf{u}$.

Let x', y' be the coordinates of the test particle in S' . The coordinates of a general event at the test particle are therefore $(t', x', y', 0)$. Using the Lorentz transformation, such an event is at

$$t = \gamma(t' + vx'/c^2), \quad x = \gamma(vt' + x'), \quad y = y'. \quad (6.8)$$

We are interested in the force on the test particle in its rest frame S' , so we pick the time $t' = 0$ since then the origins of the two reference frames coincide so the source particle is at the origin of S' . This is useful because at this moment in S' the coordinates x', y', z' represent the position of the test particle *relative to the source particle*, not just relative to the origin. Eq. (6.8) then tells us that the event at which we want to evaluate the force is at

$$x = \gamma x', \quad y = y'.$$

This takes care of the issue we illustrated by the spacetime diagram in figure ??.

In frame S we have Coulomb's law, giving

$$f_{\parallel} = f_x = f \frac{x}{r}, \quad f_{\perp} = f_y = f \frac{y}{r} \quad (6.9)$$

where $f = qQ/(4\pi\epsilon_0 r^2)$. Now apply the force transformation (4.6):

$$\begin{aligned} f'_{\parallel} &= f'_x = \frac{f \frac{x}{r} (1 - vu/c^2)}{1 - uv/c^2} = \frac{fx}{r} \\ f'_{\perp} &= f'_y = \frac{f \frac{y}{r}}{\gamma(1 - uv/c^2)} = \frac{\gamma fy}{r} \end{aligned} \quad (6.10)$$

where we used $\mathbf{f} \cdot \mathbf{u} = f_x u$ and $\mathbf{v} = \mathbf{u}$. Expressing this result in the primed coordinates, including

$$r = (x^2 + y^2 + z^2)^{1/2} = (\gamma^2(x')^2 + (y')^2 + (z')^2)^{1/2}$$

we obtain

$$\begin{aligned} f'_x &= \frac{qQ\gamma x'}{4\pi\epsilon_0(\gamma^2(x')^2 + (y')^2 + (z')^2)^{3/2}}, \\ f'_y &= \frac{qQ\gamma y'}{4\pi\epsilon_0(\gamma^2(x')^2 + (y')^2 + (z')^2)^{3/2}}. \end{aligned} \quad (6.11)$$

We can gather these two equations together into the single vector result:

Electric field of point charge moving with constant velocity

$$\mathbf{E}' = \frac{\gamma Q \mathbf{r}'}{4\pi\epsilon_0(\gamma^2(x')^2 + (y')^2 + (z')^2)^{3/2}}. \quad (6.12)$$

where \mathbf{r}' is the vector (x', y', z') .

The general field transformation equations (6.1) give the same result, which you can see immediately because they would lead directly to eqs. (6.10).

The magnetic field of a moving point charge could be obtained by similar methods, but for brevity let's use eqs. (6.1), relying on the proof in section 6.3.1. We thus obtain

$$\mathbf{B}' = \frac{\mathbf{v} \wedge \mathbf{E}'}{c^2} \quad (6.13)$$

(this correctly matches both $\mathbf{B}'_{\parallel} = 0$ and $\mathbf{B}'_{\perp} = \gamma \mathbf{v} \wedge \mathbf{E}/c^2$ because the cross product only involves \mathbf{E}_{\perp} , and $\mathbf{E}'_{\perp} = \gamma \mathbf{E}_{\perp}$.) In the limit of low velocities, eqs. (6.12) and (6.13) lead to the Biot-Savart law.

Equation (6.12) has some remarkable properties. For one thing, it says the electric field due to a moving source particle is in a direction radially outward from the particle, see figure 6.4. This

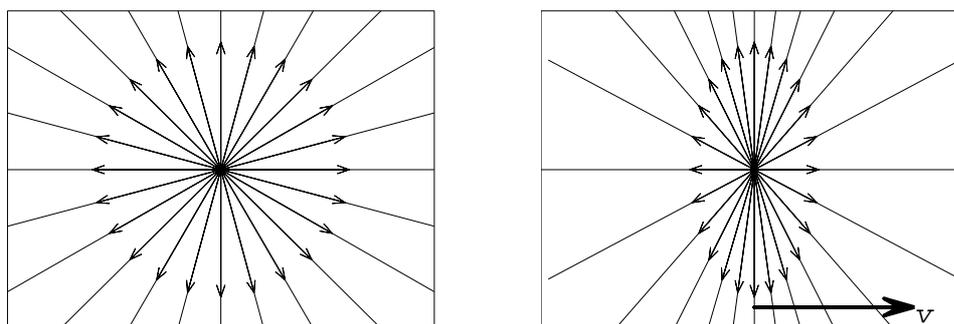


Figure 6.4: Electric field lines due to a stationary charge (left) and a moving charge (right). The lines are along the field direction; their density (per unit area in 3 dimensions) represents the field strength. A remarkable property is that the right diagram (moving charge) could be obtained by applying a Lorentz contraction to the left diagram (stationary charge).

seems sensible at first, but on reflection, one realises that the field has no business pointing outwards from the *present* location of the particle! The field at x', y', z' at time $t' = 0$ can only 'know about' or be caused by what the source particle was doing earlier on, in the past light cone. If one had to guess, one might guess that the field at any event t', x', y', z' would point in the direction away from the source's *earlier* position, not from where it is now. But instead the field seems to 'know' where the moving source is *now*. Of course we are discussing a *uniformly* moving source, so the information on where the source is going to be is contained in its past history, assuming the uniform motion continues. That the result should turn out so simple is however important. If the field were not radial from the present position, then a system of two particles moving uniformly abreast would exert a non-zero net total force on itself, leading to a self-acceleration in the absence of external forces. This would violate momentum conservation. The equations succeed in avoiding that situation. It is as if the source gives its 'marching orders' to the field in the form 'line yourself up on my future position, assuming that I will continue at constant velocity'. We shall re-examine this point in section 6.5.3.

Eq. (6.13) says that the magnetic field has a similar forward-back symmetry. It loops around the direction of motion of the charge, with a maximum strength at positions to the side, falling to zero in front and behind (figure 6.6).

We already noticed that the electric field is diminished in front and behind the moving particle, and enhanced at the sides. The next remarkable feature is that the size of these changes is just as if the field lines of a stationary particle had been 'squeezed' by a Lorentz contraction, see figure 6.4. The field lines from a point source transform like rigid spikes attached to the source. You should not deduce that this is a universal feature of electric field lines: just add a magnetic field in the first reference frame and this behaviour is lost. However, the picture does give a good insight into the way the Lorentz contraction of moving objects is brought about and embodied by the fields inside them.



Figure 6.5: “B of the Bang”, a sculpture in Manchester, England, designed by Thomas Heatherwick. The sculpture draws its inspiration from the explosive start of a sprint race at the “Bang” of the starting pistol, but to a physicist it is also reminiscent of the electric field due to a fast moving charged particle: perhaps a muon arriving in Manchester from a cosmic ray event. [Photo by Nick Smale.]

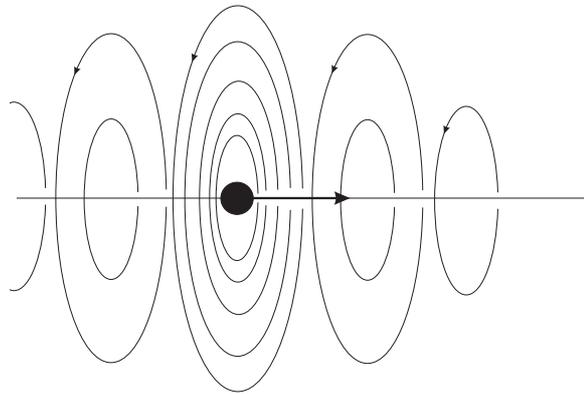


Figure 6.6: Magnetic field due to a uniformly moving point charge. The field lines loop around the line of motion of the charge. There is no magnetic field directly in front of or behind the charge.

In the ‘relativistic limit’, i.e. as the speed approaches c , a charged particle such as an electron appears like a stealthy pancake with a mighty force field around it. There is little sign of its approach, but as it whizzes by it exerts, for a moment, a powerful lateral force, like a shock wave. However, because this force appears in a short burst, the net impulse delivered is not enhanced, but varies in proportion to $1/v$ (see exercises).

6.3 Covariance of Maxwell’s equations

We already stated that Maxwell’s equations are ‘Lorentz covariant’: they take the same form in one reference frame as they do in another. However, when written down in the standard way, eqs. (6.2), this covariance is far from obvious. Now we shall develop some concepts that allow the covariance to be easily seen.

Any textbook of electromagnetism will tell you that the electric and magnetic fields can be obtained from two potentials ϕ and \mathbf{A} called the scalar and vector potential, through

$$\begin{aligned}\mathbf{E} &= -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \wedge \mathbf{A}.\end{aligned}\tag{6.14}$$

It is not hard to see where this idea comes from. If you look at M2 (the second Maxwell equation, (6.2)ii) you see that \mathbf{B} has zero divergence. This implies that \mathbf{B} can be written as the curl of something, so we write it that way and call the ‘something’ a ‘vector potential’ \mathbf{A} . You should also see that another vector $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi$ —for any scalar field χ —would be just as good, because it has the same curl: more on that in a moment. Next turn to Faraday’s law M3. Now it looks like

$$\nabla \wedge \mathbf{E} = -\frac{\partial}{\partial t} \nabla \wedge \mathbf{A}.$$

The order of differentiation with respect to time and space can be reversed, so this can be written

$$\nabla \wedge \left(\mathbf{E} + \frac{\partial\mathbf{A}}{\partial t} \right) = 0.$$

The combination in the bracket has zero curl, therefore it can be written as the gradient of something. We write the something $-\phi$ with ϕ called the ‘scalar potential’ (the minus sign comes in for convenience: it means this potential behaves like a potential energy per unit charge in electrostatics).

By using the potentials \mathbf{A} and ϕ , and eqs (6.14), we guarantee that, no matter what functional form we put into \mathbf{A} and ϕ , two of the Maxwell equations will be automatically satisfied! Our

work is reduced because now we only have to find four potential functions (ϕ and the three components of \mathbf{A}) instead of six field components.

When looking for solutions for \mathbf{A} and ϕ it proves to be very useful to keep in mind that we have some flexibility, as we already noted. We can add to \mathbf{A} any field with zero curl, without in the least affecting the \mathbf{B} field that is obtained from it, eq. (6.14)ii. However since \mathbf{A} influences \mathbf{E} as well we need to check what goes on there. You can easily confirm that we can keep the flexibility if both potentials are changed together, as

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi, \quad \tilde{\phi} = \phi - \frac{\partial\chi}{\partial t} \quad (6.15)$$

where χ is an arbitrary function. If the potentials are changed in this way, the derived fields are not changed at all. This is no more mysterious than the well-known fact that the gradient of a function does not change if you add a constant to the function, it is just that in three dimensions the possibilities are more rich. The change from \mathbf{A}, ϕ to $\tilde{\mathbf{A}}, \tilde{\phi}$ given in (6.15) goes by the fancy name of a ‘gauge transformation’. We say the electric and magnetic fields are ‘invariant under gauge transformations’. A simple example is to shift the scalar potential by a constant: $\tilde{\phi} = \phi + V_0$. This is a gauge transformation with $\chi = -V_0t$.

Now, anyone studying Relativity who comes across a vector paired with a scalar, and who sees eq. (6.15), begins to suspect that we have a 4-vector in play. Let’s see if it works. We form the ‘4-vector potential’

$$\mathbf{A} \equiv (\phi/c, \mathbf{A}) \quad (6.16)$$

and note that the gauge transformation equation (6.15) can be written

$$\tilde{\mathbf{A}} = \mathbf{A} + \square\chi. \quad (6.17)$$

We haven’t yet proved that \mathbf{A} is a four-vector, but the fact that we can write the gauge transformation in four-vector notation is promising.

Next we shall plug the forms (6.14) into Maxwell’s equations M1 and M4 (eqs 6.2i and iv). One obtains

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = \frac{\rho}{\epsilon_0}, \quad (6.18)$$

$$c^2\nabla(\nabla \cdot \mathbf{A}) + \frac{\partial}{\partial t}\nabla\phi + \frac{\partial^2\mathbf{A}}{\partial t^2} - c^2\nabla^2\mathbf{A} = \frac{\mathbf{j}}{\epsilon_0}. \quad (6.19)$$

As things stand this does not look very simple! However, the second equation is suggestive. The last two terms look like $-c^2\square^2$ acting on \mathbf{A} (recall that the d’Alembertian \square^2 was defined in

(5.26)). The trouble is that we also have the first two terms, which together form the 3-gradient of $(c^2 \nabla \cdot \mathbf{A} + \partial\phi/\partial t)$. Now we take a clever step. We are going to take advantage of the idea of gauge transformation. We recall that we have some flexibility in picking the potential functions, and we propose that by taking advantage of this flexibility it is always possible to arrange that

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial\phi}{\partial t}. \quad [\text{Lorenz gauge}] \quad (6.20)$$

When we impose this condition, the first two terms in (6.19) cancel and the equation reduces to the simple form

$$\square^2 \mathbf{A} = \frac{-\mathbf{j}}{c^2 \epsilon_0}. \quad (6.21)$$

You can also confirm that (6.18) becomes

$$\square^2 \phi = \frac{-\rho}{\epsilon_0}. \quad (6.22)$$

Equation (6.20) is called the *Lorenz gauge condition*³ and imposing it is called ‘choosing the Lorenz gauge’. One needs to be aware that once such a gauge choice has been made, results based on it no longer have the full flexibility offered by eqs. (6.15). However that is merely a statement about the potentials. The fields that are obtained through any given choice of gauge are completely valid and ‘care nothing’ about how they were calculated.

Before commenting on the beautifully simple (6.21) and (6.22) we need to check that it is always possible to impose the Lorenz gauge condition. To this end, first suppose we have some arbitrary \mathbf{A} and ϕ not necessarily in the Lorenz gauge. They have

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial\phi}{\partial t} = f(\mathbf{r}, t)$$

for some function f . Let’s try a gauge transformation and see what happens:

$$\nabla \cdot \tilde{\mathbf{A}} + \frac{1}{c^2} \frac{\partial\tilde{\phi}}{\partial t} = f(\mathbf{r}, t) + \nabla^2 \chi - \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2}.$$

It follows that we can achieve the Lorenz condition as long as χ can be chosen such that it satisfies the equation

$$\frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} - \nabla^2 \chi = f.$$

³The gauge condition (6.20) was derived and exploited by Ludvig Lorenz in 1867. However it is commonly named the Lorenz gauge, after Hendrik Lorentz (1853–1928). It seems somehow unfair to Lorenz to perpetuate that terminology; see Jackson’s book for further comments.

This is a wave equation with f as source. The important point is that it is known that there always exist solutions to this equation, no matter what form the source function f takes. The method of solution is explained in section 6.5.2. It follows that we can always adjust the potentials so that they satisfy the Lorentz gauge condition.

Equations (6.21) and (6.22) are beautiful because they are uncoupled (you can solve them for ϕ on its own, and then for \mathbf{A} on its own) and because they are both wave equations with a source term, for which powerful methods of solution exist. Furthermore, they open the way to writing down Maxwell's equations in a 4-vector notation that makes their Lorentz covariance explicit and obvious.

We already learned in chapter 5 that for a conserved quantity such as electric charge, the combination $(\rho c, \mathbf{j})$ is a 4-vector. We can write all the formulae leading up to (6.21) and (6.22) in 4-vector notation. We have

$$\mathbf{J} = (\rho c, \mathbf{j}), \quad \mathbf{A} = (\phi/c, \mathbf{A}).$$

The Lorentz gauge condition is $\square \cdot \mathbf{A} = 0$, and the final result is

Maxwell's equations

$$\square^2 \mathbf{A} = \frac{-1}{c^2 \epsilon_0} \mathbf{J}, \quad \text{with } \square \cdot \mathbf{A} = 0. \quad (6.23)$$

This equation does two jobs at once. First it shows that \mathbf{A} is indeed a 4-vector as we suspected (because we already know that \mathbf{J} is a 4-vector, c^2 and ϵ_0 are constants, and we know \square^2 is a Lorentz scalar operator). Secondly, it expresses *all* of Maxwell's equations in one go, in explicitly Lorentz covariant form! I say 'all' because we already noted that two of the equations were already taken care of when adopting the potentials, so there are only two left to worry about. The point is that we can see immediately that a change of reference frame will give the equation $\square'^2 \mathbf{A}' = -\mathbf{J}'/(c^2 \epsilon_0)$, i.e. the same equation with primed symbols, and therefore, by reversing the argument, we would obtain Maxwell equations in their 3-vector form just as we claimed in eqs (6.3).

Coulomb gauge

We picked the Lorentz gauge above because it leads to a simple statement of Maxwell's equations. For some calculations, another choice of gauge (i.e. choice of constraint to impose on \mathbf{A}) can be more convenient. There is an infinite variety of constraints one could choose. One that has proved sufficiently useful to earn a name is the **Coulomb gauge**, also called *radiation gauge*, where the constraint is

$$\nabla \cdot \mathbf{A} = 0, \quad [\text{Coulomb gauge} \quad (6.24)$$

i.e. the divergence of the 3-vector potential is zero. Note, this is a three-vector equation. Therefore if the potentials are in Coulomb gauge in one inertial frame, they are not guaranteed to be in Coulomb gauge in all inertial frames. This does not make the calculations invalid: the fields are obtained correctly, no matter what gauge is adopted.

If the scalar potential is independent of time then the potentials can satisfy both Lorentz and Coulomb gauge conditions.

The proof that it is always possible to find a gauge transformation so as to satisfy the Coulomb gauge condition is treated in the exercises. In the Coulomb gauge, the first Maxwell equation (6.18) becomes Poisson's equation

$$\nabla^2 \phi = -\rho/\epsilon_0.$$

This is the same equation as one would obtain in electrostatics, but now we are treating general situations! If ρ changes with time, the influence on ϕ happens instantaneously in the Coulomb gauge. However, the influence on the *fields* is not instantaneous: once the contribution of both the scalar and the vector potential is taken into account, one gets the same result as one would in any other gauge, i.e. light-speed-limited cause and effect.

6.3.1 Transformation of the fields: 4-vector method

[Section omitted in lecture-note version.]

6.4 Electromagnetic waves

We have already referred repeatedly to the phenomena of electromagnetic radiation. Next we shall look briefly at the relationship between electromagnetic waves and Maxwell's equations, and derive some properties of the fields.

First we shall derive the existence of electromagnetic plane waves, assuming the Maxwell equations as a starting point. The quickest way is simply to present them as trial functions and prove that they are solutions.

It is convenient to write a general electromagnetic plane wave using the complex number notation

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (6.25)$$

where \mathbf{E}_0 and \mathbf{B}_0 are constant vectors, independent of both time and space, as is \mathbf{k} , the wave vector. It is understood that the physical fields are given by the real part of this solution,

$\mathbf{E}_{\text{observed}} = \Re[\mathbf{E}]$, $\mathbf{B}_{\text{observed}} = \Re[\mathbf{B}]$. If the constant vectors \mathbf{E}_0 and \mathbf{B}_0 are real then the plane waves are linearly polarized; if one allows \mathbf{E}_0 and \mathbf{B}_0 to be complex then one can treat any type of polarization. The waves are plane because we are assuming \mathbf{k} is constant, so the wavefronts are flat and the direction of propagation is everywhere the same.

It is very easy to ‘plug’ this trial solution into Maxwell’s equations if one once learns (e.g. by exhaustive coordinate analysis) that for vectors \mathbf{a} , \mathbf{k} that are independent of time and position (i.e. they are constants) and constant ω :

$$\frac{\partial}{\partial t} \left(\mathbf{a} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right) = -i\omega \mathbf{a} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (6.26)$$

$$\nabla \cdot \left(\mathbf{a} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right) = i\mathbf{k} \cdot \mathbf{a} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \quad (6.27)$$

$$\nabla \wedge \left(\mathbf{a} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right) = i\mathbf{k} \wedge \mathbf{a} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (6.28)$$

It is useful to learn these, and they are easy to remember. They are saying that, in the case of the function “position-independent vector times $\exp(i\mathbf{k}\cdot\mathbf{r})$ ” the ∇ operator performing a div or curl acts just like the vector \mathbf{k} producing a scalar or vector product. This makes the process of putting our trial solution in to Maxwell’s equations in free space extremely easy. In the case of waves in free space (zero charge and current density), we find by using the above and dividing out the exp function:

M1:	$i\mathbf{k} \cdot \mathbf{E}_0 = 0.$	\mathbf{E} is orthogonal to the wave vector.
M2:	$i\mathbf{k} \cdot \mathbf{B}_0 = 0.$	\mathbf{B} is orthogonal to the wave vector.
M3:	$i\mathbf{k} \wedge \mathbf{E}_0 = i\omega \mathbf{B}_0$	\mathbf{E} , \mathbf{B} mutually orthogonal, $E_0 = (\omega/k)B_0$
M4:	$ic^2 \mathbf{k} \wedge \mathbf{B}_0 = -i\omega \mathbf{E}_0$	$\omega = kc$, $E_0 = cB_0$

The last equation (M4) on its own gives a statement about the mutual directions and it says the sizes are related by $c^2 k B = \omega E$. The directions are consistent with M3, and the sizes agree with M3 as long as $c^2 k = \omega c$, leading to the conclusion $\omega = kc$ and $E_0 = cB_0$ that has been given on the last line of the table.

Since the above are all mutually consistent, they confirm that the trial solution is indeed a solution, and we find the constraints on the plane waves: they must be transverse (with \mathbf{E} , \mathbf{B} , \mathbf{k} forming a right-handed set) the sizes of the fields must be ‘equal’, i.e. related by $|E_0| = c|B_0|$, and the phase velocity ω/k must be equal to c .

In terms of the 4-vector potential, the Maxwell equations (6.23) in free space ($\mathbf{J} = 0$) give the wave equation, so it is no surprise that there are plane wave solutions

$$\mathbf{A} = \mathbf{A}_0 e^{i\mathbf{K}\cdot\mathbf{X}}$$

where \mathbf{A}_0 is a constant 4-vector amplitude. In order to get the simple form $\square^2 \mathbf{A} = 0$ for the Maxwell equations we must use the Lorenz gauge $\square \cdot \mathbf{A} = 0$ (eq. (6.20)), which means we have

the constraint

$$\square \cdot \mathbf{A} = i\mathbf{K} \cdot \mathbf{A} = 0 \quad \Rightarrow \quad \mathbf{K} \cdot \mathbf{A}_0 = 0. \quad (6.29)$$

Therefore in Lorenz gauge the waves of \mathbf{A} are ‘transverse’ in spacetime. Often a polarization 4-vector ε is introduced, such that

$$\mathbf{A} = A\varepsilon, \quad (6.30)$$

and then the Lorenz condition is

$$\varepsilon \cdot \mathbf{K} = 0 \quad \Rightarrow \quad \varepsilon^0 = \varepsilon \cdot \mathbf{k} \frac{c}{\omega} = \varepsilon \cdot \hat{\mathbf{k}}. \quad (6.31)$$

Note that ε can have a component along \mathbf{k} . This possibility is called *longitudinal polarization*. It does not mean the fields have longitudinal polarization: they remain transverse.

In free space we can always choose that the scalar potential is zero, $\phi = 0$ (in addition to the Lorenz gauge condition) since there exists a gauge transformation within the Lorenz gauge that accomplishes this (see below). Then $\square \cdot \mathbf{A} = \nabla \cdot \mathbf{A}$ so the Coulomb gauge condition is satisfied as well. In this case the polarization vector has $\varepsilon^0 = 0$ and then (6.31) implies that ε is transverse (i.e. $\mathbf{A} \cdot \mathbf{k} = 0$).

Problem. A plane wave in free space is described by a 4-vector potential $\mathbf{A} = \mathbf{A}_0 \exp(i\mathbf{K} \cdot \mathbf{X})$ satisfying the Lorenz gauge condition, with $A^0 = \phi/c \neq 0$. Find a gauge change $\mathbf{A} \rightarrow \tilde{\mathbf{A}}$ that results in a 4-potential still in Lorenz gauge, but with $\tilde{\phi} = 0$.

Solution. Since we want to get rid of ϕ , we suggest the gauge function $\chi = \int \phi dt$, so that $\partial\chi/\partial t = \phi$. In order to stay in the Lorenz gauge we need this χ to satisfy the wave equation. It does, because $\square^2\chi = \square^2 \int \phi dt = \int \square^2\phi dt$ which is zero because here ϕ satisfies the wave equation.

More generally, a change of 4-polarization by

$$\varepsilon \rightarrow \varepsilon + a\mathbf{K}, \quad (6.32)$$

where a is an arbitrary constant, amounts to a gauge change and therefore does not affect the fields. Since \mathbf{K} is null the 4-potential remains within the Lorenz gauge. In this way one can always arrange that one of the components of ε is zero. The Lorenz condition gives a further

constraint, and therefore there remain just two independent components of the polarization 4-vector.

We have already discovered some of the kinematics of these plane wave solutions, through our study of the headlight effect and the Doppler effect, and the energy falling into a bucket. A Lorentz transformation applied to the 4-wave-vector, and equations (6.1) to transform the fields, must reproduce all those effects. For example, suppose a linearly polarized plane wave has its electric field along the y direction, its magnetic field along the z direction, and propagates along the x direction. In another reference frame S' in standard configuration with the first, one finds

$$\begin{aligned} E'_x = E'_z = 0, & & E'_y &= \gamma(E_0 - vB_0) e^{i\varphi} = \gamma(1 - \beta)E_0 e^{i\varphi} \\ B'_x = B'_y = 0, & & B'_z &= \gamma(B_0 - vE_0/c^2) e^{i\varphi} = \gamma(1 - \beta)B_0 e^{i\varphi} \end{aligned}$$

where the phase $\varphi = kx - \omega t = k'x' - \omega't'$ is an invariant. Notice the similarity with the longitudinal Doppler effect: the field amplitudes transform in the same way as frequency.

We shall show in section 12.2 that the intensity (power per unit area) is proportional to $\mathbf{E} \wedge \mathbf{B}$, so we have $I' = \gamma^2(1 - \beta)^2 I$, in agreement with eq. (5.30).

6.5 Solution of Maxwell's equations for a given charge distribution

We shall now use the potentials to get some more information about electromagnetic fields. The idea is not to attempt a full presentation of electromagnetism, but to investigate how it relates to Special Relativity. A common type of problem would be of the form, "given that there are charges here and here, moving thus, what can you tell me about the fields?" That is, we would like to solve the equations in such a way that we can obtain the fields from the given information about the charges and currents.

An important example is the case of no charge and no current. One possible solution for this case is zero field everywhere, but that is not the only solution: putting zero on the right hand side of (6.23) results in a wave equation. This has many rich solutions, in the form of waves of ϕ and \mathbf{A} propagating around at the speed of light. Therefore in vacuum the fields also can have forms that propagate as waves at the speed of light, as we saw in the previous section.

Another simple case is that of a single point charge in uniform motion. We studied this in section 6.2. It will serve as a useful introduction to methods based on potentials.

6.5.1 The four-vector potential of a uniformly moving point charge

As in section 6.2 we suppose a point charge is at rest in one reference frame and therefore moving in another. However here we will choose the primed frame S' to be the one in which the source particle is at rest, instead of S as before. We are not being perverse, it is simply that we are preparing now for a more general treatment in which we want to learn the potentials in a given reference frame in terms of the charge and current distribution in that frame. It will save a lot of clutter if we adopt unprimed symbols for the reference frame that is the 'final destination' of our calculation.

So, suppose a charge q is at rest in frame S' , and this frame is in standard configuration with S . Then the charge is moving along the x -axis of S with speed v . The potentials for the case of a point charge at rest are

$$\phi' = \frac{q}{4\pi\epsilon_0 r'}, \quad \mathbf{A}' = 0. \quad (6.33)$$

By applying an inverse Lorentz transformation to the 4-vector \mathbf{A}' we obtain

$$\begin{aligned} \phi &= \gamma(\phi' + vA'_x) = \gamma \frac{q}{4\pi\epsilon_0 r'} \\ A_x &= \gamma(v\phi'/c^2 + A'_x) = v\phi/c^2, \\ A_y &= A_z = 0. \end{aligned} \quad (6.34)$$

Now

$$r' = ((x')^2 + (y')^2 + (z')^2)^{1/2} = (\gamma^2(x - vt)^2 + y^2 + z^2)^{1/2}$$

(by Lorentz transformation of the coordinates) so

$$\begin{aligned} \phi &= \frac{q}{4\pi\epsilon_0} \frac{\gamma}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{1/2}}, \\ \mathbf{A} &= \mathbf{v}\phi/c^2. \end{aligned} \quad (6.35)$$

Note that the source particle is located at $\mathbf{r}_0 = (vt, 0, 0)$ at any given time t in S .

Now we apply eqs. (6.14) to find the fields. One obtains

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\gamma(\mathbf{r} - \mathbf{r}_0)}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}} \quad (6.36)$$

in agreement with (6.4), and⁴

$$\mathbf{B} = \frac{q}{4\pi\epsilon_0 c^2} \frac{\gamma \mathbf{v} \wedge (\mathbf{r} - \mathbf{r}_0)}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}}. \quad (6.37)$$

One can notice that $\mathbf{B} = \mathbf{v} \wedge \mathbf{E}/c^2$, as previously remarked.

So what have we learned from this? We knew the fields already (section 6.2), though perhaps the new method of calculation is (marginally) simpler. The more important point is that we have the potentials, eqs. (6.35). They will prove to be very useful in what follows.

6.5.2 The general solution

So far we have mentioned two types of solution to the Maxwell equations: the waves in free space, and the field due to a uniformly moving point charge. Next we shall consider the general solution for the type of problem where the distribution of charge and current is known.

Our aim is to solve equations (6.21) and (6.22), which we shall rewrite here for convenience:

$$\square^2 \phi = \frac{-\rho}{\epsilon_0}, \quad \square^2 \mathbf{A} = \frac{-\mathbf{j}}{c^2 \epsilon_0}. \quad (6.38)$$

There are four equations (3 for the components of \mathbf{A} and 1 for ϕ) but they are all of the same form,

$$\frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} - \nabla^2 f = s(\mathbf{r}, t). \quad (6.39)$$

This equation is called the ‘inhomogeneous wave equation’ or ‘wave equation with a source term’. We want to solve such equations for the unknown function $f(\mathbf{r}, t)$ when the source function s has been given.

Treatment of the wave equation

To get the general idea, first consider the situation of electrostatics, i.e. there are just fixed charges and no currents, with no time dependence. In this case the vector potential is zero,

⁴The vector in the numerator of \mathbf{B} is found to be $(0, -z, y)$ multiplied by v ; here owing to the fact that the source travels through the origin, \mathbf{r}_s and \mathbf{v} are parallel so one can write this either as $\mathbf{v} \wedge \mathbf{r}$ or as $\mathbf{v} \wedge (\mathbf{r} - \mathbf{r}_0)$. A shift of origin must not affect the result, however, so the latter form is more general.

and equation (6.38) for the scalar potential becomes the Poisson equation

$$\nabla^2 \phi = \frac{-\rho}{\epsilon_0} \quad (6.40)$$

since $\partial\phi/\partial t = 0$. We know that the potential due to a non-moving point charge is $\phi = q/4\pi\epsilon_0 r$ where r is the distance from the charge to the point where the potential is to be evaluated. We say r is the distance from the *source point* to the *field point*. The potential due to a set of charges can be obtained simply by adding the contributions from each charge. This follows from the fact that the Poisson equation is linear. We can consider any charge distribution ρ to be made of many tiny elements, each containing an amount of charge $dq = \rho dV_s$ where dV_s is a volume element at the source point. Therefore the solution for the potential can be written

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}_s)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_s|} dV_s. \quad (6.41)$$

This method of solution, by dividing up the source function ρ into many tiny pieces, is called Green's method, and one can see that it will work whenever the differential equation is linear. The function

$$\frac{-1}{4\pi |\mathbf{r} - \mathbf{r}_s|}$$

is called the Green function (or Green's function) for Poisson's equation. It is the solution of (6.40) when the right hand side takes the form of a sharp spike having unit volume, i.e. a δ -function.

The inhomogeneous wave equation is linear, so it can be tackled by Green's method. To use the full method, we would start by finding the solution of the wave equation when the source term is concentrated in a tiny region of both space and time. However it saves a little working if we use some general knowledge of waves to jump straight to a solution where the source is unrestricted in time. That is, we suppose the function s on the right hand side of (6.39) can have any time dependence, but it is zero everywhere except near one spatial point, which we may as well take to be the origin. This means that elsewhere, away from the origin, the differential equation is just the wave equation in free space. We already know that this has plane wave solutions, but they are not the solutions we need here because they won't have the right behaviour near our source at $r = 0$. However, another type of wave is the spherical wave, which has the general form

$$f = \frac{g(t - r/c)}{r} \quad (6.42)$$

and this does have a non-trivial behaviour near $r = 0$. You can check that this is a solution of $\square^2 f = 0$ for any function g , except at the origin: see box.

Spherical waves

We seek a spherically symmetric solution to the wave equation $\square^2 f = 0$. For spherical symmetry, the function f does not depend on angles, so the Laplacian reduces to

$$\nabla^2 f \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) = \frac{2}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rf).$$

Now let $u = rf$ and substitute into $\square^2 f = 0$. We have

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2} = 0.$$

This is the one-dimensional wave equation. Its general solution is $u(r, t) = g(t - r/c) + h(t + r/c)$ where g, h are arbitrary functions. The general spherically-symmetric solution of the 3-dimensional problem is therefore

$$f = \frac{g(t - r/c)}{r} + \frac{h(t + r/c)}{r}.$$

This solves the wave equation everywhere except at the origin ($r = 0$) which requires special consideration: see main text. The $t - r/c$ dependence means that g gives waves propagating towards positive r , i.e. outwards from the origin; h gives waves propagating inwards towards the origin. These are also called the *retarded* and *advanced* parts of the solution, respectively. For a situation in which the waves are caused by a source at the origin, the h function is zero: the solution is purely retarded.

Physically this corresponds to waves excited by a point source that oscillates with some time dependence described by the function g . The waves travel outwards from the source, with speed c and spherical wavefronts. The $1/r$ factor means they diminish in amplitude as they go, thus ensuring energy conservation. Another solution is $h(t + r/c)/r$ for any function h : this corresponds to waves collapsing in towards the origin.

We would like to evaluate $\square^2 f$. We already know that this is zero for $r \neq 0$ (see box—but you might like to evaluate it explicitly as a check), but now we want to examine the region $r \rightarrow 0$. To this end, let's take a look at the first derivatives with respect to r and t :

$$\begin{aligned} \frac{\partial}{\partial r} \left(\frac{g(t - r/c)}{r} \right) &= -\frac{g(t - r/c)}{r^2} - \frac{1}{c} \frac{g'(t - r/c)}{r}, \\ \frac{\partial}{\partial t} \left(\frac{g(t - r/c)}{r} \right) &= \frac{g'(t - r/c)}{r}, \end{aligned}$$

where g' refers to the first derivative of the function g . In the limit $r \rightarrow 0$, the g/r^2 term dominates all the others, unless g' tends to infinity. We will assume g varies smoothly, never having an abrupt change, and therefore g' is finite and hence we only need to keep the g/r^2

term. Applying this argument again to the second derivative, we have

$$\lim_{r \rightarrow 0} \square^2 \left(\frac{g(t-r/c)}{r} \right) = \lim_{r \rightarrow 0} \square^2 \left(\frac{g(t)}{r} \right) = g(t) \lim_{r \rightarrow 0} \nabla^2 \left(\frac{1}{r} \right) \quad (6.43)$$

where in the second step we used that the infinite spatial derivative must dominate the finite time derivative⁵.

Now let's take a look at $\nabla^2(1/r)$:

$$\begin{aligned} \nabla^2 \left(\frac{1}{r} \right) &= \nabla \cdot \left(\nabla \frac{1}{r} \right) = \nabla \cdot \left(\frac{-1}{r^2} \hat{\mathbf{r}} \right) = -\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = -\nabla \left(\frac{1}{r^3} \right) \cdot \mathbf{r} - \frac{\nabla \cdot \mathbf{r}}{r^3} \\ &= \frac{3r}{r^4} - \frac{3}{r^3} = \frac{3}{r^3} \left(\frac{r}{r} - 1 \right). \end{aligned}$$

It is extremely tempting to evaluate the bracket as equal to zero, and hence obtain $\nabla^2(1/r) = 0$. This is correct almost everywhere. However, at $r = 0$ it is not legitimate and our expression is simply ill-defined. We can figure out what is going on at $r = 0$ by noticing that we are now handling a problem exactly the same as the one that arises in Poisson's equation. Here is the comparison:

$$\begin{array}{lll} \text{Poisson} & \epsilon_0 \nabla^2 \phi = -\rho, & \epsilon_0 \phi = \frac{q}{4\pi r}, \quad q = \int \rho dV. \\ \text{wave} & \nabla^2 f = ?, & f = \frac{g}{r}, \quad g = ? \end{array}$$

where ϕ gives the solution for a charge density ρ that is concentrated in a δ -function 'spike' at the origin. By making the comparison, we deduce that g must correspond to a 'charge', and s to a 'charge density', and $\nabla^2(1/r)$ will be not zero but infinite at the origin.

There is a nice trick which serves to bring this out. Rather than working with the troublesome $\nabla^2(1/r)$ directly, we integrate it over a volume of space and then apply Gauss' theorem:

$$\int \nabla^2 \left(\frac{1}{r} \right) dV = \int \nabla \cdot \left(-\frac{\hat{\mathbf{r}}}{r^2} \right) dV = \int \frac{-1}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{S}.$$

Now choose the volume integrated over to be a sphere centred at the origin. The surface integral then evaluates to the surface area of the sphere, and we find

$$\int \nabla^2 \left(\frac{1}{r} \right) dV = -4\pi. \quad (6.44)$$

⁵We can replace $g(t-r/c)$ by $g(t)$ here because the spatial dependence of $g(t-r/c)$ only introduces terms of the form g'/r , which we just showed are negligible.

The volume integral thus ‘tames’ the function, and we conclude that $\nabla^2(1/r)$ is not zero at the origin, but takes such a value there that its volume integral is finite and equal to -4π .

Substituting this solution into the wave equation (6.39) which was our starting point, we have

$$g(t) \lim_{r \rightarrow 0} \nabla^2 \left(\frac{1}{r} \right) = -s(t). \quad (6.45)$$

Therefore, using (6.44), we have

$$4\pi g(t) = \int s(t) dV. \quad (6.46)$$

This relates the function g appearing in our solution ($f = g(t - r/c)/r$) to the source term $s(t)$ in the equation we are trying to solve.

It is convenient to absorb the 4π factor by defining $q \equiv 4\pi g$, then we have $f = q(t - r/c)/4\pi r$. The overall conclusion is as follows.

If the source in the inhomogenous wave equation is concentrated at a point in space but has an arbitrary time dependence $s(t)$ of total strength

$$q(t) = \int s(t) dV,$$

then a solution of (6.39) is

$$f(\mathbf{r}, t) = \frac{q(t - r/c)}{4\pi r}. \quad (6.47)$$

This solution looks just like the Coulomb potential, except instead of evaluating the ‘charge’ q at the time t , it is evaluated at the ‘retarded’ time $t - r/c$. The interpretation is that the potential at a given position receives waves from the source, and they take time to get there. This makes sense, it is the mathematical expression of the cause–effect relationship between the source and the potential, with a finite speed for signals.

Another solution exists, with ‘advanced’ time $t + r/c$, but this corresponds to waves moving in towards the source, so it does not correspond to the physical situation we are treating.

We can now complete the Green method and deduce that for any given source function (now spread out in space and time), the solution to the wave equation (6.39), with retarded potentials,

is

$$f(\mathbf{r}, t) = \int \frac{s(\mathbf{r}_s, t - |\mathbf{r} - \mathbf{r}_s|/c)}{4\pi|\mathbf{r} - \mathbf{r}_s|} dV_s. \quad (6.48)$$

Application to Maxwell's equations

Using (6.48), we are now in a position to write down the solutions we wanted, for given charge and current distributions in Maxwell's equations. The complete story is given in the box.

Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \wedge \mathbf{E} &= -\frac{d\mathbf{B}}{dt}, & c^2 \nabla \wedge \mathbf{B} &= \frac{\mathbf{j}}{\epsilon_0} + \frac{d\mathbf{E}}{dt}. \end{aligned}$$

Their solution:

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{B} &= \nabla \wedge \mathbf{A}, \\ \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}_s, t - r_{sf}/c)}{r_{sf}} d^3\mathbf{r}_s \\ \mathbf{A}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0 c^2} \int \frac{\mathbf{j}(\mathbf{r}_s, t - r_{sf}/c)}{r_{sf}} d^3\mathbf{r}_s \end{aligned} \quad (6.49)$$

where $r_{sf} = |\mathbf{r} - \mathbf{r}_s|$.

One can verify that the potentials written here do satisfy the Lorenz gauge condition (6.20).

It might seem to be unwarranted to call (6.49) 'the solution' of Maxwell's equations, because it still leaves some work to do: we have to carry out the integrals, and having done that we have to differentiate to get the fields. However, in principle an integral is nothing more nor less than adding up lots of tiny bits, and the equation tells us precisely what has to be added up: the amount of charge (for ϕ), or of current (for \mathbf{A}) at the event $(t - r_{sf}/c, \mathbf{r}_s)$, divided by r_{sf} , and we have to sum over all source points \mathbf{r}_s . Differentiation is even more straightforward. This is an explicit set of instructions, as opposed to the very different sort of demand "solve this partial differential equation".

To write down the integral, we had to pick a reference frame in order to allow us to talk about things like distance, volume, and charge density. Obviously the integral is designed to tell you what the potentials are *in that reference frame*, but it doesn't matter what reference frame you pick. This fact can be made self-evident by writing the whole problem, and its solution, in 4-vector notation. The second box, eqs. (6.50), shows this. To write the relationship between the

fields and the potentials we used the “field tensor” \mathbb{F} that will be introduced in chapter 12: don’t worry about it yet; it is included here for future reference. Its relation to the potential takes care of the second and third Maxwell equations; the other two are given by the $\square^2 A$ equation (recall section 6.3). The integral used to calculate the 4-vector potential is still written in (6.50) in terms of (frame-dependent) distances rather than 4-vector intervals. This is to preserve clarity. You should notice that the integral is not simply an ‘integral over position’, because the source position \mathbf{r}_s is being examined at the retarded time $t - r_{sf}/c$. The set of events contributing to the integral are therefore all on the past light cone of the field point. This set of events has nothing to do with any choice of reference frame. The distances r_{sf} will admittedly change from one reference frame to another, as will the volume element. Therefore we rely on the fact that we already know A is a 4-vector to feel satisfied that the result obeys the Principle of Relativity.

The last equation (6.51) illustrates this by supplying the result of the integral when the source is a single point charge. This will be derived in the next section.

Maxwell’s equations:^a

$$\begin{aligned}\mathbb{F} &= \square \wedge A, \\ \square^2 A &= -\mu_0 J \quad (\text{for } \square \cdot A = 0).\end{aligned}$$

Their solution:

$$A = \frac{1}{4\pi\epsilon_0 c} \int \frac{J(\mathbf{r}_s, t - r_{sf}/c)}{r_{sf}} d^3\mathbf{r}_s \quad (6.50)$$

where $r_{sf} = |\mathbf{r} - \mathbf{r}_s|$.

\Leftrightarrow For an arbitrarily moving point charge:

$$A = \frac{q}{4\pi\epsilon_0 c} \frac{\mathbf{U}}{(-\mathbf{R} \cdot \mathbf{U})} \quad (6.51)$$

where \mathbf{U} is its 4-velocity at the source event, and \mathbf{R} is the (null) 4-vector from the source event to the field event.

^aIn index notation, the first equation is $\mathbb{F}^{ab} = \partial^a A^b - \partial^b A^a$.

6.5.3 The Liénard-Wiechart potentials

We are now in a position to find the potential and field of an *arbitrarily* moving point charge, i.e. one that may accelerate, and change its acceleration, etc., not just maintain a constant velocity. This is a wonderful possibility, because *all* fields come from point charges moving somehow or other (or at least we can model them that way), so we can encapsulate a great deal of insight into electromagnetism into one small but powerful result. We can get it because we have in eq. (6.50) all the information we need.

Consider first the zeroth element of J , i.e. ρ/c , and look at its formula in eqs. (6.49). Faced

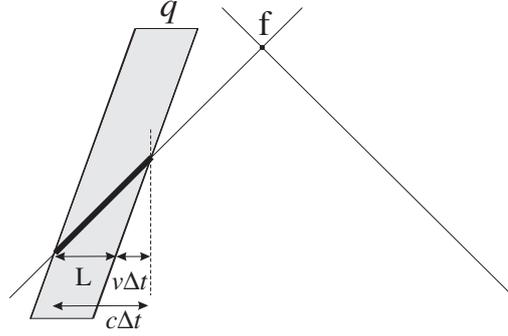


Figure 6.7: Spacetime diagram to help calculate the potential at the field event f due to the charged particle q . We must allow the particle a finite spatial extent and take the limit as this becomes small compared to all other distances. The diagonal lines show the past light cone of f . The events contributing to the integral are those shown bold. Suppose we want to calculate ϕ in the reference frame whose lines of simultaneity are horizontal in the diagram. Then the (spatial) length of the contributing line of events is $s = c\Delta t$ where Δt is the time taken for a light pulse to travel $s = L + v\Delta t$ while the lump of charge travels $v\Delta t$, where L is equal to the length of the lump. Eliminating Δt we find $s = L/(1 - v/c)$. Thus the moving charge contributes as much to the integral as a non-moving charge of the same density but longer length would contribute. This leads to the ‘enhancement’ factor $1/(1 - v/c)$ where v is the component of velocity towards the field point.

with the integral in (6.49) and the desire to evaluate it in the case of a point charge, most of us would note that since ρ is then a sharply peaked function, the $1/r_{sf}$ can be brought outside the integral, and then we would take the volume integral of ρ to be the charge q , thus obtaining

$$\phi \stackrel{?}{=} \frac{q}{4\pi\epsilon_0[r_{sf}]} \quad (\text{wrong})$$

It is what one might think, but it is wrong. The reason is because this does not correctly treat the time-dependant nature of the integrand when the charge is moving. Figure 6.7 explains the problem and its solution. The correct answer is

$$\phi = \frac{q}{4\pi\epsilon_0[r_{sf}(1 - v_r/c)]} = \frac{q}{4\pi\epsilon_0[r_{sf} - \mathbf{v} \cdot \mathbf{r}_{sf}/c]} \quad (6.52)$$

The square brackets in the denominator serve as a reminder that whereas we are evaluating the potential at the field point at some time t , the \mathbf{r}_{sf} and \mathbf{v} appearing in the formula are understood to mean $\mathbf{r}_{sf}(t_s)$ and $\mathbf{v}(t_s)$, i.e. their values at the source event which occurred at time $t_s = t - r_{sf}/c$.

The same reasoning applies to all the terms in the 4-vector form (6.50), so we obtain

$$\mathbf{A} = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{\mathbf{v}}{r_{sf} - \mathbf{v} \cdot \mathbf{r}_{sf}/c} \right] \quad (6.53)$$

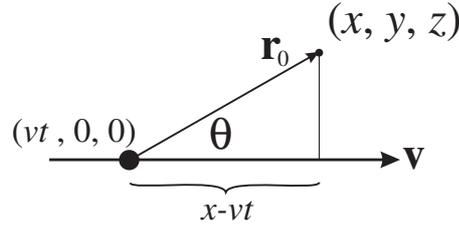


Figure 6.8: Vectors and angle used to express the potential due to a uniformly moving charge.

The potentials for a point charge as in equations (6.52) and (6.53) are called the *Liénard-Wiechart potentials*.

The tricky integration was perhaps a lesson in the caution that is needed in dealing with δ -functions. For further confidence, it would be useful if we could derive the potentials another way. We can check the answer for the case of a uniformly moving charge, of course, because we already know that, see eqs. (6.35). We shall show that they agree in a moment. However, by some clever reasoning, we can do much better. An important feature of the integrand in eqs. (6.50) is that it involves the *velocity* of some distribution of charge (giving the current density \mathbf{j}), but not its *acceleration*. It follows that the value of the integral for a point charge will only depend on its velocity at the source event, not its acceleration. But that means we can get it by Lorentz transformation!

So far in this book we have often approached the ‘frame hopping’ type of argument by writing down what we know to be the case in one frame, and then applying \mathcal{L} . However, where possible we should use another type of reasoning that can save a lot of trouble. Rather than laboriously transforming from one frame to another, we simply express the result in terms of 4-vectors that correctly produce it in the starting frame, and then we use physical reasoning to show that no further terms could appear in other frames, i.e. terms that just happened to cancel or vanish in the starting frame. This is the generalization of the ‘method of invariants’ (section 3.6). It is now a ‘method of 4-vectors’.

We are familiar with this type of reasoning in the case of 3-vectors. To take an example, consider the expressions (6.35) for the potentials of a uniformly moving point charge. In the denominator we have a term

$$(\gamma^2(x-vt)^2 + y^2 + z^2).$$

This expression clearly depends on the choice of coordinate system. However, by inspection of figure 6.8 you can easily convince yourself that the same result can be written down by substituting $(x-vt) = r_0 \cos \theta$ and $(y^2 + z^2)^{1/2} = r_0 \sin \theta$ where \mathbf{r}_0 is the vector from the charge at time t to the field point at time t , and θ is the angle between this vector and the velocity \mathbf{v} of the charge. Thus the expression is

$$(\gamma^2(x-vt)^2 + y^2 + z^2) = r_0^2(\gamma^2 \cos^2 \theta + \sin^2 \theta), \quad (6.54)$$

$$\text{with } \mathbf{r}_0 \cdot \mathbf{v} = r_0 v \cos \theta. \quad (6.55)$$

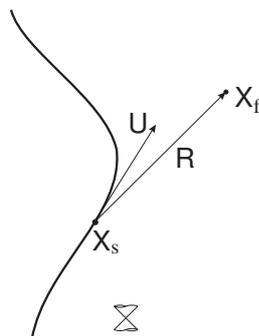


Figure 6.9: Definition of 4-vectors \mathbf{R} and \mathbf{U} for the calculation of the 4-potential of an arbitrarily moving charge.

We know for sure the vector form of the expression is valid in the coordinate system we started from, and we can see that there is no reason for things to stray from this form in other coordinate systems. Therefore eq. (6.35)i can be written more generally as

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r_0(\gamma^2 \cos^2 \theta + \sin^2 \theta)^{1/2}}. \quad [\text{constant velocity}] \quad (6.56)$$

The use of vectors saves us the trouble of applying rotation matrices to the original formula.

Now let's apply this type of reasoning to the 4-vector potential of an arbitrarily moving charge. First we write down the form for a charge at rest: what could be more simple? It is

$$\phi = q/(4\pi\epsilon_0 r), \quad \mathbf{A} = 0. \quad [\text{at rest}] \quad (6.57)$$

Next we reason that the field in the general case can only depend on what the source is doing at the source event. That is, the distance r in (6.57) has to be 'read' as the distance from the retarded position $\mathbf{r}_s(t_s)$ to the field point, see figure 6.9. How can we write it in terms of 4-vectors? One 4-vector that naturally presents itself is the one from the source event to the field event. Let this 4-vector be \mathbf{R} . In a general reference frame it has components (ct, \mathbf{r}) , where $t = r/c$ is the light travel time from $\mathbf{r}_s(t_s)$ to the field point \mathbf{r}_f . We have dropped the subscripts on \mathbf{r}_{sf} because we hope by now it is clear that this is the centrally important displacement vector. We are therefore writing \mathbf{r}_f for the field point. Another 4-vector that must be important is \mathbf{U} , the 4-velocity of the particle. In the rest frame it is $(c, 0)$ and in the general frame it is $\mathbf{U} = (\gamma c, \gamma \mathbf{v})$. The denominator of (6.57) is a scalar, so we try

$$\mathbf{R} \cdot \mathbf{U} = (ct, \mathbf{r}) \cdot (\gamma c, \gamma \mathbf{v}) = \gamma(-rc + \mathbf{r} \cdot \mathbf{v}). \quad (6.58)$$

This is promising, because it evaluates to $-rc$ in the rest frame, so it will give the correct $1/r$ Coulomb potential if it is in the denominator. Therefore we propose the solution

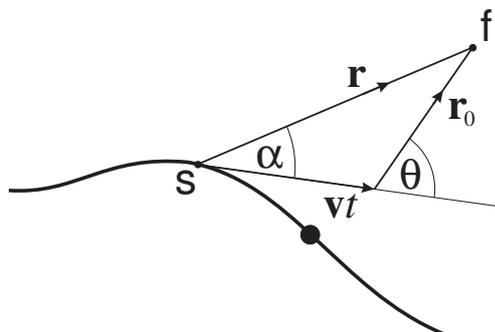


Figure 6.10: Defining the ‘projected position’. At the moment when the field is to be calculated at the field point f , the particle (large blob) has moved to some position of no interest. The field at f is caused by what went on at the source point s . We can express it in a useful way in terms of the vector r_0 between the projected position and the field point. The time $t = r/c$ is the time taken for the influence from s to reach f .

4-vector potential of a charge in arbitrary motion

$$A = \frac{q}{4\pi\epsilon_0} \frac{U/c}{(-R \cdot U)}. \quad (6.59)$$

This has the properties (1) it is a 4-vector, (2) it reproduces the known result (6.57) in the rest frame, (3) it only depends on quantities at the source event and on the source-to-field interval. If you were to Lorentz transform (6.57), therefore, you would certainly get (6.59). If you were happy with the 3-vector example leading to eq. (6.54), then you should be similarly convinced of (6.59).

The final piece of this argument is to claim that (6.59) is the complete solution for an *arbitrarily* moving charge, not just a constant velocity one, because we knew from (6.50) that the answer in the general case was going to depend only on the position and velocity of the charge at the source event, not its acceleration or rate of change of acceleration etc.

Using (6.58), it is straightforward to confirm that (6.59) agrees with the Liénard-Wiechart potentials (6.52), (6.53), with $\mathbf{r} \equiv \mathbf{r}_{sf}$. In other words, A in eq. (6.59) *is* the Liénard-Wiechart potentials. Thus we have derived them without needing to perform an integral.

We are now in a position to understand how the wonderful ‘magic’ of the electric field pointing away from the uniformly moving charge (figure 6.4) comes about. For a charge in an arbitrary state of motion, we focussed attention on two positions: that of the source event and that of the field event. We can also take an interest in another position: the “projected position.” This is the position the particle would have ‘now’ (i.e. at the time of the field event, in our chosen reference frame) *if* it were to continue on from the source event at the velocity it then had. The “projected position” is not usually on the particle’s trajectory: the particle doesn’t go there (unless of course its velocity happens to be constant), but it is a well-defined place that we

can take an interest in if we like. So, define the vector \mathbf{r}_0 to be the vector from the projected position to the field event. It is the vector that appeared in our formula (6.56) for the uniformly moving case, but now we are considering the general case. Using $\mathbf{r} = \mathbf{v}(r/c) + \mathbf{r}_0$ (figure 6.10) we obtain

$$\mathbf{r}_0 = \mathbf{r} - \mathbf{v}r/c. \quad (6.60)$$

We shall now write the general potential again, but expressing \mathbf{r} in terms of \mathbf{r}_0 and \mathbf{v} . We have $\mathbf{r} \cdot \mathbf{v} = rv \cos \alpha$ and using figure 6.10 you can see that $r \sin \alpha = r_0 \sin \theta$. So after using $\cos^2 \alpha = 1 - \sin^2 \alpha$ we have

$$(\mathbf{r} \cdot \mathbf{v})^2 = r^2 v^2 (1 - \sin^2 \alpha) = r^2 v^2 - v^2 r_0^2 \sin^2 \theta.$$

Using this result you can easily confirm that

$$\left(r - \frac{\mathbf{r} \cdot \mathbf{v}}{c} \right)^2 = r_0^2 \left(1 - \frac{v^2}{c^2} \sin^2 \theta \right).$$

Next replace 1 by $\cos^2 \theta + \sin^2 \theta$ on the right hand side and multiply by γ^2 , to obtain

$$\gamma(r - \mathbf{r} \cdot \mathbf{v}/c) = r_0(\gamma^2 \cos^2 \theta + \sin^2 \theta)^{1/2}.$$

Therefore

$$(\phi/c, \mathbf{A}) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\gamma(c, \mathbf{v})}{r_0(\gamma^2 \cos^2 \theta + \sin^2 \theta)^{1/2}}. \quad (6.61)$$

What is this? It is the same expression we got for the uniformly moving charge, of course (c.f. eq. (6.56)). We have confirmed that all our derivations are mutually consistent, and although the field for the case of uniform motion has the interesting form we noticed, we have confirmed that it is caused to assume that pattern by means of light-speed-limited communication.

With hindsight, one could now reason backwards from the potential of a particle at constant velocity (which is very easily derived by using the knowledge that \mathbf{A} is a 4-vector) to the Liénard-Wiechart potentials, by introducing a change of ‘position of interest’ from the projected position back to the source event. Since the fields can then be derived from the potentials, even for an arbitrarily moving charge, people sometimes claim that all of electromagnetism can be derived from Coulomb’s law and Lorentz transformations. Such a claim is wrong, however, because much more is needed. For example, we need to know that the potentials form a 4-vector, and how the fields relate to the potentials, and we need to know the non-trivial fact that the potentials only depend on the position and velocity of the charge at the source event, not on its acceleration. This is far from obvious: after all, the *fields* do depend on the acceleration. We also need to know that only properties at the source event are important, not some kind of integral over the history of the particle up to the source event.

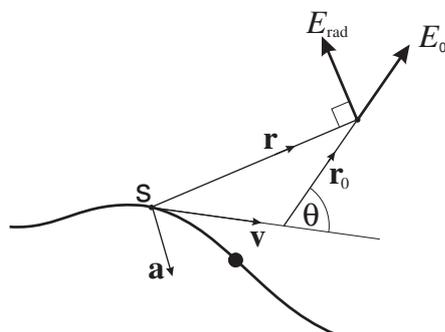


Figure 6.11: The electric field due to an arbitrarily moving charge, illustrating the directions of the non-radiative and radiative parts of the field.

The attempt to derive electromagnetism from Coulomb's law and Lorentz covariance therefore fails. However, the goal of developing fundamental theories from a minimal set of assumptions is valid and important. In chapter 12 we shall exhibit a construction of electromagnetic theory, i.e. Maxwell's equations and the Lorentz equation, based on a set of assumptions that we state explicitly, and that we try to make as small and simple as possible. This theme will also reemerge in chapter 18.

6.5.4 The field of an arbitrarily moving charge

The electric and magnetic fields of an arbitrarily moving charge can be obtained directly from the Liénard-Wiechart potentials, by applying the relations $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla \wedge \mathbf{A}$ (eq. (6.14)). Carrying out the differentiations with respect to time and space is a lot of work, however. The effort is reduced (though not to nothing) by some modest use of tensor methods to be described in chapter 9. The steps are outlined in appendix 2, which you should consult after reading chapter 9. One finds

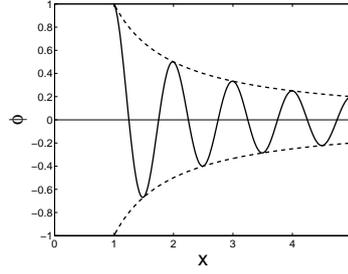


Figure 6.12: The scalar potential $\phi(t, x, y, z)$ plotted as a function of distance at some instant of time in an inertial frame, for a case where the source charge has been undergoing oscillatory motion about the origin. The dashed lines show the characteristic $1/r$ decay of the potential of a stationary charge. At large r the gradient of the potential at the zero crossings is approximately proportional to this envelope, hence $\nabla\phi$ varies as $1/r$ not $1/r^2$.

Field of a moving charge:

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0(r - \mathbf{r} \cdot \mathbf{v}/c)^3} \left(\frac{(\mathbf{r} - \mathbf{vr}/c)}{\gamma^2} + \frac{\mathbf{r} \wedge [(\mathbf{r} - \mathbf{vr}/c) \wedge \mathbf{a}]}{c^2} \right) \\ &= \frac{q}{4\pi\epsilon_0\kappa^3} \left(\frac{\mathbf{n} - \mathbf{v}/c}{\gamma^2 r^2} + \frac{\mathbf{n} \wedge [(\mathbf{n} - \mathbf{v}/c) \wedge \mathbf{a}]}{c^2 r} \right)\end{aligned}\quad (6.62)$$

where $\mathbf{n} = \mathbf{r}/r$, $\kappa = 1 - v_r/c = 1 - \mathbf{n} \cdot \mathbf{v}/c$

$$\mathbf{B} = \mathbf{n} \wedge \mathbf{E}/c \quad (6.63)$$

$$= \frac{\mathbf{v} \wedge \mathbf{E}}{c^2} \text{ when } \mathbf{a} = 0 \quad (6.64)$$

where $\mathbf{r} = \mathbf{r}_f - \mathbf{r}_s$; the source event is at $(t_s = t - r/c, \mathbf{r}_s)$; \mathbf{v} , \mathbf{a} are velocity and acceleration of the charge at the source event.

In terms of the displacement $\mathbf{r}_0 = \mathbf{r} - \mathbf{vr}/c$ from the projected position,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r_0^3 (\gamma^2 \cos^2 \theta + \sin^2 \theta)^{3/2}} \left(\gamma \mathbf{r}_0 + \frac{\gamma^3}{c^2} \mathbf{r} \wedge [\mathbf{r}_0 \wedge \mathbf{a}] \right) \quad (6.65)$$

where θ is the angle between \mathbf{r}_0 and \mathbf{v} .

Alternative form (Feynman):

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left(\frac{\mathbf{n}}{r^2} + \frac{r}{c} \frac{d}{dt} \left(\frac{\mathbf{n}}{r^2} \right) + \frac{1}{c^2} \frac{d^2}{dt^2} \mathbf{n} \right) \quad (6.66)$$

Examining eq. (6.62), we see two terms. The first term is independent of the acceleration and can be recognised as the field due to a uniformly moving charge. Its form is brought out by the version (6.65). The second term is proportional to the acceleration. It varies as $1/r$ not $1/r^2$, so it dominates at large r . This is the radiation field. Its electric field vector is at right angles to \mathbf{r} and in the plane containing the vectors \mathbf{r}_0 and \mathbf{a} (by using the triple vector product rule, $\mathbf{r} \wedge (\mathbf{r}_0 \wedge \mathbf{a}) = (\mathbf{r} \cdot \mathbf{a})\mathbf{r}_0 - (\mathbf{r} \cdot \mathbf{r}_0)\mathbf{a}$).

We started with a $1/r$ potential, eq. (6.52), so how can it come about that the radiation field varies as $1/r$, when differentiation of $1/r$ ought to give $1/r^2$? The answer to this is illustrated by figure 6.12. Owing to the propagation of the waves, a time dependence at the source is converted into a spatial dependence in the potential around it. For a sinusoidally oscillating source, for example, over any given wavelength in space, the potential varies up or down by an amount or order $1/r$. The wavelength is independent of r , so the *slope* of the potential, at the positions of maximum slope, must be falling off approximately as $1/r$ not $1/r^2$.

The alternative form (6.66), due to Feynman, brings out some further features. It has three terms. The first is the familiar Coulomb field, but evaluated, N.B., at the retarded position and time. The second term says we need to correct the retarded Coulomb field. We multiply the rate of change of that field by r/c , which is just the retardation time; this is like a linear extrapolation from the retarded time to the present time. For a slowly changing field that extrapolation turns out to be a very good approximation, but it is clear that it can't be exactly right. The last term corrects it. This term varies as $1/r$; it contains all the radiation effects.

A simple but useful and correct insight into the connection between radiation and acceleration is contained in the following argument (see figure 6.13). Suppose a particle moves at constant velocity for a while, then at event A it starts to accelerate and shortly after, at event B , it assumes a constant velocity again. Then the electromagnetic field for field points whose source event is either before A or after B is easy to write down: it is just the one associated with constant velocity motion (eq. 6.12). This provides the information about the field throughout most of spacetime. The part in between lightcones through events A and B can be obtained exactly from eq. (6.62), or approximately by simply joining up the field lines already obtained, since the total field is divergenceless (away from the charge). Figure (6.13) shows the result, for an example case in which the charge begins and ends at the same velocity, so the accelerated motion included both a speeding up and a slowing down part.

Close to the particle, the electric field lines are pointing radially outwards from the current position \mathbf{r} of the particle; they are given by (6.12). They do this until they reach a distance $c(t - t_B)$ from the location of event B . At the surface of that sphere they change direction. Now consider field points further out: at a distance further than $c(t - t_A)$ from A the field has no 'knowledge' that either event A or event B happened. It is directed radially outwards from the position the particle would now have, if it had never changed its state of motion. The field is given by eq. (6.12) again, but with the initial velocity and the projected position based on that velocity. We have thus obtained the field throughout most of space. Using the general idea that 'field lines' are continuous (a property of fields of zero divergence), we can connect them up in the middle region and thus get some idea of what the field is doing there. You can immediately see that there is a 'kink' in the field lines, that this kink propagates outwards at the speed of light, that the propagating part of the field is transverse (so as to introduce the observed change in direction of the field lines), and that it falls to zero along the line of the acceleration. This propagating pulse is the part of the total field that we call electromagnetic radiation.

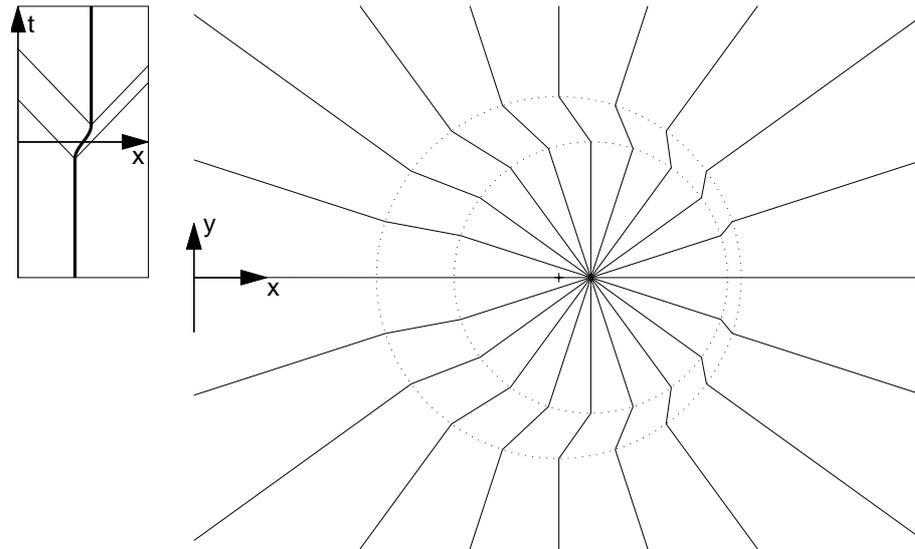


Figure 6.13: A point charge is ‘nudged’ to the right. That is, the charge moves uniformly, then undergoes a brief period of acceleration and deceleration, then moves uniformly again at the original velocity. The inset shows the worldline; the main figure shows the lines of electric field in a plane containing the acceleration vector, in the initial (and final) rest frame, at some moment shortly after the acceleration ceased. The circles show the current position of two light spheres that propagate outwards from source events at the beginning and end of the nudge. Near the charge the field is that of a uniformly moving charge, which points radially outwards from the current position of the charge (eq. (6.12)). Beyond the second light sphere the field is again that of a uniformly moving charge, but now pointing outwards from the projected position (the position the charge would now have, had it not accelerated), shown by a cross. In between the light spheres the field has a bound part and a radiative part. The radiative part at any point is transverse to the light sphere passing through that point.

Identifying the radiation

The claim that the term proportional to \mathbf{a} signifies “radiation” while the term without \mathbf{a} does not merits some attention. By “radiation” or a “radiative field” we mean a field that, once it is produced, can be regarded as a separate entity independent of the source. It propagates outwards at the speed of light, carrying a well-defined amount of energy and momentum with it, and it can be assigned its own energy-momentum 4-vector. The latter point is not self-evident because we are talking about an extended entity.

According to eq. (6.62) we can always separate an electromagnetic field at any given event into two parts:

$$\mathbf{E}_I \equiv \frac{q}{4\pi\epsilon_0\kappa^3} \frac{\mathbf{n} - \mathbf{v}/c}{\gamma^2 r^2}, \quad \mathbf{E}_{II} \equiv \frac{q}{4\pi\epsilon_0\kappa^3} \frac{\mathbf{n} \wedge ((\mathbf{n} - \mathbf{v}/c) \wedge \mathbf{a})}{c^2 r}, \quad (6.67)$$

$$\mathbf{B}_I \equiv \mathbf{n} \wedge \mathbf{E}_I/c, \quad \mathbf{B}_{II} \equiv \mathbf{n} \wedge \mathbf{E}_{II}/c. \quad (6.68)$$

In order to make this separation we would have to identify the source event and thus \mathbf{v} , \mathbf{a} and all the other parts of these formulae. This may not always be easy (perhaps we have a field in our lab but we don’t know what produced it in the past), but in principle it could be done by an all-knowing investigator. We should like to propose that \mathbf{E}_I , \mathbf{B}_I (hereafter called EM_I) should be identified as a ‘bound’, non-radiative field, which may be regarded as a field owned by or in permanent interaction with the source, while \mathbf{E}_{II} , \mathbf{B}_{II} (hereafter called EM_{II}) is a radiative field having an independent existence, possessing a well-defined energy-momentum. Can we prove such a statement?

First note that \mathbf{B}_{II} is perpendicular to \mathbf{E}_{II} , and since \mathbf{E}_I is perpendicular to \mathbf{n} , their sizes are related by $B = E/c$. A field with these properties is called **light-like**.

The formula for \mathbf{E}_I looks just like the formula for the field of a non-accelerating charge. In fact, it doesn’t just look like it, it is precisely the formula for the field of a non-accelerating charge (eq. (6.65) makes this clear). However, the ‘position vector’ \mathbf{r} is not here the position in space at some given time in a reference frame, it is a position vector *on a light cone* from the source event. In some reference frame at a given time, for fixed values of the rest of the parts of the formula, \mathbf{r} picks out positions on the surface of a light sphere centred on the source event. The bound field at other positions is given by a different source event, where the charge may have had a different velocity. Therefore the whole bound field at any given reference frame time is *not* simply the field of a charge in uniform motion. In fact, one may show that it is not even a solution of Maxwell’s equations! For example, for $a \neq 0$ one finds $\nabla \cdot \mathbf{E}_{II} \neq 0$ and therefore $\nabla \cdot \mathbf{E}_I \neq 0$ in empty space (but then we have $\nabla \cdot \mathbf{E}_I = -\nabla \cdot \mathbf{E}_{II}$ of course, since the total field *is* a solution of Maxwell’s equations). For this reason the separation of the field into type I and type II has to be interpreted with care. It turns out to be a useful way to consider energy movements in the field.

As we follow \mathbf{E}_{II} , \mathbf{B}_{II} out along the light cone of a given source event, we see their sizes diminishing as $1/r$, whereas \mathbf{E}_I , \mathbf{B}_I diminish as $1/r^2$. These statements are not about the dependence on position at any given time, they describe the dependence on the radii of a succession of light

spheres all centred at the same source event. Clearly, except in the direction along \mathbf{a} (where \mathbf{EM}_{II} vanishes but \mathbf{EM}_I does not), the \mathbf{EM}_{II} field dominates at large r , and furthermore if the energy content of the field goes as the square of the field amplitudes (as we shall show in chapter 12), the total amount of energy in the \mathbf{EM}_{II} field is *undiminished* as it propagates out, while the energy in the \mathbf{EM}_I field, in a spherical shell of fixed thickness, falls to zero. This enables one to identify the energy content of the \mathbf{EM}_{II} field purely from the behaviour of the total field on a huge light sphere in the distant future. Therefore a large enough light sphere offers information about the division of the field into two parts without requiring knowledge of the sources. The far field is sometimes called the ‘radiation zone’ or ‘wave zone’.

Note, the total energy movement in the field is caused by both contributions. For example, if the net energy flow is zero it does not necessarily imply there is no radiative part; rather it implies that the contributions to the total energy flow are balanced. (This point was widely misunderstood in the first half of the twentieth century, and is still a possible area of confusion for students.) There are three contributions to $\mathbf{E} \wedge \mathbf{B}$:

$$\mathbf{E} \wedge \mathbf{B} = \mathbf{E}_I \wedge \mathbf{B}_I + \mathbf{E}_{II} \wedge \mathbf{B}_{II} + (\mathbf{E}_I \wedge \mathbf{B}_{II} + \mathbf{E}_{II} \wedge \mathbf{B}_I).$$

An example where the $\mathbf{E}_{II} \wedge \mathbf{B}_{II}$ term is equal and opposite to the rest occurs in the case of a charge in hyperbolic motion.

Another important property of the \mathbf{EM}_{II} field of a given charge is that it can be zero. It is zero for all field events for which there is no acceleration at the source event. Therefore, if we assume the particle has not been undergoing permanent acceleration from the distant past until now, then at any given instant in a given frame, the non-zero part of \mathbf{EM}_{II} is completely contained in a finite region of space.

Thus \mathbf{EM}_{II} has the following properties:

- At any moment, it is completely contained in a finite region of space, not necessarily including the point where the particle is located.
- Its total energy content is constant when the particle is not accelerating.

We shall discuss the energy flow in more detail in section 12.2, and show that the total energy and momentum of \mathbf{EM}_{II} transform in the right way to form a 4-vector. This allows us to conclude that it is legitimate to call \mathbf{EM}_{II} the *radiative field*⁶. It also follows that, when observed in an inertial reference frame, accelerated charges always radiate, and radiation fields always have their source in accelerated (not constant velocity) motion.

⁶In the far field, i.e. far from the source event, one may say the field is ‘only’ the \mathbf{EM}_{II} part since it dominates, and this is sufficient for examining the interaction of the field with other things such as detectors. However, even though \mathbf{EM}_I is small, its divergence is not small compared to that of \mathbf{EM}_{II} (they are equal and opposite); this is because the divergence of \mathbf{EM}_I involves a cancellation of terms of opposite sign: they almost balance but not quite. The weaker \mathbf{EM}_I field has a larger divergence relative to its size, and can supply a matching contribution.

6.5.5 Two example fields

The far field of a slowly oscillating dipole

The most important type of light source, or source of electromagnetic radiation in general, is the oscillating dipole. Most of the light we see around us is sourced by oscillating electric dipoles in atoms and molecules. Radio waves are produced by antennas that may be treated as dipoles to first approximation.

We shall obtain the form of the electromagnetic field of an oscillating dipole as simply as possible, by assuming the speed of motion of the charge is small compared to c , and the dipole is itself small compared to the distance to the field point. This covers most cases of practical importance, and is the first step to treating more general cases.

Consider a dipole made of two charges $\pm q$ separated by a displacement \mathbf{x}_q , so the dipole moment is

$$\mathbf{d} = q\mathbf{x}_q. \quad (6.69)$$

We suppose the $-q$ charge is fixed and the q charge moves with velocity $\mathbf{v} = \dot{\mathbf{x}}_q$. We shall obtain the fields from the 4-vector potential. We could start with the electric field, but it turns out that the calculation is easier if we first obtain the magnetic field, which only depends on the 3-vector potential \mathbf{A} .

To calculate the magnetic field we only need to consider the contribution to \mathbf{A} due to the moving charge. Starting from eq. (6.59), using $\mathbf{U} = \gamma(c, \mathbf{v})$ and $\mathbf{R} = (c(t - t_s), \mathbf{r}_{\text{sf}})$ we obtain for the moving charge

$$\mathbf{A} = \frac{q}{4\pi\epsilon_0} \frac{(c, \mathbf{v})}{c(r_{\text{sf}}c - \mathbf{r}_{\text{sf}} \cdot \mathbf{v})}. \quad (6.70)$$

This is true in general.

Now we make an approximation: we treat a ‘slowly’ oscillating dipole, meaning the speed of movement of the charge is small compared to c , i.e. $v \ll c$. For sinusoidal oscillation, this implies that the wavelength of the emitted radiation is large compared to size of the dipole. For example, for a dipole of atomic dimensions we are restricted to treating radiation in the electromagnetic spectrum from radio waves to soft X-rays. With this approximation we have

$$\mathbf{A} \simeq \frac{1}{4\pi\epsilon_0 c^2} \frac{(c, \dot{\mathbf{d}}[t - r_{\text{sf}}/c])}{r_{\text{sf}}}. \quad (6.71)$$

where we used $q\mathbf{v} = \dot{\mathbf{d}}$ and we have explicitly indicated the fact that this has to be evaluated at the source time $t_s = t - r_{\text{sf}}/c$. For example, for a sinusoidally oscillating source,

$$\begin{aligned} \mathbf{d} &= q\mathbf{x}_0 \sin \omega t, \\ \Rightarrow \dot{\mathbf{d}}[t - r_{\text{sf}}/c] &= \omega q\mathbf{x}_0 \cos(\omega t - kr_{\text{sf}}) \end{aligned} \quad (6.72)$$

where k is the wave vector.

We calculate the \mathbf{B} field from

$$\mathbf{B} = \nabla \wedge \mathbf{A} \simeq \frac{1}{4\pi\epsilon_0 c^2} \nabla \wedge \left(\frac{\dot{\mathbf{d}}[t - r_{\text{sf}}/c]}{r_{\text{sf}}} \right). \quad (6.73)$$

At this stage it is helpful to introduce a further approximation, namely to set $r_{\text{sf}} \simeq r$ where r is the distance from the origin to the field point (for a dipole oscillating about the origin). This is allowable as long as two conditions hold. First the field point must be far from the dipole, $r \gg x_q$, and also the speed of movement of the charge must be small, $v \ll c$ (as we already assumed). The second condition arises because the derivative of r_{sf} with respect to (for example) x involves $\partial \mathbf{r}_s / \partial x$, i.e. the change in the source point position when we ask about a change in field point by dx . If you think about the light cones you should see that this amounts to asking how far the source moves during a time dx/c . Clearly the movement of the source is small compared to dx when $v \ll c$, so under this condition we may take $dr_{\text{sf}} \simeq dr$.

Having made both the low speed and the far field approximations, we now have

$$\mathbf{B} \simeq \frac{1}{4\pi\epsilon_0 c^2} \nabla \wedge \left(\frac{\dot{\mathbf{d}}[t - r/c]}{r} \right) \quad (6.74)$$

which is reasonably straightforward to evaluate. Assume the motion of the dipole is linear, and chose the z axis to be along \mathbf{d} . Then \mathbf{A} is along z , so for B_x we only need to evaluate

$$B_x = \frac{\partial A_z}{\partial y} = \frac{1}{4\pi\epsilon_0 c^2} \left(\left(\frac{-y}{r^3} \right) \dot{d} - \frac{y}{cr^2} \ddot{d} \right) \quad (6.75)$$

using $\partial r / \partial y = y/r$ twice (and dropping the \simeq). The expression for B_y can be calculated similarly (it is given by the same formula with the substitution $-y \rightarrow x$).

The first term in (6.75) is the same as the result of the Biot-Savart law, except $\dot{d} = qv$ is to be evaluated at the source time (=retarded time) not the field time t . It falls off as $1/r^2$, whereas

the second term varies as $1/r$. Since we are here interested in the far field, we drop the first term, and the result is (bringing together all three components)

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0 c^3} \frac{\ddot{\mathbf{d}}[t-r/c] \wedge \mathbf{r}}{r^2} = \frac{\omega^2 d_0}{4\pi\epsilon_0 c^3} \frac{\sin\theta}{r} \sin(kr - \omega t) \hat{\phi} \quad (6.76)$$

where the first version treats a general time-dependence of the source, and the second version gives the result for a sinusoidally oscillating dipole on the z axis (using $\hat{\mathbf{z}} \wedge \mathbf{r} = \sin(\theta) \hat{\phi}$).

To get the electric field, one can go via the potentials again⁷ but for the far field we don't need to. We have found that the far field is purely a radiation field (it falls as $1/r$ and is proportional to $\ddot{\mathbf{d}}$, hence to the acceleration of the source). Therefore we know it is light-like, i.e. $E = cB$ and \mathbf{E} , \mathbf{B} and \mathbf{r} form a right-handed set. Hence

$$\mathbf{E} = c\mathbf{B} \wedge \mathbf{r}/r = \frac{1}{4\pi\epsilon_0 c^2} \frac{(\ddot{\mathbf{d}}[t-r/c] \wedge \mathbf{r}) \wedge \mathbf{r}}{r^3}. \quad (6.77)$$

The vector product implies that the sizes of B and E vary with direction as

$$E = cB \propto \frac{\sin\theta}{r}, \quad (6.78)$$

where θ is the angle between \mathbf{d} and \mathbf{r} . This pattern of the strength of the radiation field is called a 'dipole pattern'. For example, the sinusoidally oscillating dipole (6.72) gives

$$E = cB = \frac{\omega^2 d_0}{4\pi\epsilon_0 c^2} \frac{\sin\theta}{r} \sin(kr - \omega t) \quad (6.79)$$

with \mathbf{E} and \mathbf{B} directed around the surface of the light sphere, \mathbf{E} in the θ direction, \mathbf{B} in the ϕ direction. This case is plotted in figure ??.

Antenna

The combination $\omega d_0 = \omega q l$ can be recognized as Il where I is the current in a short segment of wire of length l . Therefore we can write (6.79) as

$$E = cB = \frac{-iI}{2\epsilon_0 c} \frac{l \sin\theta}{\lambda r} e^{i(kr - \omega t)}$$

⁷In a calculation of the electric field from the potentials, one finds that the approximation $r_{\text{sf}} \simeq r$ is inadequate for the scalar potential: higher order terms are needed. However, one can avoid this difficulty by adopting the Lorenz gauge and obtaining ϕ from $\partial\phi/\partial t = -c^2 \nabla \cdot \mathbf{A}$.

where the complex notation is convenient in order to signal that the fields are a quarter cycle out of phase with the current.

An *antenna* is a short length of wire carrying an oscillating current and intended for use in either broadcasting or receiving electromagnetic waves. In that application we are interested in maximising the transmitted or received power. Consider for example an antenna that is fed in the middle. Then the current oscillations are maximal at the centre of the antenna and zero at the ends. For a short antenna of length L we can approximate the current distribution as roughly linear, $I = I_0(1 - 2|z|/L)$. Integrating this along the antenna gives $I_0L/2$, so the emitted power varies as L^2 . This suggests there is interest in using longer antennas. However, to calculate the field correctly we should allow for the phase lag, i.e. the fact that the distance from a current element on the antenna to the field point is also a function of z . This is very much like a diffraction calculation in optics. The essential point is that once the antenna is longer than about $\lambda/2$, further increases in length alter the directional distribution of the radiated field significantly.

A center-fed antenna of length $L = \lambda/2$ is called a *half-wave dipole antenna*. We can model the current distribution roughly as $I = I_0 \cos kz$ (this falls to zero at $z = \pm\lambda/4$, i.e. the ends of the antenna). Then $\int Idz = I_0\lambda/\pi$, and therefore (ignoring the diffraction effects) the fields are given approximately by

$$E = cB \simeq \frac{-iI_0}{2\pi\epsilon_0 c} \frac{\sin\theta}{r} e^{i(kr - \omega t)}. \quad (6.80)$$

(for a more accurate result, see box). The dependence on wavelength has now dropped out. The constant $(2\pi\epsilon_0 c)^{-1}$ has the value 59.96 ohms; it is equal to $Z_0/2\pi$ where $Z_0 \equiv \mu_0 c$ is the characteristic impedance of free space.

A charge in hyperbolic motion*

[Section omitted in lecture-note version.]

6.6 Radiated power

It is very useful to have a formula for the power in the radiation part of the field. Such a formula can be obtained for a charge in an arbitrary state of motion.

To calculate the power in the emitted radiation, for convenience choose the frame that is the instantaneous rest frame of the particle at the source event, so $\mathbf{v} = 0$. Then the radiation field

Far field of an antenna. The approximation $I = I_0 \cos kz$ is quite good for a half-wave antenna, but not exact because the radiation itself extracts power from the antenna. Within this approximation an accurate expression for the far field can be obtained by integrating along the antenna, allowing for the phase (just as in a Fraunhofer diffraction calculation). Each element on the antenna contributes $d\mathbf{E}$ to the field. First we make the approximation that the field point P is sufficiently far away that the *directions* of all these contributions agree, so

$$\mathbf{E} = \frac{-iI_0 \sin \theta}{2\epsilon_0 c \lambda} \int \frac{1}{r'} \cos(kz) e^{i(kr' - \omega t)} dz \hat{\boldsymbol{\theta}}$$

where $r'(z)$ is the distance from each current element to P (we wish to reserve the symbol r for the distance from the centre of the antenna to P). The assumption of far field allows us to use the Fraunhofer approximation

$$r' = r - z \cos \theta$$

so the integral in the expression above is

$$\frac{1}{r} e^{i(kr - \omega t)} \int_{-\lambda/4}^{\lambda/4} \cos(kz) e^{-ikz \cos \theta} dz$$

where we brought $1/r'$ outside the integral since the variation of r' is negligible except through its effect on the phase. The integration can now be carried out easily by writing $\cos kz = (e^{ikz} + e^{-ikz})/2$. One finds

$$\frac{\sin(\frac{\pi}{2}(\cos \theta + 1))}{k(\cos \theta + 1)} + \frac{\sin(\frac{\pi}{2}(\cos \theta - 1))}{k(\cos \theta - 1)} = \frac{2 \cos(\frac{\pi}{2} \cos \theta)}{k \sin^2 \theta}$$

where in the last step we used $\sin(A + B) = 2 \sin A \cos B$ and added the two terms. Upon multiplying the various factors together, one power of $\sin \theta$ cancels and we have $k\lambda = 2\pi$, so the field is

$$E = \frac{-iI_0}{2\pi\epsilon_0 c} \frac{\cos(\frac{\pi}{2} \cos \theta)}{r \sin \theta} e^{i(kr - \omega t)}.$$

This expression is the more accurate replacement for (6.80).

in eq. (6.62) reduces to

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{a})}{r} \right].$$

The energy flux is given by the Poynting vector $\mathbf{N} \equiv \epsilon_0 c^2 \mathbf{E} \wedge \mathbf{B}$ (see chapter 12). It is allowable to calculate the Poynting vector of the radiative field alone (rather than the total field) since in any case, for large enough r this part of the field contains all the energy crossing the light

sphere of radius r . We obtain:

$$\mathbf{N} = \epsilon_0 c \mathbf{E}_{\text{rad}} \wedge (\mathbf{n} \wedge \mathbf{E}_{\text{rad}}) = \epsilon_0 c E_{\text{rad}}^2 \mathbf{n}. \quad (6.81)$$

A solid angle $d\Omega$ on a sphere around the source event receives this flux onto an area $r^2 d\Omega$ at normal incidence, so the power radiated per unit solid angle is

$$\frac{d\mathcal{P}}{d\Omega} = N r^2 = \frac{q^2}{4\pi\epsilon_0} \frac{a^2 \sin^2 \theta}{4\pi c^3} \quad (6.82)$$

where θ is the angle between \mathbf{n} and \mathbf{a} . This exhibits a characteristic $\sin^2 \theta$ dependence, called a ‘dipole pattern’. The radiation is emitted primarily to the sides, i.e. in directions orthogonal to the acceleration.

The total power emitted is obtained by integrating (6.82) over all solid angle, giving

$$\mathcal{P}_L = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0} \frac{a^2}{c^3}. \quad (6.83)$$

This is Larmor’s formula for the power emitted by a nonrelativistic accelerating charge.

Now we should like to generalize this to all velocities. It is not necessary to re-do the calculation, because we can argue that \mathcal{P}_L is a Lorentz invariant quantity. The argument hinges on the idea that we can regard the radiated part of the field as an ‘isolated system’ whose total energy and momentum form the components of a 4-vector. This is not obvious (it is not true of the non-radiative part of the field, for example); but it is valid because after the charge stops accelerating the radiation field continues to propagate outwards, so that it can be completely contained in a region of space where there are no charged particles in interaction with it. A more thorough discussion involves a consideration of the momentum flow in the field, this is provided in chapter 12.

Let $d\mathcal{E}$ be the total energy emitted into the radiation field in the instantaneous rest frame during some small time $d\tau$ (this is a proper time), then $\mathcal{P}_L = d\mathcal{E}/d\tau$. Since the radiation is emitted equally in opposite directions in the rest frame, the total momentum of the radiation field is zero in that frame, and since this energy and momentum form a 4-vector, we know how they transform. Clearly the energy will be $d\mathcal{E}' = \gamma d\mathcal{E}$ in some other frame, and the time interval $dt' = \gamma d\tau$, therefore $d\mathcal{E}'/dt' = d\mathcal{E}/d\tau$, so the power is Lorentz invariant.

We can now find the general formula for the power by writing down a Lorentz scalar quantity that depends only on velocity, acceleration and proper time, and that reduces to (6.83) in the rest frame. The unique answer (Heaviside 1902) is

Power emitted by an accelerating charge

$$\mathcal{P}_L = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3} \dot{\mathbf{U}} \cdot \dot{\mathbf{U}} = \frac{2}{3} \frac{q^2 a_0^2}{4\pi\epsilon_0 c^3} \quad (6.84)$$

where the dot signifies $d\mathbf{U}/d\tau$ (we have not used \mathbf{A} for the 4-acceleration here in order to avoid confusion with the 4-vector potential; the mention of proper time in $\dot{\mathbf{U}}$ does not change the fact that \mathcal{P}_L is an energy per unit reference frame time). In order to use the formula in practice it can be helpful to have it expressed in terms of 3-velocity and 3-acceleration at the source event, using eq. (3.61):

$$\mathcal{P}_L = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3} \gamma^6 \left(a^2 - \frac{(\mathbf{v} \wedge \mathbf{a})^2}{c^2} \right). \quad (6.85)$$

This version is associated with Liénard (1898).

To prepare for discussions of momentum in chapter (12), we shall quote also the 4-vector giving the rate at which 4-momentum is carried away by the radiation (Abraham 1903):

$$\frac{d\mathbf{P}}{d\tau} = \frac{\mathcal{P}_L \mathbf{U}}{c^2} = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^5} (\dot{\mathbf{U}} \cdot \dot{\mathbf{U}}) \mathbf{U}. \quad (6.86)$$

This is obtained by arguing that the energy and momentum of the radiation field form a 4-vector (see above), so \mathcal{P}_L is part of a 4-vector, and the radiation pattern is symmetric in the rest frame of the particle at the source event, so that no 3-momentum is generated from the given source event in that frame. Hence \mathbf{U} is the relevant 4-vector.

6.6.1 Linear and circular motion

For linear acceleration, i.e. \mathbf{a} parallel to \mathbf{v} , we have from (6.85)

$$\mathcal{P}_L = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3} (\gamma^3 a)^2 = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 m^2 c^3} \left(\frac{dp}{dt} \right)^2$$

For fixed rest mass, the rate of change of momentum is equal to the change of energy per unit distance, $dp/dt = dE/dx$, so for linear motion the power radiated depends only on the externally provided force (potential energy gradient), not on the actual energy or momentum of the particle.

First let us consider the cases of a linear accelerator and a dipole oscillator. Writing $dE/dx =$

$(dE/dt)(dt/dx)$ we find the ratio of radiated power to supplied power is

$$\frac{\mathcal{P}_L}{dE/dt} = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 m^2 c^3} \frac{1}{v} \frac{dE}{dx}. \quad (6.87)$$

The infinity for $v \rightarrow 0$ here is quite interesting: it says that if a particle accelerates through $v = 0$ then there is a moment at which it continues to emit radiation even though the externally applied forces are not providing any energy! We shall investigate this in chapter 12, and argue that the bound field provides the energy. For high velocity particles ($v \rightarrow c$), the result shows that energy losses by radiation are negligible unless an energy equal to the rest energy of the particle is provided in a distance $q^2/(4\pi\epsilon_0 mc^2)$. For an electron this distance is 2.8×10^{-15} m; the acceleration would have to reach 10^{14} MeV/m before losses were significant. Radiation loss in linear particle accelerators (on Earth) is utterly insignificant.

For an electric dipole oscillator of dipole moment $d(t) = qz = d_0 \cos \omega t$ we have $a = d^2 z/dt^2 = -\omega^2(d_0/q) \cos \omega t$ so the instantaneous emitted power is

$$\mathcal{P}_L = \frac{2/3}{4\pi\epsilon_0 c^3} (\gamma^3 \omega^2 d_0 \cos \omega t)^2.$$

Taking the non-relativistic limit $\gamma \simeq 1$, and taking the average over a cycle (the average value of the \cos^2 function is $1/2$), we find the average power emitted is

$$\bar{\mathcal{P}}_L = \frac{1}{3} \frac{\omega^4 d_0^2}{4\pi\epsilon_0 c^3} = \frac{2\pi^2}{3} \frac{\omega d_0^2}{\epsilon_0 \lambda^3}. \quad (6.88)$$

This gives an important general insight into power radiation by small oscillators: the ω^4 dependence shows that, for an oscillator of given size, energy is much more rapidly emitted via high frequency than low frequency oscillation. This explains why mobile phones have to use microwave not radiowave technology. It also explains why the ultra-violet transitions in atoms and molecules are typically much stronger than the visible or infra-red ones. This general insight lies behind the much-beloved problem of explaining why the sky is blue. Molecules and dust particles in the atmosphere scatter light from the sun; owing mainly to (6.88) they do so more efficiently for blue than for red light; we receive the scattered light—except during a sunrise or sunset when we see primarily the remaining non-scattered part.

The d_0^2 term in (6.88) is also significant. It shows why radio masts are tall. Its cousin in gravitational wave physics is the reason why no gravitational waves have ever been detected by detectors of modest (a few metres) size.

For circular motion, the acceleration is perpendicular to the velocity and in synchrotrons it is typically much larger than in linear accelerators since a given force can cause a much larger transverse than longitudinal acceleration (by a factor γ^2 , see eq. (4.13)). Using $|\mathbf{v} \wedge \mathbf{a}| = va$

and $a = v^2/r$ for motion around a circle of radius r , eq. (6.85) gives

$$\mathcal{P}_L = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3} \frac{\gamma^4 v^4}{r^2}. \quad (6.89)$$

The radiative loss per revolution is therefore

$$\Delta E = \frac{q^2}{3\epsilon_0 r} \gamma^4 (v/c)^3.$$

For electrons the quantity $e^2/(3\epsilon_0 r)$ is 6×10^{-9} eV when $r = 1$ metre. A 10 GeV electron synchrotron has $\gamma = E/(mc^2) \simeq 2 \times 10^4$ so $\Delta E \simeq 880$ MeV if the radius is 1 metre. At Cornell such a synchrotron was built with $r = 100$ m, giving a loss per turn of 8.8 MeV.

6.6.2 Angular distribution

[Section omitted in lecture-note version.]

6.7 Exercises

[Section omitted in lecture-note version.]

Chapter 9

Tensors and index notation

In this chapter we shall introduce the methods of tensor algebra, which are needed to take the subject further. The study of tensors and their manipulation is a rich field of mathematics in its own right, and this can be daunting for a physics student meeting the ideas for the first time. For Special Relativity, however, we don't need to invoke all the methods. In this chapter I will take a 'gentle' approach that is intended to bridge the gap between the 4-vectors we have met so far, and the complicated multi-dimensional objects whose treatment requires a whole new notation. We shall find our way by using first of all the vector and matrix methods we have used up till now. This will help to clarify the meaning of some of the more general tensor results that are hard to read when one first sees them written down. It is also a good way to find out why some new concepts, such as covariant and contravariant vectors, can be useful.

For introductory Special Relativity it is sufficient to learn how to handle scalars, 4-vectors and the next class of object, the second rank tensor. However one should know that these are part of a more general mathematical structure. That more general structure is one of the starting points of General Relativity.

In this chapter we shall use the symbol Λ for the Lorentz transformation, following the same practice we introduced in section 5.8 on the Lorentz group.

9.1 Introducing tensors

So far we have introduced the idea of a 4-vector as 'something which transforms under a change of reference frame as:'

$$A' = \Lambda A. \tag{9.1}$$

We also introduced some scalar quantities that do not change from one reference frame to another, such as electric charge and rest mass, and some that do, such as energy.

We also met the idea of the ‘scalar product’ $\mathbf{A} \cdot \mathbf{B}$, which is defined

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T g \mathbf{B} \quad (9.2)$$

(c.f. eq. (3.49)) where g is the metric tensor. It is worth remembering why the definition included the g in the middle: it is not simply the form $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$ that would apply for 3-vectors. The reason is that we deliberately designed the scalar product so that the scalar quantity that it produces will be a Lorentz invariant. Let’s remind ourselves of the proof:

$$\begin{aligned} \mathbf{A}' &= \Lambda \mathbf{A}, \quad \mathbf{B}' = \Lambda \mathbf{B} \\ \Rightarrow \quad \mathbf{A}' \cdot \mathbf{B}' &= (\Lambda \mathbf{A})^T g (\Lambda \mathbf{B}) = \mathbf{A}^T \Lambda^T g \Lambda \mathbf{B} \\ &= \mathbf{A}^T g \mathbf{B}. \end{aligned}$$

The crucial step is the last one: it makes use of the *defining property* of the Lorentz transformation, namely

$$\Lambda^T g \Lambda = g. \quad (9.3)$$

The quantity $\mathbf{A}^T \mathbf{B}$, by contrast, gives a scalar such as $c^2 t^2 + \mathbf{r}^2$ or $E^2/c^2 + \mathbf{p}^2$. These are well-defined mathematical quantities, but they are of little interest to us.

With scalar invariants and 4-vectors in hand, it is natural to start to consider whether we can generalize towards higher-dimensional objects such as matrices. What kind of role could they play?

Consider first of all the sort of situation where a matrix might be used for the analysis of 3-vectors. We already know of one sort of matrix: the rotation matrix, and we have a spacetime equivalent of that: a general Lorentz transformation. However another type of matrix also arises. Consider for example electric conduction and Ohm’s law. If an electric field is applied to an ordinary conducting material which has some non-negligible resistance, then Ohm’s law $V = IR$ can be written in the form

$$\mathbf{j} = \sigma \mathbf{E} \quad (9.4)$$

where \mathbf{E} is the applied electric field and σ is the conductivity¹. This is correct for an amorphous material, that has no crystalline structure or preferred direction. However, for many crystalline

¹The resistance of a cylinder of cross-sectional area A and length L is $R = L/(\sigma A)$ so (9.4) says $\mathbf{j}AR = L\mathbf{E}$ which is $IR = V$.

materials, current flows more readily in some directions than others. When an applied electric field pushes on the conduction electrons, the crystal lattice pushes too, and the net result can be a flow in a direction not parallel to \mathbf{E} . For small fields, the current density \mathbf{j} will still be proportional to \mathbf{E} , but its direction need not agree. We can express this more general relationship by

$$\mathbf{j} = C\mathbf{E} \quad (9.5)$$

where C is a 3×3 matrix called the ‘conductivity tensor’. If the size of current flow remains proportional to the applied field, then we have succeeded in writing a formula so that the dependence on \mathbf{E} is taken care of, and therefore C depends only on the conducting object: we have ‘packaged into it’ all the information about conduction, including which directions are preferred. The 2, 3 element of C answers the question ‘how much current in the y direction do you get for a given amount of applied field in the z direction?’ and so on.

3×3 matrices arise in many areas of physics where the situation is not isotropic. Other examples are the electric polarizability tensor (\mathbf{E} field creates an electric polarization \mathbf{P}), magnetic susceptibility tensor (\mathbf{B} field creates a magnetization \mathbf{M}) and moment of inertia tensor (angular velocity $\boldsymbol{\omega}$ gives rise to angular momentum \mathbf{L}).

The components of a matrix such as C can be found by experimental measurement, but our interest here is to ask what sort of mathematical object C is. The equation (9.5) gives an instructive hint: it says that the product of this object with a vector, following the standard rules of matrix multiplication, gives another vector. This observation suffices to *define* a class of mathematical object. A ‘thing that when multiplied onto a vector gives another vector, not necessarily parallel to the first’ is called a *second rank tensor*. A vector is then said to be a ‘first rank tensor’ and an invariant scalar is called a ‘tensor of rank zero.’

Similarly, in tensor analysis in 4-dimensional spacetime, we define a 2nd rank tensor to be an object that can multiply a 4-vector so as to give another 4-vector, but just as we needed to be careful to include the metric g in the definition of the scalar product, we shall need to work out if and when g comes into the use of 2nd rank tensors.

9.1.1 Outer product

Every physics student is familiar with the scalar product that is written $\mathbf{a} \cdot \mathbf{b}$ for 3-vectors. We already noted that if we assume the vectors are always column vectors, then the scalar product can be expressed as

$$\mathbf{a}^T \mathbf{b},$$

where it is understood that the notation means a multiplication following the standard rules of matrix multiplication. Since the objects being multiplied here are of dimensions 1×3 and

3×1 , the product is valid (because the 3's match) and the answer will be a 1×1 matrix, i.e. a scalar. This is also called an inner product.

Physics students are usually much less familiar with another simple way to combine two vectors: the outer product, which can be expressed (again, assuming we are dealing with column vectors)

$$\mathbf{a}\mathbf{b}^T.$$

This is a legal matrix multiplication, because it is a product of a 3×1 'matrix' (i.e. a column vector) with a 1×3 'matrix' (a row vector). The result is a 3×3 matrix. You can easily confirm that the elements of this matrix are given by

$$\mathbf{a}\mathbf{b}^T = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix}.$$

That is, we just write out all possible combinations of an element of \mathbf{a} with an element of \mathbf{b} , arranged in the right order.

Both inner and outer product are much used in quantum theory, where in Dirac notation they are expressed $\langle \phi | \psi \rangle$ and $|\phi\rangle \langle \psi|$.

Let's apply the same idea to 4-vectors. We define a mathematical object

$$\mathbb{M} = \mathbf{A}\mathbf{B}^T. \tag{9.6}$$

How does such an object transform under Lorentz transformations? Let's see:

$$\mathbb{M}' = (\Lambda\mathbf{A})(\Lambda\mathbf{B})^T = \Lambda\mathbf{A}\mathbf{B}^T\Lambda^T = \Lambda\mathbb{M}\Lambda^T. \tag{9.7}$$

That is, under a Lorentz transformation the tensor is pre-multiplied by Λ and post-multiplied by Λ^T .

What happens when \mathbb{M} multiplies a 4-vector? Employing the associative rule for matrix multiplication, we have

$$\mathbb{M}\mathbf{C} = (\mathbf{A}\mathbf{B}^T)\mathbf{C} = \mathbf{A}(\mathbf{B}^T\mathbf{C}). \tag{9.8}$$

Danger! This is not a 4-vector. It is easy to see why not: the second form is a 4-vector \mathbf{A} multiplying a scalar ($\mathbf{B}^T\mathbf{C}$) that is not Lorentz invariant. It is also easy to see what we need to do to fix the problem. When multiplying \mathbb{M} onto a 4-vector we need to introduce the metric:

$$\mathbb{M}g\mathbf{C} = \mathbf{A}\mathbf{B}^T g\mathbf{C} = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}). \tag{9.9}$$

Now all is well.

We can now make a satisfactory definition of 2nd rank tensors in general. We *define* a second rank tensor \mathbb{F} in general to be an object which transforms, under a change of reference frame, as

$$\mathbb{F}' = \Lambda \mathbb{F} \Lambda^T. \quad (9.10)$$

We *allow* multiplication of 2nd rank tensors onto 1st rank tensors (4-vectors) as long as the metric g is inserted, as in a tensor equation such as

$$\mathbf{B} = \mathbb{F} g \mathbf{A}. \quad (9.11)$$

Finally, there are two ways to obtain Lorentz scalars (that is, scalar invariants) from 2nd rank tensors:

$$\text{Tr}(g\mathbb{F}) \quad \text{and} \quad \text{Tr}(\mathbb{F}^T g \mathbb{G} g) \quad (9.12)$$

where Tr signifies the *trace*, i.e. the sum of the diagonal elements. If you write $\mathbb{F} = \mathbf{A}\mathbf{B}^T$ and $\mathbb{G} = \mathbf{C}\mathbf{D}^T$ then you can confirm that the first of these gives $\mathbf{A}\cdot\mathbf{B}$ and the second gives $(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})$, which confirms that they are Lorentz scalars. (The trace is not affected by the order of matrix multiplication.) N.B. to work out the second invariant in (9.12) it is not necessary to perform matrix multiplication: see the comment after eq. (9.34).

Not all tensors can be written as an outer product (those that can are called ‘pure’) but they can always be written as a sum of outer products, so the outer product is sufficient to tell us how they behave.²

9.1.2 The vector product

We still have not exhibited a 4-vector quantity similar to the well-known vector product $\mathbf{a}\wedge\mathbf{b}$ for 3-vectors. The reason is connected with the fact that $\mathbf{a}\wedge\mathbf{b}$ is not quite a ‘perfectly proper’ vector. It is (quite rightly) called a vector because it behaves the right way under rotations, but it gets up to no good when you try reflections or inversions through the origin (parity transformation). Consider a rotating object and its angular momentum $\mathbf{L} = \sum \mathbf{r} \wedge \mathbf{p}$ for example. The angular

²*Alternative notations.* Sometimes the outer product is written $\mathbf{A} \otimes \mathbf{B}$ and sometimes you see simply $\mathbf{A}\mathbf{B}$. In the latter form it is to be understood that the outer product is intended. The outer product is also called ‘dyadic product’. The symbol \otimes is also used, in other contexts, for a tensor product, and sometimes you will find the dyadic product called a ‘tensor product’, but strictly that is an abuse of terminology.

momentum vector is defined *by convention* to point along the axis of rotation, with a direction such that the rotation is right-handed. Now imagine placing an ordinary arrow-shaped rod next to a rotating wheel, with the rod pointing in the direction of the angular momentum \mathbf{L} of the wheel. Put a mirror next to them. First suppose that the axis of rotation of the wheel is vertical and so is the mirror surface. Now look in the mirror: the arrow, seen in reflection, is still pointing in the same direction, but what has happened to the wheel? Its reflection is rotating in the opposite sense, so its \mathbf{L} vector has reversed direction! The angular momentum vector and the arrow rod have done opposite things: over reversed direction, the other did not.

Now lay the mirror flat, in a horizontal plane. This time the arrow rod changes direction but the rotation does not.

We have in $\mathbf{r} \wedge \mathbf{p}$ a quantity that behaves like a vector under rotations, but has exactly the ‘wrong’ behaviour under reflections. Such a quantity is called a *pseudovector*. Alternatively, the ordinary vectors are called *polar* vectors, and ones like angular momentum are called *axial* vectors. A polar vector is one that changes sign under parity transformations; an axial vector is one that does not. That is, under an inversion of all three coordinate axes, an ordinary vector changes sign—what you would expect—but an axial vector does not.

Axial vectors might seem to be an invention that should have been avoided, but once you are aware of them you will find them throughout physics. We already mentioned one important example, the angular momentum, and another is the magnetic field vector \mathbf{B} . The electric field, on the other hand, is a ‘straightforward’ polar vector. The vector product of two polar vectors (e.g. $\mathbf{r} \wedge \mathbf{p}$) gives an axial vector. The scalar product of a polar vector with an axial vector produces a scalar that changes sign under parity inversions; it is called a *pseudoscalar*.

We mentioned this business of polar and axial vectors in order to introduce the fact that the vector product has to be reconsidered before we can generalize it to more than 3 dimensions.

If we examine the vector product

$$\mathbf{r} \wedge \mathbf{p} = (r_y p_z - r_z p_y)\mathbf{i} + (r_z p_x - r_x p_z)\mathbf{j} + (r_x p_y - r_y p_x)\mathbf{k}, \quad (9.13)$$

we find first a ‘ yz thing’, then a ‘ zx thing’, then a ‘ xy thing’. This suggests we could arrange the pieces into a matrix, by putting the L_x term in the ‘ yz ’ position (2nd row, 3rd column) of the matrix, etc.:

$$\mathbb{L} = \begin{pmatrix} \cdot & L_z & \cdot \\ \cdot & \cdot & L_x \\ L_y & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & r_x p_y - r_y p_x & \cdot \\ \cdot & \cdot & r_y p_z - r_z p_y \\ r_z p_x - r_x p_z & \cdot & \cdot \end{pmatrix}$$

where we haven’t filled in the rest of the matrix yet. However, the arrangement is suggestive, because you can see that it could be obtained from the difference of two outer products:

$$\mathbb{L} = \mathbf{r}\mathbf{p}^T - \mathbf{p}\mathbf{r}^T.$$

Now we have a matrix that is obviously, from its construction, a 2nd rank tensor, and also it behaves the same way under parity changes as other 2nd rank tensors: it does not change sign. Picking out some elements of this 2nd rank tensor and calling them a vector is a ‘trick’ that only works in 3 dimensions. It works because those elements do transform as a vector under rotations. This is partly because an antisymmetric 2nd rank tensor in 3 dimensions has just 3 non-zero independent elements. In 4 dimensions we can construct the tensor, it is antisymmetric and so now has 6 independent non-zero elements, but that is 2 too many to have any hope of making a 4-vector out of them! Instead we can find two 3-vectors, one polar and one axial: more of this later.

The 4-vector generalization of the vector product is, then, a 2nd rank tensor defined by

$$\mathbf{A} \wedge \mathbf{B} \equiv \mathbf{A}\mathbf{B}^T - \mathbf{B}\mathbf{A}^T. \tag{9.14}$$

9.1.3 Differentiation

We already defined the 4-gradient $\square\phi$, the 4-divergence $\square \cdot \mathbf{F}$, and the d’Alembertian $\square^2 \equiv \square \cdot \square$. Two more derivatives naturally suggest themselves:

$$\square\mathbf{A}^T \quad \text{and} \quad \square \wedge \mathbf{A} \equiv \square\mathbf{A}^T - (\square\mathbf{A}^T)^T. \tag{9.15}$$

The first of these should be read as a sort of ‘gradient’ of a vector field, but now the gradient has to say how every component of the vector changes in every direction. By writing out the elements of the tensor in full, you may recognize it as a Jacobian matrix:

$$\square\mathbf{A}^T = \begin{pmatrix} -\frac{1}{c} \frac{\partial A^t}{\partial t} & -\frac{1}{c} \frac{\partial A^x}{\partial t} & -\frac{1}{c} \frac{\partial A^y}{\partial t} & -\frac{1}{c} \frac{\partial A^z}{\partial t} \\ \frac{\partial A^t}{\partial x} & \frac{\partial A^x}{\partial x} & \frac{\partial A^y}{\partial x} & \frac{\partial A^z}{\partial x} \\ \frac{\partial A^t}{\partial y} & \frac{\partial A^x}{\partial y} & \frac{\partial A^y}{\partial y} & \frac{\partial A^z}{\partial y} \\ \frac{\partial A^t}{\partial z} & \frac{\partial A^x}{\partial z} & \frac{\partial A^y}{\partial z} & \frac{\partial A^z}{\partial z} \end{pmatrix}. \tag{9.16}$$

For example, you should confirm that $\square\mathbf{X}^T = g$.

The second quantity in (9.15), a sort of ‘4-curl’, gives an antisymmetric tensor, therefore a set of 6 independent non-zero elements. The second part is written with a double transpose because we want the differential operator to be to the left of \mathbf{A} . You can read the result as $\square \wedge \mathbf{A} = \text{“(thing) - (transpose of thing)”}$ which makes it clear that the outcome is antisymmetric.

The ‘gradient of a vector’ idea is quite useful in 3-vector analysis too. Compare, for example, the horrible

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \wedge (\nabla \wedge \mathbf{v}) + \mathbf{v} \wedge (\nabla \wedge \mathbf{u}) \tag{9.17}$$

with the much more elegant

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\nabla \mathbf{u}^T) \mathbf{v} + (\nabla \mathbf{v}^T) \mathbf{u}. \quad (9.18)$$

(You can prove the latter without much difficulty by converting to components in a rectangular coordinate system).

9.2 Contravariant and covariant

All the 4-vectors and 2nd rank tensors we have been using up till now are termed ‘contravariant’. This term means that they change in the standard way under a change of reference frame, as in eqs. (9.1) and (9.10). We can also construct objects that transform instead by the inverse transformation. That is, if a given change of reference frame is described by Λ , so that our familiar 4-vectors transform as $\mathbf{A}' = \Lambda \mathbf{A}$, then we can construct objects that are just like 4-vectors except they transform as $\underline{\mathbf{A}}' = (\Lambda^{-1})^T \underline{\mathbf{A}}$. Such objects are termed *covariant* 4-vectors.

In order to do calculations in Special Relativity we don’t need to introduce covariant 4-vectors or covariant tensors, because they always have contravariant versions, but they can be useful for reducing clutter, especially once we introduce the index notation in the next section. The terminology seems strange at first: why are the ‘ordinary’ ones called contravariant, and the ‘contrary’ ones called covariant? The reason is partly historical accident, but it is connected to the way the all-important metric tensor behaves. Up till now we have taken it for granted that the metric tensor g does not change from one reference frame to another, but suppose we allow that it might. The metric g is *defined* to be that tensor which allows the invariant ‘distance’ (i.e. spacetime interval in Special Relativity) to be calculated by the form

$$\mathbf{X}^T g \mathbf{X}, \quad (9.19)$$

where \mathbf{X} is the 4-vector displacement between neighbouring events. The requirement that this scalar be invariant is

$$\mathbf{X}^T g \mathbf{X} = \mathbf{X}'^T g' \mathbf{X}' = \mathbf{X}^T \Lambda^T g' \Lambda \mathbf{X}, \quad (9.20)$$

therefore

$$g = \Lambda^T g' \Lambda \quad \Rightarrow \quad g' = (\Lambda^{-1})^T g \Lambda^{-1}. \quad (9.21)$$

By comparing this with eq. (9.10) you can see that g is covariant. So the word ‘covariant’ has the connotation ‘transforming in the same way as the metric tensor’. For the Lorentz transformation the definition (9.3) results in $g' = g$.

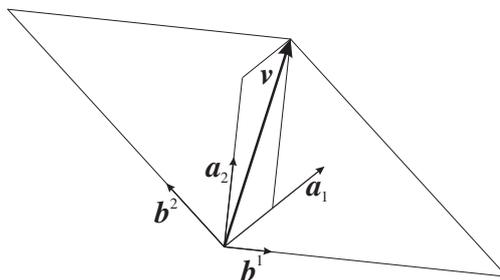


Figure 9.1: Contravariant and covariant components. A given vector \mathbf{v} can be expressed in more than one way: either using the set of basis vectors $\{\mathbf{a}_i\}$, or the set of basis vectors $\{\mathbf{b}^i\}$. Thus $\mathbf{v} = v^1\mathbf{a}_1 + v^2\mathbf{a}_2 = v_1\mathbf{b}^1 + v_2\mathbf{b}^2$ where $\{v^i\}$ and $\{v_i\}$ are the respective sets of components. Neither set of basis vectors need itself be orthonormal. However, the i 'th basis vector in the second set (\mathbf{b}^i) is chosen to be orthogonal to all of the first set except the i 'th member (\mathbf{a}_i), and it is given a length such that its inner product with \mathbf{a}_i is 1. Thus $\mathbf{a}_i \cdot \mathbf{b}^j = \delta_{i,j}$. This permits the inner product between any pair of vectors \mathbf{u}, \mathbf{v} to be written $\sum_{\lambda} u^{\lambda}v_{\lambda}$ as you can confirm by expanding $(u^1\mathbf{a}_1 + u^2\mathbf{a}_2) \cdot (v_1\mathbf{b}^1 + v_2\mathbf{b}^2)$. If \mathbf{a}_i are along the coordinate axes then the components v^i are said to be contravariant. The other set of components, v_i , are said to be covariant.

For other transformations, such as in General Relativity, we could have $g' \neq g$ without losing the role of the metric in constructing an invariant, i.e. one still has $\mathbf{X}^T g \mathbf{X} = \mathbf{X}'^T g' \mathbf{X}$ when \mathbf{X} is an infinitesimal displacement.

A covariant 4-vector can be obtained from a contravariant one by pre-multiplying by g . Let $\underline{\mathbf{A}} = g\mathbf{A}$. Then

$$\underline{\mathbf{A}}' = g'\mathbf{A}' = g'\Lambda\mathbf{A} = (\Lambda^{-1})^T g \Lambda^{-1} \Lambda \mathbf{A} = (\Lambda^{-1})^T (g\mathbf{A}), \quad (9.22)$$

hence $\underline{\mathbf{A}}$ is covariant. You can show similarly that if $\underline{\mathbf{B}}$ is covariant then $(g\underline{\mathbf{B}})$ is contravariant. It follows that the invariant scalar $\mathbf{A} \cdot \mathbf{B}$ can be written either

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T g \mathbf{B} = \mathbf{A}^T \underline{\mathbf{B}}$$

or

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T g \mathbf{B} = (g\mathbf{A})^T \mathbf{B} = \underline{\mathbf{A}}^T \mathbf{B}.$$

This underline notation is not recommended, however, because we are about to replace it with a better one. Its purpose was merely to comment on covariant 4-vectors; it has now fully served its purpose.

There is a simple geometric interpretation of the contravariant and covariant sets of components of a vector, illustrated in figure 9.1. This makes it clear that one must regard the contravariant and covariant forms as two versions of the *same* object, not two different objects.

Contravariant vector or contravariant components?

Some types of argument require care to distinguish between a vector and the set of components which may be used to describe it. Be warned: this can be confusing. My advice is, don't worry about it but just learn the rules of index notation. However, for General Relativity, greater clarity is required. Here is a brief discussion.

A vector is not described by a set of components alone, but by a set of components and another set of vectors, namely the basis vectors. The notation A^a refers to each of the components when the basis is the standard one, i.e. unit vectors along the directions of the axes of some chosen reference frame. The notation A_a refers to a different set of components for the *same* vector (namely A); this is possible because a given vector can be expressed in terms of more than one basis. When using index notation, one does not need to know what this second basis is; it suffices to know that $A^\lambda A_\lambda$ is a Lorentz scalar. However, if you want to take an interest in the basis vectors, consult figure 9.1. When we change reference frame, any given 4-vector such as a 4-momentum does not change, but the basis vectors do change, and usually we would prefer to know the components in terms of the new basis vectors. A matrix equation such as $A' = \Lambda A$ should be regarded as a shorthand for the index notation version, $A'^a = \Lambda^a_\lambda A^\lambda$. This makes it clear that we are here talking about each component of the vector, not the vector itself. The idea of “a contravariant 4-vector” or “a covariant 4-vector” is meaningless, according to this stricter use of terminology. Rather, the set of components A^a is contravariant, and the set A_a is covariant.

The metric tensor can now be understood to play two roles. Its primary role is to show how to calculate an invariant ‘distance’ in spacetime; its secondary role is to allow easy conversion between contravariant and covariant forms of tensors.

9.3 Index notation and tensor algebra

The 4-vector and tensor notation we have employed so far is adequate for most purposes in Special Relativity. However, it becomes more and more awkward as tensors of higher rank are introduced, and some manipulations can be made clearer by a change of notation. This change is to the ‘index notation’ that we shall now introduce. We have taken the trouble to exhibit the matrix methods chiefly in order to help you make the transition to index notation more easily. It can be confusing at first, and to find your way I recommend using both notations ‘side by side’ for a while.

In index notation, a tensor of the k 'th rank is written by exhibiting a representative element, such as A^{ab} , where the number of indices indicates the rank. Thus A^a is a 4-vector, F^{ab} is a 2nd rank tensor, etc. The logic is that whereas the symbol F^{ab} looks like just a single element of a tensor, any formula in which it enters should be taken to be valid no matter which element is picked, so it gives a statement about the whole tensor.

The symbol δ_b^a is the *Kronecker delta* which equals 1 when $a = b$ and 0 when $a \neq b$. In matrix notation this is represented by the identity matrix.

Indices can be superscript or subscript, and the distinction is significant: it is the distinction between contravariant and covariant. Thus A^a is a contravariant 4-vector, A_a is a covariant 4-vector. F^{ab} is a contravariant 2nd rank tensor. A symbol with both types of index, as G^a_b is said to be *mixed*. We say a tensor has *valence* (m, n) if it has m upper (contravariant) indices and n lower (covariant) indices. You are only allowed to sum tensors of the same valence (see later).

Prime notation. It is useful to indicate two different coordinate systems (associated with two different reference frames) by the use of a prime, as in $\{t, x, y, z\}$ and $\{t', x', y', z'\}$. So far when referring to a 4-vector quantity in either frame we have used \mathbf{A} and \mathbf{A}' . In index notation we have two choices: the prime can be attached to the main letter ('kernel') as in A'^a or to the index as in $A^{a'}$. The latter choice is arguably more logical, since when transforming from one coordinate system to another the 4-vector does not itself change, but its components change because the basis vectors change. Therefore we will use $A^{a'}$ in the following. However we will make the other choice (attaching the prime to the kernel) in some situations, where it reduces clutter.

Transformation matrix. For any function f depending on coordinates $\{t, x, y, z\}$ we may write

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

In index notation this is written

$$df = \sum_{\lambda} \frac{\partial f}{\partial x^{\lambda}} dx^{\lambda}. \quad (9.23)$$

If we introduce a second coordinate system $\{t', x', y', z'\}$ then as one explores spacetime each of the primed coordinates is some function of all the unprimed coordinates, to which the above result can be applied, and therefore

$$dx^{a'} = \sum_{\lambda} \frac{\partial x^{a'}}{\partial x^{\lambda}} dx^{\lambda}. \quad (9.24)$$

The partial derivatives on the right hand side form a set of coefficients that characterize the transformation of coordinates $\{x^a\} \rightarrow \{x^{a'}\}$. They may be considered to form a matrix Λ defined by

$$\Lambda^{a'}_b \equiv \frac{\partial x^{a'}}{\partial x^b}. \quad (9.25)$$

For Special Relativity, for example, this set constitutes the Lorentz transformation. The Lorentz transformation matrix is symmetric, so we don't need to separate the indices horizontally in order to know which refers to the column which the row: it doesn't matter; more generally however (in General Relativity, for example) the transformation matrix need not be symmetric (e.g. consider the case $y' = y + z$, $z' = z$).

A *contravariant rank 1 tensor* (or 'vector' for short) is defined as an object which can be specified by giving its components A^a in any given reference frame, and which transforms as

$$A^{a'} = \sum_{\lambda} \frac{\partial x^{a'}}{\partial x^{\lambda}} A^{\lambda} = \sum_{\lambda} \Lambda_{\lambda}^{a'} A^{\lambda}. \quad (9.26)$$

The summation in (9.26) results in precisely the same result as the matrix multiplication in (9.1).

Inverse transformation matrix. A moment's thought should convince you that

$$\frac{\partial x^a}{\partial x^{b'}} = \delta_b^a \quad \text{and} \quad \frac{\partial x^{a'}}{\partial x^{b'}} = \delta_{b'}^{a'}.$$

Applying this to (9.24) we have

$$\sum_{\lambda} \frac{\partial x^{a'}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial x^{b'}} = \delta_{b'}^{a'}. \quad (9.27)$$

This result can be 'read' as saying that, considered as matrices, $\partial x^{a'}/\partial x^a$ and $\partial x^a/\partial x^{a'}$ are inverses of one another. The set of equations (9.26) can be inverted to find expressions for A^a in terms of $A^{\lambda'}$. In matrix notation, this would involve premultiplying both sides of the equation by Λ^{-1} ; in index notation we multiply by $\partial x^a/\partial x^{a'}$ and sum over a' , then make use of (9.27):

$$\sum_{a'} \frac{\partial x^a}{\partial x^{a'}} A^{a'} = \sum_{a'} \sum_{\lambda} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{a'}}{\partial x^{\lambda}} A^{\lambda} = \sum_{\lambda} \delta_{\lambda}^a A^{\lambda} = A^a.$$

In the last step notice how the Kronecker delta symbol has the effect of changing the index from λ to a . Since the index a' on the left is summed over, we can change its name (it is a 'dummy') so the result can be written

$$A^a = \sum_{\lambda'} \frac{\partial x^a}{\partial x^{\lambda'}} A^{\lambda'}.$$

This is exactly what we would have to write if we applied the definition (9.26) directly to the transformation from primed to unprimed system of coordinates. Therefore the concepts are consistent, and so is the notation.

So far we have added nothing to our knowledge of Special Relativity, but we have shown that the concepts apply to arbitrary coordinate transformations, which is useful in General Relativity.

Covariant tensor. A *covariant rank 1 tensor* is defined as an object which can be specified by giving its components B_a in any given reference frame, and which transforms in such a way that

$$\sum_{\lambda} A^{\lambda} B_{\lambda} \quad (9.28)$$

is invariant. A suitable transformation rule for covariant rank 1 tensors is therefore

$$A_{a'} = \sum_{\lambda} \frac{\partial x^{\lambda}}{\partial x^{a'}} A_{\lambda}. \quad (9.29)$$

Proof:

$$\begin{aligned} \sum_{\lambda'} A^{\lambda'} B_{\lambda'} &= \sum_{\lambda'} \sum_{\mu} \sum_{\nu} \frac{\partial x^{\lambda'}}{\partial x^{\mu}} A^{\mu} \frac{\partial x^{\nu}}{\partial x^{\lambda'}} B_{\nu} \\ &= \sum_{\mu} \sum_{\nu} \delta_{\mu}^{\nu} A^{\mu} B_{\nu} \quad [\text{Using (9.27)}] \\ &= \sum_{\nu} A^{\nu} B_{\nu} \quad \text{QED.} \end{aligned} \quad (9.30)$$

If we write the right hand side of (9.29) as $\sum_{\lambda} K_{a'}^{\lambda} A_{\lambda}$ where

$$K_{a'}^a \equiv \frac{\partial x^a}{\partial x^{a'}}$$

then the transpose of K is the inverse of Λ . This is the same as the result we found in section 9.2 for the transformation of covariant 4-vectors.

Summation convention. The above proof could have been written more conveniently in matrix notation (exercise for the reader). So far the index notation is not very attractive: it seems to be turning nice clean and simple matrix products into multiple sums and an unwieldy collection of indices. However, bear with me and I think you will come to like it: everyone else does!

It is standard practice to adopt the *Einstein summation convention*, which is the convention that the sum is understood to be carried out whenever an index is repeated. Thus we would write (9.26) as

$$A^{a'} = \Lambda_{\lambda}^{a'} A^{\lambda} \quad (9.31)$$

and the invariant (9.28) is written simply $A^\lambda B_\lambda$. Notice that the sum always involves one up-index and one down-index. The summation over a repeated index is very much like matrix multiplication, but to make the connection you have to arrange the matrices in the right order, because matrix multiplication is not commutative, and you must pay attention to the ordering of the indices. By contrast, a product in index notation is just a product of scalars (in the sense of single elements of tensors) so the order does not matter. The matrix multiplication rule (eq. (2.10)) is that the indices being summed over are adjacent. Thus

$$A^{a\lambda} g_{\lambda\mu} B^{\mu b} \leftrightarrow \mathbb{A}g\mathbb{B} \quad \text{but} \quad A^{\lambda a} g_{\lambda\mu} B^{b\mu} \leftrightarrow \mathbb{B}g\mathbb{A}.$$

We get the second result by using that g is symmetric so $g_{\lambda\mu} = g_{\mu\lambda}$ then writing the product as $B^{b\mu} g_{\mu\lambda} A^{\lambda a}$ so that the summed indices are next to each other (the second index of B and the first index of g ; the second index of g and the first index of A). Then the conversion to matrix notation is straightforward.

Tensors of any rank. It is now straightforward to define tensors of any rank. They are defined to be objects that can be written down in any given reference frame, and which transform in the same way as outer products of rank 1 tensors. For example a 2nd rank contravariant tensor transforms as

$$A^{a'b'} = \Lambda_{\lambda}^{a'} \Lambda_{\mu}^{b'} A^{\lambda\mu}. \quad (9.32)$$

You should confirm that the implied sums in (9.32) represent precisely the same formula as the matrix multiplications in (9.10).

A tensor of rank 0 is a scalar invariant: it does not change at all under a change of reference frame.

Nontensors. The transformation matrix Λ_b^a is not itself a tensor: it cannot be written down in any one reference frame, but rather it acts as the ‘bridge’ between reference frames. There can be other matrix-like quantities that are not tensors (because they don’t transform in the right way) but are nonetheless useful.

It is interesting to ask whether the Kronecker delta δ_b^a is a tensor. If it is, then the placement of the indices implies that it is of mixed rank, so it ought to transform as

$$\delta_{b'}^{a'} = \Lambda_{\mu}^{a'} \Lambda_{b'}^{\nu} \delta_{\nu}^{\mu} = \Lambda_{\nu}^{a'} \Lambda_{b'}^{\nu} = \frac{\partial x^{a'}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{b'}} = \delta_{b'}^{a'} \quad (9.33)$$

which is correct, so δ_b^a is a (mixed, second-rank) tensor. This can also be proved from the quotient rule.

Quotient rule. In section 9.1 we introduced the second rank tensor by defining it as something which produces a 4-vector when it multiplies a 4-vector, whereas in this section we defined it

as something which transforms as in eq. (9.32). We can now prove that these two definitions are equivalent. The *quotient rule* states that if an expression of the form

$$B^{a\lambda}C_\lambda$$

yields a 4-vector whenever C is a 4-vector, then B must be a tensor (of the type indicated by the placement of its indices), and similar statements apply at all ranks. e.g. if $(B_{d\mu}^{abc}C_g^{f\mu})$ is a tensor for all tensor C_g^{fh} then B_{de}^{abc} is a tensor. The proof is the one given in (9.30), extended in an obvious way.

9.3.1 Rules for tensor algebra

There are four basic legal operations in tensor algebra: *sum*, *outer product*, *contraction* and *index permutation*.

The *sum* of two tensors of the same valence is defined

$$C_{cd\dots}^{ab\dots} = A_{cd\dots}^{ab\dots} + B_{cd\dots}^{ab\dots},$$

i.e. just add corresponding elements. It is easy to prove that this is a tensor if A and B are being evaluated at the same event. Note, however, that when summing tensors at different points in the coordinate space (i.e. different events in spacetime) the sum is a tensor when the transformation is linear, as for example the Lorentz transformation, but not always if it is non-linear, as in General Relativity.

The *outer product* of two tensors is obtained by forming the product of their representative components, as in

$$A^a B^b = C^{ab} \quad \text{or} \quad A_b^a B_{de}^c = C_{bde}^{ac}.$$

Contraction consists in replacing one superscript and one subscript by a dummy index, and summing over it. For example the scalar product of a pair of 4-vectors is obtained by first forming their outer product, and then contracting, so as to obtain:

$$A^\lambda B_\lambda, \quad \text{c.f.} \quad A^T g B = A \cdot B.$$

More generally one could have combinations such as

$$C^{ab} = A^{a\lambda} B_\lambda^b \quad \text{c.f.} \quad \mathbb{C} = \mathbb{A} g \mathbb{B}$$

and

$$B_c^{ab} = A_{\lambda c}^{a\lambda b}.$$

Contraction reduces each valence by 1, and therefore the rank of the tensor by 2. Contracting all the way down to a scalar results in an invariant, so this is an important operation. The two

invariants noted in eq. (9.12) were in fact identified by using index notation, where they take the simple forms

$$F_{\lambda}^{\lambda} \quad \text{and} \quad F^{\lambda\mu}G_{\lambda\mu}. \quad (9.34)$$

This shows that to calculate the second, you just need to multiply corresponding elements and sum.

Index permutation consists in reordering *either* the upper *or* the lower indices (of all terms in a sum). This is a generalized form of the transpose operation (for a 2nd rank tensor, it is equivalent to a transpose of its matrix representation).

Metric. A metric is a ‘way of measuring distance’ in a tensor space; we met it first with the Minkowski metric in chapter 3 and then in a more general form in chapter 7. The essential idea is that an invariant scalar can be formed from the quadratic equation

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \quad (9.35)$$

where g_{ab} is a set of coefficients. These coefficients may be constant (e.g. Minkowski metric in rectangular coordinates) but in general they need not be. However since there is a sum involved we can always take $g_{ab} = g_{ba}$, i.e. the matrix of coefficients is symmetric. Using the quotient rule, we can also deduce that g_{ab} is a tensor, and is covariant. Therefore, for any contravariant 4-vector A^a , the combination $g_{a\lambda}A^{\lambda}$ is a covariant 4-vector, which we may conveniently call A_a . In other words

$$A_a = g_{a\lambda}A^{\lambda}. \quad (9.36)$$

Thus, forming an inner product with the metric tensor has the effect of lowering one index. This is just as we saw in (9.22) for 4-vectors, and you can now prove that it applies more generally to tensors of higher rank.

Equations (9.36) represent a set of equations for A_a in terms of A^a . These can be solved for A^a , yielding another set of equations

$$A^a = g^{a\lambda}A_{\lambda} \quad (9.37)$$

where the matrix g^{ab} is clearly the inverse of the matrix g_{ab} . In index notation the proof is

$$A_a = g_{a\lambda}A^{\lambda} = g_{a\lambda}g^{\lambda b}A_b,$$

which is true for all A_a , and therefore

$$g_{a\lambda}g^{\lambda b} = \delta_a^b. \quad (9.38)$$

This equation can be regarded as the definition of g^{ab} . The object on the left hand side could also be written g_a^b since it can be read as g^{ab} with the first index lowered, so we have

$$g_a^b = \delta_a^b.$$

and therefore $g_\lambda^\lambda = \delta_\lambda^\lambda = 4$. In Special Relativity using rectangular coordinates, one finds that g_{ab} and g^{ab} are represented by the same matrix. More generally, this need not be the case, but (9.38) is always true. We met some examples in chapter 7. The first metric in eq. (7.44) reads $g_{ab} = \text{diag}(-h^2, 1, 1, 1)$. Taking the inverse, we find $g^{ab} = \text{diag}(-1/h^2, 1, 1, 1)$.

Index lowering completes the connection with matrix notation. In that notation, if we only ever write contravariant tensors, then in order to multiply a 2nd rank tensor onto a 4-vector we must introduce the metric, as in

$$F^{a\lambda} g_{\lambda\mu} A^\mu = F^{a\lambda} A_\lambda. \quad (9.39)$$

The right hand side shows that this amounts to lowering an index and contracting. Thus the rules of the index method ensure that we respect the requirement we expressed by (9.11), where we already noted the presence of g .

It is useful to extend the definition of the dot \cdot symbol when using matrix notation:

The form $A \cdot B$ for contravariant tensors A, B of any rank means the combination $A^{\dots\lambda} g_{\lambda\mu} B^{\mu\dots}$. That is, the first index of B is lowered and then a contraction is performed over the last index of A and the first index of B .

N.B. it is understood in this definition that we only ever write contravariant tensors when using matrix notation.

A summary of some of the merits or otherwise of the two notations is given in the following table. Note that both notations allow you to write nonsense such as $A^a \stackrel{?}{=} B^{a\lambda} C^\lambda$ or $A \stackrel{?}{=} \mathbb{B}C$ (in both cases the symbol on the left is a 4-vector but the combination on the right is not). To avoid nonsense it is up to you to obey the rules!

index notation	vectors and matrices
number of indices tells you the rank	font or underline tells you the rank
lots of fiddly indices	less clutter
use further labels with caution	labels are ok, e.g. $P_{\text{tot}} = \sum_i P_i$
upper, lower index to take care of g	use \cdot or remember g
all ranks	only rank 0 to 2
handles everything	restricted
identify invariants easily	invariants less obvious
longer derivations easier	good for the simplest derivations

Conventions in index notation. A long-standing convention in index notation has been to use Greek letters such as μ, ν when the index runs over all components $\{0 \dots 3\} = \{t, x, y, z\}$ and Roman letters such as i, j when the index runs over only the spatial components $\{1, 2, 3\} = \{x, y, z\}$. Thus $T^{\mu\nu}$ is immediately recognisable as a 4-tensor, whereas T^{ij} would be used to signify a 3-tensor (such as, for example, the spatial part of $T^{\mu\nu}$). However, in calculations which deal exclusively in 4-tensors this issue does not arise and we can use whatever alphabet we like. I find “ $T^{ab}J_c$ ” slightly clearer than “ $T^{\alpha\beta}J_\gamma$ ” or “ $T^{\mu\nu}J_\sigma$ ”. The practice adopted in this book is to use Roman letters for free indices and Greek letters for dummy (i.e. summed-over) indices. There is one special case: we reserve the letters i, j for use in 3-tensors: indices labelled by i or j indicate just the 1, 2, 3 ($=x, y, z$) components.

9.3.2 Index notation for derivatives

We introduced 4-vector differentiation by means of the contravariant differential operator \square , and we noted that its components are $(-(1/c)\partial/\partial t, \nabla)$. The minus sign is needed to make sure that it produces an ‘ordinary’, i.e. contravariant, 4-vector when it is used to take the 4-gradient of a scalar potential.

We can now interpret the minus sign another way: the differential operator $((1/c)\partial/\partial t, \nabla)$ must be covariant. The proof is precisely the one we gave in 5.2 where we showed that $\square V$ transforms as a contravariant 4-vector. It is instructive to examine this again in the context of more general tensor notation.

With the summation convention, eq. (9.23) is

$$df = \frac{\partial f}{\partial x^\lambda} dx^\lambda.$$

We take an interest in the partial derivatives of f with respect to some other set of coordinates $\{t', x', y', z'\}$. To this end, divide the equation by $dx^{a'}$ while holding $dx^{\mu' \neq a'}$ constant:

$$\frac{\partial f}{\partial x^{a'}} = \frac{\partial f}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x^{a'}}$$

Since this is true for all f , we have

$$\frac{\partial}{\partial x^{a'}} = \frac{\partial x^\lambda}{\partial x^{a'}} \frac{\partial}{\partial x^\lambda}. \quad (9.40)$$

The right hand side is the transformation appropriate to covariant vectors (c.f. eq. (9.29)), so we have proved that for any invariant scalar function ϕ , $\partial\phi/\partial x^a$ is a covariant 4-vector. The gradient operator $\partial/\partial x^a$ is said to be *naturally covariant*.

Transformation notation. The coordinate transformation matrix $\Lambda^a_{b'}$ can now be written a number of ways. We have used $\Lambda^a_{b'} = \partial x^a / \partial x^{b'}$ which we may also write either $\partial_{b'} x^a$ or $\partial'_{b'} x^a$. In comma notation, it would be $x^a_{,b'}$.

A convenient notation to convey these facts is achieved by the symbols ∂^a and ∂_a , defined by

$$\partial_a \equiv \frac{\partial}{\partial x^a}, \quad \partial^a = \frac{\partial}{\partial x_a} = g^{a\lambda} \partial_\lambda. \quad (9.41)$$

The use of ∂ naturally brings to mind partial differentiation, and in the first definition the x^a on the *bottom* of the partial derivative gives a reminder that the object one obtains should be exhibited with a *lower* index. (9.40) could now be written $\partial_{a'} = K_{a'}^\lambda \partial_\lambda$.

As usual with derivative operators, the order of symbols matters: $\partial_a u^b v^c$ is not the same as $u^b \partial_a v^c$. [A practice that can be useful when a lot of operators are in play is to introduce a comma notation after all the indices: further indices after the comma indicate partial derivatives. Thus a result such as

$$\partial_d(u^a_b v^c) = (\partial_d u^a_b) v^c + u^a_b (\partial_d v^c)$$

would be written

$$(u^a_b v^c)_{,d} = u^a_{b,d} v^c + u^a_b v^c_{,d}.$$

This notation restores full freedom in the order of writing the symbols. We shall not be using it in this book, however, because we don't want to require you to learn new notation unnecessarily.]

The partial derivative operators commute among themselves because $\partial^2 f / \partial x \partial y = \partial^2 f / \partial y \partial x$, etc., (assuming the functions are single-valued). So, for example:

$$\partial_a \partial_b u^c = \partial_b \partial_a u^c.$$

This can be useful in simplifying expressions.

The quantity

$$\partial_\lambda \mathbb{A}^{\lambda b} \quad \text{or} \quad \square \cdot \mathbb{A}$$

is a sort of 'divergence of a tensor', it yields a 4-vector.

9.4 Some basic results

In this section we shall use the word 'tensor', unqualified, to mean '2nd rank tensor'.

In order to ‘read’ a tensor M^{ab} as a matrix, it is helpful to think of the indices a, b as a two-digit number, and to ‘read’ the matrix in the way one reads text in most western languages, i.e. across the top row from left to right, then down to the next row, etc. As the ‘two digit number’ a, b increments, the second digit b changes fastest, and this corresponds to moving along a row in the matrix.

Lowering a first index of a tensor corresponds to premultiplying the matrix by g , thus changing the sign of the first *row*. Lowering a second index corresponds to post-multiplying by g , thus changing the sign of the first *column*. Lowering both indices changes the sign only of the time-space part (i.e. the 0,0 element and the lower right block are unaffected).

It is usually best to derive any non-trivial new results by using index notation, to avoid making mistakes, but it is often clearer to display the outcome in matrix notation.

For example,

$$\begin{aligned}
 A^a &\leftrightarrow A \text{ or } A^T \text{ (depending on context)} \\
 A_a &\leftrightarrow gA \\
 A^\lambda B_\lambda &\leftrightarrow A \cdot B \\
 A^a B^b &\leftrightarrow AB^T \\
 A^\lambda \mathbb{F}^a_\lambda &\leftrightarrow \mathbb{F} \cdot A \\
 A^\lambda \mathbb{F}^a_\lambda &\leftrightarrow A \cdot \mathbb{F} \text{ where the transpose of } A \text{ is understood} \\
 A^\lambda \mathbb{F}_{\lambda a} &\leftrightarrow \text{raise } a \text{ then use the above} \\
 \mathbb{F}^{a\lambda} \mathbb{G}_\lambda^b &\leftrightarrow \mathbb{F} \cdot \mathbb{G} \\
 \mathbb{F}^{\lambda a} \mathbb{G}_\lambda^b &\leftrightarrow \mathbb{F}^T \cdot \mathbb{G} \text{ (since } \mathbb{F}^{\lambda a} = (\mathbb{F}^T)^{a\lambda}) \\
 A_\lambda \mathbb{F}^{\lambda a} &\equiv \mathbb{F}^{\lambda a} A_\lambda \leftrightarrow A \cdot \mathbb{F} \equiv \mathbb{F}^T \cdot A
 \end{aligned}$$

The necessity or otherwise of a transpose operation can require some thought. In the dot notation we take it for granted that a 4-vector will be transposed as necessary in order for the multiplication or equality to make sense. Thus the last expression can be understood as a shorthand for $A^T g^T \mathbb{F} \equiv (\mathbb{F}^T g A)^T$. In the penultimate result, on the other hand, the transpose of the 2nd-rank tensor has to be indicated explicitly since $\mathbb{F} \cdot \mathbb{G}$ and $\mathbb{F}^T \cdot \mathbb{G}$ are both legal but are not the same (unless \mathbb{F} is symmetric).

The combination $A \cdot \mathbb{F} \cdot B$ is a scalar and therefore is unaffected by a transpose, hence

$$A \cdot \mathbb{F} \cdot B = B \cdot \mathbb{F}^T \cdot A. \tag{9.42}$$

The order of the indices of a given tensor in index notation does matter and must be respected. For example \mathbb{A}^{ab} is not necessarily equal to \mathbb{A}^{ba} , and \mathbb{A}^a_b is not necessarily equal to \mathbb{A}_b^a . This

Tips for manipulating tensor equations

1. Name your indices sensibly; make repeated indices easy to spot.
2. Look for scalars. e.g. $\mathbb{F}_{\lambda\mu} \mathbb{A}_b^a \mathbb{F}^{\lambda\mu}$ is $s \mathbb{A}_b^a$ where $s = \mathbb{F}_{\lambda\mu} \mathbb{F}^{\lambda\mu}$.
3. You can always change the names of dummy (summed over) indices; if there are two or more, you can swap names.
4. The ‘see-saw rule’

$$A_\lambda B^\lambda = A^\lambda B_\lambda \quad (\text{works for any rank})$$

5. ∂_a behaves like $\partial/\partial x$
6. In the absence of ∂_a , everything commutes.

point is sometimes treated rather loosely in the literature. The only exception is when a tensor is symmetric, and this is why we can write δ_b^a without bothering to indicate which index is first, which second.

The box presents some basic tips for manipulating equations involving index notation. The ‘see-saw’ rule is obvious for 4-vectors, but it is true more generally. It comes from the fact that inserting two g ’s so as to lower one index and raise the other is an identity operation when the indices are the same. Proof: we have $\mathbb{A}_{a\dots} = g_{a\lambda} \mathbb{A}^{\lambda\dots}$ and $\mathbb{B}^{a\dots} = g^{a\mu} \mathbb{B}_\mu^{\dots}$ for any \mathbb{A} , \mathbb{B} , where the dots signify other indices (which may more generally be up or down and in any order). Therefore

$$\mathbb{A}_{a\dots} \mathbb{B}^{a\dots} = \mathbb{A}^{\lambda\dots} \mathbb{B}_\mu^{\dots} g_{\lambda a} g^{a\mu} = \mathbb{A}^{\lambda\dots} \mathbb{B}_\mu^{\dots} g_\lambda^\mu = \mathbb{A}^{\lambda\dots} \mathbb{B}_\lambda^{\dots}$$

where the first step used that g is symmetric, and the last step used the fact that $g_\lambda^\mu = \delta_\lambda^\mu$.

The product rule for differentiation reads, for some generic tensors A and B ,

$$\partial^\bullet (A^{\bullet\bullet} B_{\bullet\bullet\bullet}) = B_{\bullet\bullet\bullet} (\partial^\bullet A^{\bullet\bullet}) + A^{\bullet\bullet} (\partial^\bullet B_{\bullet\bullet\bullet}) \quad (9.43)$$

where the dots signify any combination of indices, not necessarily repeated. It is just like taking the derivative of a product of scalars, because, after all, each element of a tensor *is* just a scalar (in the sense of a single number, not a Lorentz scalar).

Let’s examine some basic examples of this. First, consider a scalar product of two 4-vectors:

$$\partial^a (U^\lambda V_\lambda) = (\partial^a U^\lambda) V_\lambda + (\partial^a V^\lambda) U_\lambda. \quad (9.44)$$

$$\epsilon_{01cd} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}, \quad \epsilon_{02cd} = \begin{pmatrix} 0 & & & \\ & 0 & & -1 \\ & & 0 & \\ & 1 & & 0 \end{pmatrix}, \quad \text{etc.}$$

Table 9.1: Evaluating the Levi-Civita symbol.

When $\mathbf{U} = \mathbf{V}$ we obtain

$$\begin{aligned} \partial^a(\mathbf{U}^\lambda \mathbf{U}_\lambda) &= 2(\partial^a \mathbf{U}^\lambda) \mathbf{U}_\lambda, \\ \text{i.e.} \quad \square(\mathbf{U} \cdot \mathbf{U}) &= 2(\square \mathbf{U}) \cdot \mathbf{U}. \end{aligned} \tag{9.45}$$

A useful case to remember is when the *size* of a 4-vector \mathbf{U} is constant (i.e. independent of time and space), so that $\square(\mathbf{U} \cdot \mathbf{U}) = 0$. Then (9.45) says that each row of its gradient tensor $\partial^a \mathbf{U}^\lambda$ is orthogonal to \mathbf{U} .

Another useful application of the product rule is observed in expressions such as

$$A_{\mu\nu} \partial_a A^{\mu\nu}.$$

Noting the repeated indices, we should like to think we have a scalar, which is almost right. We just need to spot that

$$A_{\mu\nu} \partial_a A^{\mu\nu} = \frac{1}{2} \partial_a s \quad \text{where} \quad s = A_{\mu\nu} A^{\mu\nu}. \tag{9.46}$$

(check it by applying (9.43) to the right hand side). This is the generalisation of the familiar $(d/dx)(f^2) = 2f(df/dx)$.

Some relationships between tensors can be found by using the *Levi-Civita symbol* or ‘permutation symbol’ ϵ_{abcd} . This is defined

$$\epsilon_{abcd} = \begin{cases} +1 & \text{if } abcd \text{ is an even permutation of } 0123 \\ -1 & \text{if } abcd \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases} \tag{9.47}$$

This is a 4-dimensional object (there are versions for any number of dimensions), but only $4! = 24$ of its elements are non-zero, half of them $+1$ and half -1 , see table 9.1. It is defined to be invariant (it is what it is: it doesn’t matter what reference frame you are working in), but it is easy to prove that it always converts tensors to pseudotensors, so it is itself a pseudotensor.

Note that $\epsilon_{abcd} = -\epsilon^{abcd}$, which is a source of ambiguity in the literature: some authors chose ϵ^{0123} to be 1.

Consider the combination $\epsilon_{ab\lambda\mu}\mathbb{F}^{\lambda\mu}$. Pick for example $a = 2$, $b = 3$. You can see that the $(2, 3)$ element of the result is made from \mathbb{F}^{01} and \mathbb{F}^{10} , the latter subtracted from the former. If \mathbb{F} is symmetric then this is zero, if it is antisymmetric then this is $2\mathbb{F}^{01}$.

9.4.1 Antisymmetric tensors

Most tensors one encounters in physics are either symmetric or antisymmetric. A symmetric tensor has 10 independent elements (6 for the upper triangle, which also gives the lower triangle, plus 4 more on the diagonal). An antisymmetric tensor in 4 dimensions has 6 independent elements (if these form the upper triangle then the lower triangle is the negative of this and the diagonal is zero).

In view of what we learned about the vector product for 3-vectors (section 9.1.2), this makes it tempting to suggest that an antisymmetric tensor might be expressible as a combination of two 3-vectors. This is indeed correct. If the tensor \mathbb{F} is antisymmetric, then it can be associated with two vectors, one polar and one axial (i.e. a ‘pseudovector’), as follows:

$$\begin{aligned}\mathbf{a} &= (\mathbb{F}^{tx}, \mathbb{F}^{ty}, \mathbb{F}^{tz}) \quad (\text{polar}) \\ \mathbf{b} &= (\mathbb{F}^{yz}, \mathbb{F}^{zx}, \mathbb{F}^{xy}) \quad (\text{axial})\end{aligned}$$

i.e.

$$\mathbb{F} = \begin{pmatrix} 0 & a_x & a_y & a_z \\ -a_x & 0 & b_z & -b_y \\ -a_y & -b_z & 0 & b_x \\ -a_z & b_y & -b_x & 0 \end{pmatrix}.$$

You can then see that, for any 4-vector $U = (U^0, \mathbf{u})$,

$$\mathbb{F}^{a\lambda}U_\lambda = W^a = (\mathbf{u} \cdot \mathbf{a}, U^0\mathbf{a} + \mathbf{u} \wedge \mathbf{b}) \quad (9.48)$$

where we introduced W^a merely to emphasize that this result is contravariant. Notice that in view of the lowered index on U_λ , we needed to use g to calculate the result correctly.

By a similar calculation,

$$\partial_\lambda \mathbb{F}^{\lambda a} = \left(-\nabla \cdot \mathbf{a}, \frac{1}{c} \frac{\partial \mathbf{a}}{\partial t} - \nabla \wedge \mathbf{b} \right). \quad (9.49)$$

For two reference frames in standard configuration, the Lorentz transformation of an antisymmetric tensor has sufficiently few terms that it is worth presenting it in full:

$$\mathbb{F}'^{ab} = \begin{pmatrix} 0 & \mathbb{F}^{tx} & \gamma(\mathbb{F}^{ty} - \beta\mathbb{F}^{xy}) & \gamma(\mathbb{F}^{tz} - \beta\mathbb{F}^{xz}) \\ \cdot & 0 & \gamma(\mathbb{F}^{xy} - \beta\mathbb{F}^{ty}) & \gamma(\mathbb{F}^{xz} - \beta\mathbb{F}^{tz}) \\ \cdot & \cdot & 0 & \mathbb{F}^{yz} \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}, \quad (9.50)$$

where the dots indicate that the lower elements are to be assigned in an antisymmetric fashion. By extracting the two vectors, and recalling that the direction of relative motion is along x , one finds

$$\begin{aligned} \mathbf{a}'_{\parallel} &= \mathbf{a}_{\parallel}, & \mathbf{a}'_{\perp} &= \gamma(\mathbf{a}_{\perp} + \boldsymbol{\beta} \wedge \mathbf{b}), \\ \mathbf{b}'_{\parallel} &= \mathbf{b}_{\parallel}, & \mathbf{b}'_{\perp} &= \gamma(\mathbf{b}_{\perp} - \boldsymbol{\beta} \wedge \mathbf{a}). \end{aligned} \quad (9.51)$$

If the above results have not reminded you of electromagnetic fields, then you have not been paying attention! That application is explored in chapter 12.

For antisymmetric \mathbb{F} , the tensor (actually a pseudotensor)

$$\tilde{\mathbb{F}}_{cd} \equiv \frac{1}{2} \epsilon_{cd\mu\nu} \mathbb{F}^{\mu\nu} \quad (9.52)$$

is called the *dual* of \mathbb{F} . It does not take long to check the 6 terms and you will find that the matrix for $\tilde{\mathbb{F}}_{cd}$ looks like the one for \mathbb{F}^{cd} but with \mathbf{a} and \mathbf{b} swapped. It follows that $\tilde{\mathbb{F}}^{cd}$ can be obtained from \mathbb{F}^{cd} by the substitutions $\mathbf{a} \rightarrow -\mathbf{b}$, $\mathbf{b} \rightarrow \mathbf{a}$. For an example, see eqs. (12.5) and (??).

An important antisymmetric tensor is the angular momentum tensor defined by

$$L^{ab} \equiv X^a P^b - X^b P^a \quad (9.53)$$

for a particle whose position and momentum are given by \mathbf{X} and \mathbf{P} . This is the subject of the next chapter.

Chapter 11

Lagrangian mechanics

It is assumed that the reader has met the Principle of Least Action in classical mechanics, and the related concepts of the Lagrangian, the Hamiltonian, and the Euler-Lagrange equations. In this chapter we shall examine their Special Relativistic generalisation. We begin with a summary of the classical results, both as a reminder, and to introduce notation.

11.1 Classical Lagrangian mechanics

Students usually first meet classical mechanics in the setting of Newton's laws, and the formula

$$\mathbf{f} = \frac{d\mathbf{p}}{dt},$$

which we shall write in the form

$$-\nabla V = m \frac{d\dot{\mathbf{x}}}{dt}.$$

The basic idea of Lagrangian mechanics is to replace this vector treatment by a treatment based on a scalar quantity called the Lagrangian, which allows vector equations to be extracted by taking derivatives (just as ∇V is a vector extracted from the potential energy V). This approach proves to be more flexible and it simplifies many problems in mechanics.

At any given instant of time, the state of a physical system is described by a set of n variables q_i called *coordinates*, and their time derivatives \dot{q}_i called *velocities*. For example, these could be the positions and velocities of a set of particles making up the system, though later we shall allow a more general notion of a coordinate.

Define a function \mathcal{L} called the *Lagrangian*, given by

$$\mathcal{L} = T - V \quad (11.1)$$

where T and V are the kinetic energy and potential energies of the system. The Lagrangian is therefore a function of the positions and velocities, and it can be a function of time. This is indicated by the notation $\mathcal{L} = \mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t)$, which we shall abbreviate to $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$.

For particle motion with no external time-dependent fields, the Lagrangian has no explicit dependence on time. The phrase ‘no explicit dependence on time’ means it has no dependence on time over and above that which is already implied by the fact that q and \dot{q} may depend on time. For example, a single particle undergoing simple harmonic motion has the Lagrangian $\mathcal{L} = (1/2)m(\dot{x}^2 - \omega^2 x^2)$. An example motion of the particle is $x = x_0 \sin(\omega t)$, and for this motion the Lagrangian can also be written $(m\omega^2 x_0^2/2) \cos 2\omega t$ —a function of time. However, the latter form hides the dependence on x and \dot{x} which is what we are chiefly interested in, and furthermore in general the Lagrangian cannot be deduced from the motion, but the motion can be deduced from the Lagrangian when the latter is written as a function of coordinates and velocities. For this reason the variables $\{q_i, \dot{q}_i\}$ are said to be the ‘natural’ or ‘proper’ variables of \mathcal{L} . (A similar issue arises in the treatment of functions of state in thermodynamics).

The time integral of the Lagrangian along a path $q(t)$ is called the *action* S :

$$S[q(t)] = \int_{q_1, t_1}^{q_2, t_2} \mathcal{L}(q, \dot{q}, t) dt \quad (11.2)$$

The *Principle of Least Action* states that the path followed by the system is the one that gives an extreme value (maximum or minimum) of S with respect to small changes in the path. (The title ‘Least’ action comes from the fact that in practice a minimum is more usual than a maximum). The path is to be taken between given starting and finishing ‘positions’ q_1, q_2 at times t_1, t_2 .

To find the extremum of S , we need to ask for a zero derivative with respect to changes in all the variables describing the path. The calculus of variations may be used to show that the result is that S reaches an extremum for the path satisfying

Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i}. \quad (11.3)$$

The physical interpretation of this set of equations is found by discovering its implications. The end result of such a study may be summarized:

$$\begin{array}{ccc} \frac{d}{dt} & \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) & = & \left(\frac{\partial \mathcal{L}}{\partial q} \right) \\ \text{(rate of change of)} & \text{('momentum')} & = & \text{('force')} \end{array}$$

The ‘force’ here is called a *generalized force*, and the ‘momentum’ is called *canonical momentum*, defined

$$\tilde{p}_i \equiv \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right). \quad (11.4)$$

In the simplest cases, such as motion of a free particle, or a particle subject to conservative forces, the canonical momentum may be equal to a familiar momentum such as linear momentum or angular momentum, but this does not have to happen. A counter-example occurs for the motion of a particle in a magnetic field, as we shall see.

The *Hamiltonian* of a system is defined

$$\mathcal{H}(q, \tilde{p}, t) \equiv \sum_i^n \tilde{p}_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t) \quad (11.5)$$

where the \dot{q}_i are to be written as functions of the q_i and \tilde{p}_i so that the result is a function of coordinates and canonical momenta (the natural variables of the Hamiltonian). For conservative forces one finds that the sum in (11.5) evaluates to twice the kinetic energy, and then $\mathcal{H} = T + V$, which is clearly the total energy of the system.

The Euler-Lagrange equations imply

Hamilton’s canonical equations

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial \tilde{p}_i}, \quad \frac{d\tilde{p}_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}. \quad (11.6)$$

Thus the Hamiltonian with the canonical equations offer an alternative to the Lagrangian with the Euler-Lagrange equations. (In field theory one usually makes use of both.)

11.2 Relativistic motion

In generalising Lagrangian mechanics to Special Relativity, we shall proceed in two steps. First we ask the question, are the Euler-Lagrange equations (and their counterparts the canonical equations) still valid? The answer is *yes, as long we use the right Lagrangian*. However, such a formulation is only partially useful. It can correctly generate 3-vector equations such as $-\nabla V = \gamma m \mathbf{v}$, but it does not immediately give the 4-force. Therefore the second step will be

to reconsider the action and Lagrangian from a more thoroughly ‘4-dimensional’ (spacetime) point of view.

11.2.1 From classical Euler-Lagrange

First we consider the argument based on the classical formula for the action, eq. (11.2). We restrict attention to a single particle, and write the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \quad (11.7)$$

where $\mathcal{L}_{\text{free}}$ is the Lagrangian for a free particle, and \mathcal{L}_{int} is the part describing interaction with something else such as an electromagnetic field.

For a single particle, the complete path of the system (i.e. the specification of $q_i(t)$ for all the coordinates) is simply the worldline of the particle. In this case it is straightforward to write the action integral as an integral with respect to proper time τ along the worldline:

$$S[q(t)] = \int_{q_1, t_1}^{q_2, t_2} \mathcal{L}(q, \dot{q}, t) dt = \int_{(1)}^{(2)} \mathcal{L} \gamma d\tau \quad (11.8)$$

where we used the by now familiar $dt/d\tau = \gamma$. We already know an important property of free motion: it maximises the proper time. This suggests the Lagrangian for free motion should be such that $\gamma \mathcal{L}_{\text{free}}$ is a constant. With this hint, we propose

$$\mathcal{L}_{\text{free}} = -mc^2/\gamma = -mc^2(1 - v^2/c^2)^{1/2}. \quad (11.9)$$

You can check that this gives the canonical momenta $\partial \mathcal{L}_{\text{free}}/\partial v_i = \gamma m v_i$, i.e. the three components of the relativistic 3-momentum.

We shall next treat the case of electromagnetic interactions. We propose (or guess) the interaction term \mathcal{L}_{int} and then prove that it gives the right equation of motion of the particle. Consider then

$$\mathcal{L}_{\text{int}} = q\mathbf{U} \cdot \mathbf{A}/\gamma = q(-\phi + \mathbf{v} \cdot \mathbf{A}). \quad (11.10)$$

After adding this to $\mathcal{L}_{\text{free}}$, one obtains the three canonical momenta

$$\frac{\partial \mathcal{L}}{\partial v_i} = \gamma m v_i + q A_i \quad (11.11)$$

which can be expressed

$$\tilde{\mathbf{p}} = \gamma m \mathbf{v} + q \mathbf{A}. \quad (11.12)$$

This equation commonly causes confusion. It does *not* mean the momentum of the particle has changed. The *momentum* (i.e. that which is conserved in collisions and influenced by forces) is still $\gamma m \mathbf{v}$. The *canonical momentum* (i.e. that which has a rate of change given by the gradient of \mathcal{L}) is $\gamma m \mathbf{v} + q \mathbf{A}$.

Now write the Euler-Lagrange equations:

$$\frac{d}{dt}(\gamma m \mathbf{v} + q \mathbf{A}) = q(-\nabla \phi + \nabla(\mathbf{v} \cdot \mathbf{A})) \quad (11.13)$$

The $d\mathbf{A}/dt$ term on the left has two parts, because a change in \mathbf{A} along the worldline is made of the time change of the field, plus a part owing to the fact that the moving particle visits a different place:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{A} \quad (11.14)$$

(c.f. eq. (11.22)). Substituting this in (11.13) gives

$$\begin{aligned} \frac{d}{dt}(\gamma m \mathbf{v}) &= -q \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right) + q(\nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla) \mathbf{A}) \\ &= q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \end{aligned} \quad (11.15)$$

where we used the vector identity

$$\mathbf{v} \wedge (\nabla \wedge \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla) \mathbf{A}.$$

Equation (11.15) is the correct equation for relativistic motion in an electromagnetic field, so we have confirmed that our choice of Lagrangian was correct and also that the Euler-Lagrange equations are valid as they are: they do not need to be modified, and they take the same form in all inertial frames of reference. They are covariant, but not manifestly covariant. The only drawback of the present approach is that one must pick a frame of reference before starting the calculation of the motion in any given case. In practice the maths is often easier if one does that anyway, so it is not much of a drawback. Nevertheless, we should like to see a frame-independent formulation, i.e. a manifestly covariant formulation, if we can. That is the subject of the next section.

[Section omitted in lecture-note version.]

11.2.2 Manifestly covariant

The ‘problem’ with the Lagrangian presented in eq. (11.9) and (11.10) is that it is not a Lorentz scalar. However, it gives a hint to what Lorentz scalar Lagrangian we could try:

$$\mathcal{L}(\mathbf{X}, \mathbf{U}) = -mc(-\mathbf{U} \cdot \mathbf{U})^{1/2} + q\mathbf{U} \cdot \mathbf{A}. \quad (11.16)$$

We use this in the action integral

$$S[\mathbf{X}(\tau)] = \int_{(\mathbf{X}_1)}^{(\mathbf{X}_2)} \mathcal{L}(\mathbf{X}, \mathbf{U}, \tau) d\tau \quad (11.17)$$

which is also a Lorentz scalar.

The inclusion of $\mathbf{U} \cdot \mathbf{U}$ in (11.16) raises a subtle point that merits a comment. We know that the velocity is a ‘unit vector’ with $\mathbf{U} \cdot \mathbf{U} = -c^2$, so why not write $\mathcal{L} = -mc^2 + q\mathbf{U} \cdot \mathbf{A}$? The problem with this version is that when substituted into the relativistic Euler-Lagrange equations, it does not result in the correct equation of motion. We have lost the information about the kinetic energy of the particle. One can get around this problem in more than one way, but the most convenient is to insist on the form $mc(-\mathbf{U} \cdot \mathbf{U})^{1/2}$ and keep in mind that the Lagrangian is not to be regarded as a property of the particle, but as a function whose ‘job’ is to tell us how the action changes if there are changes in the path. We shall comment further on this at the end.

One way to handle the minimisation of the action (11.17) is to change variables back to t in the integral, and then look for a minimum with respect to variations in the path. It is immediately clear that we shall regain the same Euler-Lagrange equations as before, and the same equations of motion. Nonetheless, we shall pursue the manifestly covariant formulation a little further, to see if we can learn anything new.

By minimizing the action with respect to variations of the worldline (see box), one finds the manifestly covariant Euler-Lagrange equations

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial U^a} = \frac{\partial \mathcal{L}}{\partial X^a}. \quad (11.19)$$

Now we extract the relativistic canonical momentum

$$\frac{\partial \mathcal{L}}{\partial U^a} = \frac{mc}{(-\mathbf{U} \cdot \mathbf{U})^{1/2}} U_a + qA_a \quad (11.20)$$

Use of a parameter to minimize the action. There is an important difference between (11.17) and (11.2) although they appear at first glance to be similar. The difference is that in (11.2) we know from the outset the values of t_1, t_2 (as well as q_1, q_2) at the beginning and end of any path. This is important: the variational calculation requires that they are fixed, i.e. the same for all paths. In (11.17), as it stands, the end points are defined by two events, but the value of the integration variable τ at those events will be different from one path to another. Therefore the calculus of variations cannot be applied to the integral as it stands. This situation is handled by introducing a parameter λ that increases monotonically along the path, and whose start and end values can be fixed at some λ_1 and λ_2 . The action integral then reads

$$\int \mathcal{L}(\mathbf{X}, \dot{\mathbf{X}}, \tau) d\tau = \int_{\lambda_1}^{\lambda_2} \mathcal{L} \frac{d\tau}{d\lambda} d\lambda.$$

This version has fixed limits, and now the Lagrangian is

$$\tilde{\mathcal{L}} = \mathcal{L} \frac{d\tau}{d\lambda}$$

where $\tilde{\mathcal{L}}$ should be written and treated as a function of \mathbf{X} and $d\mathbf{X}/d\lambda$. In our case we have $d\tau^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2$ so

$$\frac{d\tau}{d\lambda} = \frac{1}{c} \left(-g_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda} \right)^{1/2}.$$

The minimization procedure can now go through and we have the Euler-Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{X}^a} = \frac{\partial \tilde{\mathcal{L}}}{\partial X^a}, \quad (11.18)$$

where the dot signifies $d/d\lambda$. Owing to the presence of $d\tau/d\lambda$ the new Lagrangian looks rather cumbersome, but fortunately by a good choice of the parameter λ we can now simplify the equations. One possible choice is to define λ as the value of τ *along the solution worldline*. For that worldline, and for that worldline only (but it is the only one we are interested in from now on), we must then find $d\tau/d\lambda = 1$ and $\tilde{\mathcal{L}} = \mathcal{L}$ and $\dot{X}^a = U^a$. Then the Euler-Lagrange equations become the very equations (11.19) that we would have written had we been ignorant of this issue!

There is one limitation to this ‘trick’ however. If the original Lagrangian has no dependence on one of the variables *and* its velocity, then the set (11.19) will include an equation reading $0 = 0$ which is true but not helpful. Then we must return to (11.18) and make some other choice of λ . For example, setting λ equal to one of the variables X^a is often a good choice.

where we used

$$\frac{\partial}{\partial U^a} (\mathbf{U} \cdot \mathbf{U}) = \frac{\partial}{\partial U^a} (U^\lambda g_{\lambda\mu} U^\mu) = g_{a\mu} U^\mu + U^\lambda g_{\lambda a} = 2U_a$$

Evaluation of $dA/d\tau$

For any function that depends on position and time we may write

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial t}\right)_{x,y,z} dt + \left(\frac{\partial f}{\partial x}\right)_{t,y,z} dx + \left(\frac{\partial f}{\partial y}\right)_{t,x,z} dy + \left(\frac{\partial f}{\partial z}\right)_{t,x,y} dz \\ \Rightarrow \frac{df}{d\tau} &= \left(\frac{\partial f}{\partial t}\right) \frac{dt}{d\tau} + \left(\frac{\partial f}{\partial x}\right) \frac{dx}{d\tau} + \left(\frac{\partial f}{\partial y}\right) \frac{dy}{d\tau} + \left(\frac{\partial f}{\partial z}\right) \frac{dz}{d\tau} \\ &= (\partial_\lambda f) \frac{dx^\lambda}{d\tau}. \end{aligned}$$

Since this result applies to all f , we may write

$$\frac{d}{d\tau} = \frac{dx^\lambda}{d\tau} \partial_\lambda \quad (11.21)$$

and this may be applied to all the components of any tensor. For example,

$$\frac{dA}{d\tau} = \frac{dx^\lambda}{d\tau} \partial_\lambda A = U^\lambda \partial_\lambda A. \quad (11.22)$$

and we assumed A is independent of U . (A is the potential *experienced* by the particle, not the one produced by the particle.)

Now we can safely replace $U \cdot U$ by $-c^2$ because we no longer need partial derivatives of this quantity with respect to components of U , so we find

$$\tilde{P}_a \equiv \frac{\partial \mathcal{L}}{\partial U^a} = mU_a + qA_a. \quad (11.23)$$

(c.f. eq (11.12)).

Notice that our manifestly covariant Lagrangian (11.16) differs by a factor γ from the one we used in the previous section, yet we get the same canonical momentum: $\tilde{\mathbf{p}}$ in eq. (11.12) is the spatial part of $\tilde{\mathbf{P}}$. The reason is that the relation between Lagrangian and action is different: in the first case we had an integral with respect to reference frame time t , now we have an integral with respect to proper time τ . This resulted in a different set of Euler-Lagrange equations: (11.19) instead of (11.3). To confirm the agreement between the two approaches, one can manipulate (11.19), replacing $d/d\tau$ on the left hand side by $(dt/d\tau)d/dt = \gamma d/dt$, and writing $U^a = \gamma dX^a/dt$, then one regains (11.3) as long as one makes the replacement $\mathcal{L} \rightarrow \mathcal{L}/\gamma$.

The right hand side of the Euler-Lagrange equation (11.19) is $q\partial_a(U^\lambda A_\lambda) = qU^\lambda \partial_a A_\lambda$ so the equation reads

$$\frac{d}{d\tau} (mU_a + qA_a) = qU^\lambda \partial_a A_\lambda.$$

This is like eq (11.13). Now use

$$\frac{dA_a}{d\tau} = U^\lambda \partial_\lambda A_a$$

(c.f. eqs (11.22) and (11.14)), giving

$$m \frac{dU_a}{d\tau} = q((\partial_a A_\lambda) - (\partial_\lambda A_a)) U^\lambda \quad (11.24)$$

$$\text{or} \quad \frac{dP}{d\tau} = q(\square \wedge A) \cdot U \quad (11.25)$$

We have found that the 4-force associated with the potential A is $q(\square \wedge A) \cdot U$. You can verify that this gives once again the correct equation for motion in an electromagnetic field, or else simply read on since this will be explored in the next chapter.

Further comment on $U \cdot U$

Since the combination $U \cdot U = -c^2$, one may wish to adopt the Lagrangian $-mc^2$ for free motion. This can be done but then the information that $U \cdot U = -c^2$ has to be incorporated into the action minimisation procedure. One has a *constrained* minimisation.

Keeping $U \cdot U$ in the Lagrangian leads to an easier solution, but one may be uneasy about the meaning of terms such as $\partial \mathcal{L} / \partial U^a$, because this quantity refers to a change in the Lagrangian when one component of U is changed *while keeping other components of U fixed*. However, one might argue, it is not possible to change one component of a 4-velocity while keeping all the other components fixed. If one component changes on its own, the size of the 4-velocity will change. To maintain the size fixed, another component must change to compensate.

This objection muddles two different things, namely path variations considered in the calculus of variations, and the evolution actually followed by the system. Consider a more familiar and simpler example: classical motion in a circle. When a particle moves in a fixed uniform magnetic field, the speed remains constant. Therefore, throughout the motion, changes in v_x are accompanied by changes in v_y , with the result that $(v_x^2 + v_y^2)$ is independent of time (for a B field in the z direction). However this does not mean that it is illegal to consider $\partial \mathcal{L} / \partial v_x$ or $\partial \mathcal{L} / \partial v_y$. By considering the effect of such ‘excursions’ while minimising the action, one arrives at the very equation (Euler-Lagrange) which ensures that the v_x and v_y changes are coupled in the right way. Similarly, in the relativistic case, one may postpone applying the constraint on the size of U , because after the whole procedure yields a prediction for the worldline, one finds that the worldline satisfies the constraint anyway!

[Section omitted in lecture-note version.]

Chapter 12

Further electromagnetism

We are now ready to ‘reinvent’ electromagnetism. The approach taken in chapter 6 was to introduce the electric and magnetic fields in terms of the forces exerted on charged particles, and to reason from Lorentz transformations, from easily analysed basic phenomena, and from the Maxwell equations. We mentioned that the electric and magnetic fields should be regarded as two parts of a single entity, but it remained less than clear what that entity might be. We are now in a position to display it clearly: it is an antisymmetric 2nd rank tensor.

At the end of section 6.5.3 we examined the claim that the whole theory of electromagnetism can be derived from Coulomb’s Law and Lorentz covariance. This claim seemed attractive at first, but on further consideration it turned out to be far too sweeping. It is based on several tacit assumptions, some of which are quite subtle. It is an important skill in physics to be able to identify what non-trivial assumptions have in fact been invoked in any given argument.

In this chapter we shall obtain the Maxwell equations and the Lorentz force equation from an explicit set of assumptions, restricting ourselves as far as possible to the simplest possible assumptions that are consistent with Lorentz covariance, and that give rise to some sort of field theory (i.e. a theory of point-like entities called particles interacting via extended entities called fields). In other words we shall show that electromagnetic theory can be considered to be one of the most simple possible field theories. The mathematical language of tensors guides us very quickly to the right formulation.

We shall present most of the equations in both index notation and matrix notation, using the former for any working that is required.

12.1 Fundamental equations

To this end, suppose we want to construct a field theory with two basic physical elements. These will be fields (whose nature is to be discovered) and material particles. By a field we mean simply something that exerts a force on a particle, and we shall assume that the particles in turn give rise to the field through a property we shall call their charge. For simplicity, we take it that the charge is a scalar invariant. Don't forget, we are in the process of inventing a theory, so we can hypothesize anything we like; we are constrained only by the language of tensors (to maintain covariance) and the policy of simplicity.

We shall therefore further assume that the force is pure, i.e. rest mass preserving: that is a great simplification if we can achieve it.

Now let's consider whether the 4-force exerted by the field at a given event might depend on anything else in addition to the charge of the particle. Suppose, for example, that it is independent of the particle's 4-velocity \mathbf{U} and 4-acceleration $d\mathbf{U}/d\tau$, etc. A field theory can be built from such an assumption, but it is not the one we are looking for because it cannot give rise to a pure force. For a pure force we require $\mathbf{F} \cdot \mathbf{U} = 0$, but if the force is independent of \mathbf{U} , then $\mathbf{F} \cdot \mathbf{U}$ can only vanish for all \mathbf{U} if \mathbf{F} is itself zero. We conclude that we shall need some dependence of \mathbf{F} on \mathbf{U} .

The next simplest assumption would seem to be that the 4-force is proportional to the charge and to the 4-velocity of the particle, but is independent of its 4-acceleration:

$$\mathbf{F} \stackrel{?}{=} q\phi\mathbf{U} \tag{12.1}$$

for some scalar field ϕ . This is no good, however: still not pure: $\mathbf{F} \cdot \mathbf{U} = q\phi\mathbf{U} \cdot \mathbf{U} \neq 0$. Next we try

electromagnetic force equation
$\mathbf{F}^a = q\mathbb{F}^{a\mu}\mathbf{U}_\mu \qquad [\mathbf{F} = q\mathbb{F} \cdot \mathbf{U} \tag{12.2}$

where \mathbb{F} is an object that describes the field. It is a 2nd rank tensor. This is the simplest thing (other than a scalar) that can take a 4-vector as 'input' and give back a 4-vector force. So is the force pure now? Let's see:

$$\begin{aligned} \mathbf{F}^\lambda \mathbf{U}_\lambda &= q\mathbb{F}^{\lambda\mu}\mathbf{U}_\mu \mathbf{U}_\lambda, \\ \text{i.e. } \mathbf{F} \cdot \mathbf{U} &= q(\mathbb{F} \cdot \mathbf{U}) \cdot \mathbf{U}. \end{aligned}$$

We require this to be zero for all \mathbf{U} . We can rearrange the right hand side to

$$q\mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U}$$

which makes it easy to see that it is an example of the type of object displayed in (9.42). The latter equation gives

$$\mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbb{F}^T \cdot \mathbf{U}.$$

This is true for any tensor \mathbb{F} . Now suppose \mathbb{F} is antisymmetric, then we would have $\mathbb{F}^T = -\mathbb{F}$ so

$$\mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U} = -\mathbf{U} \cdot \mathbb{F}^T \cdot \mathbf{U}. \quad (12.3)$$

Combining the last two results gives

$$\mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U} = -\mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U} \implies \mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U} = 0 \quad (12.4)$$

which is precisely what we want. If \mathbb{F} were not antisymmetric, on the other hand, then one could find a \mathbf{U} for which (12.3) was not true and therefore $\mathbf{U} \cdot \mathbb{F} \cdot \mathbf{U}$ was not zero. The conclusion is *requiring \mathbb{F} to be antisymmetric is both necessary and sufficient to guarantee a pure force.*

We now know our tensor is antisymmetric. That is good, because this is the least complicated type of 2nd rank tensor, and it has some nice properties that we investigated in section 9.4.1. It can be regarded as being composed of two 3-vectors, so our tensor field can be interpreted as a linked pair 3-vector fields. By comparing (12.2) with (9.48) you can see that the spatial part of (12.2) gives the Lorentz force equation, and

$$\mathbb{F}^{ab} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \quad (12.5)$$

We immediately know how the fields transform under a change of reference frame, see (9.51) which gives our old friend (6.1). This is undoubtedly the most direct route to that result. Note that, as before, we have obtained it by using the force equation without yet needing to evoke the field equations.

So far we have established how our field \mathbb{F} affects particles, and we have learned that we can, if we so chose, interpret it as a linked pair of 3-vector fields. It remains to propose how the particles might generate the field. We shall assume that some sort of differential equation is needed, so we take an interest in $\partial_\lambda \mathbb{F}^{\lambda b}$ which is a sort of divergence of the tensor field. This reduces the rank of the object from 2 to 1; it is arguably the simplest differential operator we could use. It is certainly one of the simplest anyway, so let's try it.

We already proposed that the effect of the field on the particles is proportional to their charges and their velocities. Some sort of general notion of a 'third law' (action and reaction), which

we know will be needed to respect momentum conservation, leads us to guess that the particles should in return affect the field also through their charges and their velocities, so we guess

1st field equation

$$\partial_\lambda \mathbb{F}^{\lambda b} = -\mu_0 \rho_0 \mathbb{U}^b \quad [\square \cdot \mathbb{F} = -\mu_0 \rho_0 \mathbb{U} \quad (12.6)$$

where μ_0 is a proportionality constant, and ρ_0 is the proper charge per unit volume, i.e. for any given event it is the charge density in the reference frame in which the local charge is at rest.

So far we only assumed the charge was Lorentz invariant. Our field equation (12.6) gives us something more: it can only be valid if the charge is conserved. This is the well-known connection between the completion of Maxwell's equations and the conservation of charge. To prove it, we investigate the 4-divergence of the 4-vector on the right hand side of (12.6):

$$\begin{aligned} \partial_\lambda (\rho_0 \mathbb{U}^\lambda) &= \partial_\lambda \partial_\mu \mathbb{F}^{\lambda\mu} \\ &= \partial_\mu \partial_\lambda \mathbb{F}^{\mu\lambda} && \text{swap } \lambda, \mu \\ &= -\partial_\mu \partial_\lambda \mathbb{F}^{\lambda\mu} && \text{antisymmetric } \mathbb{F} \\ &= -\partial_\lambda \partial_\mu \mathbb{F}^{\lambda\mu} && \text{commute partial differentiation} \\ \implies \partial_\lambda (\rho_0 \mathbb{U}^\lambda) &= 0. \end{aligned}$$

In the first step we simply swapped the indices: this is valid because they are dummy indices (being summed over): we can call them what we like. You can imagine that λ was first changed to σ , then μ to λ , then σ to μ . In the second step we invoked the antisymmetry of \mathbb{F} . In the third we invoked the symmetry of 2nd partial derivatives: $\partial_\lambda \partial_\mu f = \partial_\mu \partial_\lambda f$ for any well-behaved scalar f , and thus for all the elements of \mathbb{F} . The whole argument is essentially the same as the one leading to (12.4), but now we have $\square \cdot (\square \cdot \mathbb{F})$ instead of $\mathbb{U} \cdot \mathbb{F} \cdot \mathbb{U}$.

Defining the 4-vector $\mathbb{J} \equiv \rho_0 \mathbb{U}$, we can write the conclusion $\partial_\lambda \mathbb{J}^\lambda = 0$. This is the continuity equation (previously we wrote it $\square \cdot \mathbb{J} = 0$), so we have deduced that the quantity whose flow is described by \mathbb{J} —i.e. the charge—is conserved!

This conservation law greatly cheers us. In fact, one might argue that for a simple theory one should insist on such a conservation, and this is further evidence that the equation (12.6) is a unique choice: it is the only one that is remotely simple and that is consistent with charge conservation.

Now you can compare (12.6) with (9.49) and you will see that (12.6) is the Maxwell equations M1 and M4 in tensor notation.

Eq. (12.6) is our first field equation. It does not yet fully describe the field, because it is only a 4-vector equation, i.e. it contains 4 equations, while we need 6 altogether. The problem is that the divergence of a field does not in itself fully characterise the field. The natural next step is to consider the 'curl' of the field—some sort of derivative that would generate a 3rd rank tensor.

There are many possibilities. However, we can keep the problem under control by noticing that we have not yet taken advantage of another feature that can arise in field theories: the concept of a potential. Therefore we shall assume next that \mathbb{F} can be derived from a potential. We can soon convince ourselves that a scalar potential will not suffice, so we try a vector potential \mathbf{A} , and propose

$$\mathbb{F}^{ab} = \partial^a \mathbf{A}^b - \partial^b \mathbf{A}^a. \quad (12.7)$$

It follows that

2nd field equation

$$\partial^c \mathbb{F}^{ab} + \partial^a \mathbb{F}^{bc} + \partial^b \mathbb{F}^{ca} = 0, \quad (12.8)$$

as you can verify. Now, it can be shown that (12.8) is not only a necessary but also a sufficient condition that \mathbb{F} can be obtained from a 4-potential as in (12.7). Therefore we can describe either one of (12.7) or (12.8) as our second field equation. The form of (12.8) represents 64 equations, and yet only 4 of them are independent, so the index notation is introducing a lot of unwanted redundancy. However there is something attractive about having a set of equations only in terms of \mathbb{F} and the charges, and the tensor technique is still playing its crucial role of guaranteeing Lorentz covariance.

With the benefit of the assumption that \mathbb{F} is completely determined by a 4-vector potential, now our first field equation (12.6) becomes sufficient to determine the field. Substituting from (12.7) it becomes

$$\partial_\lambda \partial^\lambda \mathbf{A}^b - \partial^b \partial_\lambda \mathbf{A}^\lambda = -\mu_0 \mathbf{J}^b. \quad (12.9)$$

This is $\square^2 \mathbf{A} - \square(\square \cdot \mathbf{A}) = -\mu_0 \mathbf{J}$, which we previously wrote in component form in eqs. (6.18) and (6.19).

We now have a complete theory, consistent unto itself. It remains to extract predictions and compare with experiment, and of course we know well that we shall be richly rewarded with experimental confirmation. The foundational equations are summarized in the box. We added the equation of motion $\mathbf{F} = d\mathbf{P}/d\tau$ in order to provide a complete story: the field equations say how the fields move, the equation of motion says how the particles move. All of classical physics except gravitation is included in this box!

Electromagnetic field theory

Force equation

$$F^a = q\mathbb{F}^{a\lambda}U_\lambda$$

(pure force $\Rightarrow \mathbb{F}$ is antisymmetric.)

Field equations

$$\partial_\lambda \mathbb{F}^{\lambda b} = -\mu_0 \mathbf{J}^b \quad (12.10)$$

$$\partial^c \mathbb{F}^{ab} + \partial^a \mathbb{F}^{bc} + \partial^b \mathbb{F}^{ca} = 0. \quad (12.11)$$

The first $\Rightarrow \partial_\lambda \mathbf{J}^\lambda = 0$, charge is conserved.The second $\Rightarrow \mathbb{F}^{ab} = \partial^a \mathbf{A}^b - \partial^b \mathbf{A}^a$, the field can be derived from a potential.

Equation of motion of a test particle

$$m \frac{dU^a}{d\tau} = q\mathbb{F}^{a\lambda}U_\lambda. \quad (12.12)$$

Variations

Variations which give rise to other sensible and reasonably simple theories are mainly of two types. We can give up the requirement of a pure force and try a simpler potential, such as a scalar potential (i.e. a Lorentz scalar, not part of a 4-vector). An example of this is the Yukawa scalar meson theory, which we shall briefly discuss in section 18.1.2. Or, we can introduce further sources, to gain more symmetry between the electric and magnetic parts, at the expense however of losing the 4-potential. These further sources could, for example, be like point charges except they generate a magnetic instead of an electric field: they are magnetic monopoles.

BLAH version with monopoles

We could thus investigate what we might expect to result from the existence of such magnetic monopoles. However, they have never been found in Nature so it is believed that Maxwell's theory is the correct one.

The wonderful succinctness of eqs. (12.10), (12.11) does not mean the equations are simple: they remain precisely the full Maxwell equations, with all their complexity and richness. However, we have shown that we cannot expect to find anything much simpler than this. Furthermore, the introduction of \mathbb{F} gives us a sense that we are at last getting to grips with what the electromagnetic field really *is*. It is a 'tensor thing' that exists throughout spacetime. At each event in spacetime there is this 4-dimensional 'thing' that looks like two 3-vectors when you pick any given reference frame. (It is 4-dimensional in the same sense that a moment of inertia 3-tensor is 3-dimensional). It exerts forces and, as we shall explore in this chapter, it carries energy and momentum. It may be right to say that it is part and parcel of the structure of spacetime itself, or else that spacetime is 'made of' things like this: this is the type of question

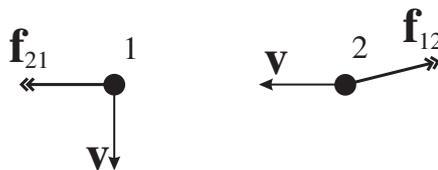


Figure 12.1: Two charged particles move on orthogonal trajectories. At the moment when one passes directly in front of the other, the forces are not in opposite directions (the electric contributions are opposed, but the magnetic contributions are not). Does this mean momentum is not conserved?

that attempts to unify quantum field theory and general relativity are trying to resolve.

12.1.1 The dual field and invariants

[Section omitted in lecture-note version.]

12.1.2 Motion of particles in a static uniform field

[Section omitted in lecture-note version.]

12.1.3 Precession of the spin of a charged particle

[Section omitted in lecture-note version.]

12.2 Electromagnetic energy and momentum

Consider two identical point charges that are released from rest at the same moment, at some modest distance from one another. They will repel one another, so fly apart with equal and opposite momenta. Thus momentum is conserved. However, consider these events from the perspective of another frame of reference moving along the line between the particles. In the new frame the release events are not simultaneous: one particle starts to move before the other one. It has changed its momentum, but the other has not, so what has happened to conservation of momentum?

Consider another scenario, depicted in figure 12.1. Two charges are moving at right angles to one another in a common plane, so that one passes in front of the other. The electric field

produced by each particle at the other is directed along the line between them, so the electric forces are in opposite directions. Since a moving charge produces no magnetic field along its line of motion, at the moment when q_2 is moving directly towards q_1 , the latter experiences no magnetic field, so experiences no further force: the net force on it is directly away from q_2 . However, it produces a non-zero magnetic field at q_2 , and the latter is in motion through this field. Therefore q_2 experiences a transverse force: the total force on it is *not* directly away from q_1 , but somewhat off to one side. So the forces are not equal and opposite! No momentum conservation again?

In both these examples an attempt was being made to talk about momentum conservation between events at separate locations. However, relativity teaches us that this is doomed to failure. A conservation law has to be *local*, not just global. That is, the conservation of a substance such as water does not mean merely that the total amount of water in the room is fixed (assuming the door is shut and no chemistry is going on): it means much more than that. If water were to disappear from a vase, it is not enough merely that an equal quantity of water should appear somewhere else such as on the window. We insist that the water *has to get there by flowing across from the one place to the other* (for example by evaporation and convection). In classical physics we might imagine that a less tangible quantity such as momentum might disappear from one place and appear in another without flowing across the intervening space, but relativity teaches us that a quantity is conserved locally or not at all. The Principle of Relativity requires that the law, if it is valid, should apply in all reference frames, and the relativity of simultaneity shows that a conservation law that relies on simultaneous behaviour at separate places cannot hold in all reference frames.

Faced with the observed behaviour of charged particles, we must either abandon the principles of conservation of energy and momentum, or else assert that something in addition to the particles, and near to them, can carry energy and momentum. The obvious candidate is the electromagnetic field (or possibly the potentials, but in view of gauge freedom, it would seem less likely that it should work out that way). We have in fact assumed this already when we allowed ourselves to talk about ‘the energy carried by a pulse of light’, and when we applied to light pulses concepts such as an energy-momentum 4-vector. Now we shall investigate whether this idea can be made precise and extended to all fields, including static ones. It turns out that it can, and it will lead to a new and more satisfactory way of understanding ‘potential energy’.

We shall start with energy, and turn to momentum afterwards, but aim to finish with a covariant treatment in terms of an energy-momentum 4-vector and associated tensors. We could restrict ourselves to covariant 4-tensor notation from the outset, but I think it is easier to understand what is going on in the more familiar language of flow through space and rate of change with time.

In what follows, we shall need to discuss the energies both of particles and of fields. It will help if you agree at the outset to abandon all talk of ‘potential energy.’ You may have been taught that a charged particle ‘possesses potential energy $q\phi$ ’ when it is in a static electric field whose scalar potential is ϕ , but we are going to show that this sort of talk is quite muddled and misleading. The only energy a particle has is its rest energy m_0c^2 and its kinetic energy $(\gamma - 1)m_0c^2$, which together make its total energy γm_0c^2 . The kinetic energy is the energy

a particle has because it is *moving*, not because of where it is, and the rest mass is constant, unchanged by interactions with the electromagnetic field.

Now let us suppose that an electromagnetic field possesses an energy per unit volume (scalar, called u) and an energy current density (vector, \mathbf{N}) which is the energy flowing across a small area, per unit area per unit time. These quantities should satisfy the continuity equation

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{N}.$$

At least, that is what we should expect for fields in free space, when there are no charged particles around. But of course electromagnetic fields can interact with particles, and presumably exchange energy with them. How does that come about? Only and wholly through the Lorentz force equation, because according to our theory that is the only ‘point of impact’ of the fields onto matter. Since we have a pure force, we can use $\mathbf{f} \cdot \mathbf{v}$ to get the rate of doing work by the force. To be precise, this is the rate of change of *kinetic energy* of the particle being pushed (that is, the rate of change of its full, relativistic kinetic energy). This is $dW/dt = q\mathbf{E} \cdot \mathbf{v}$ for a particle of charge q . We model a general distribution of charge as many small volumes dV each containing charge $q = \rho dV$. The combination $q\mathbf{v} = \rho\mathbf{v}dV$ can be recognised as $\mathbf{j}dV$ where \mathbf{j} is the current density, so the rate of doing work at some given point, per unit volume, is $\mathbf{E} \cdot \mathbf{j}$. This work is the energy being given *to* the charged particles (increasing their kinetic energy) and therefore being taken *from* the field. If the particles are being slowed then this is taken care of by the sign of $\mathbf{E} \cdot \mathbf{j}$. Therefore the conservation of energy is represented by the equation

$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{N} + \mathbf{E} \cdot \mathbf{j}. \quad (12.13)$$

The left hand side says how much energy is going out of the field in some small volume (per unit volume), the right hand side says how much is field energy but is flowing out of the region, and how much is being given to the particles. We have accounted for the total energy of field and particles, and asserted that it is conserved.

You may be concerned that in a typical electric circuit with a constant current, there is inside any resistor a field \mathbf{E} and a constant current density \mathbf{j} . The $\mathbf{E} \cdot \mathbf{j}$ says work is being done, but the constant \mathbf{j} says the particles are not in fact speeding up, so where is the energy going? Is it ‘potential energy’ after all? The answer is that the current carriers inside the resistor *are* continually being accelerated by the field, but they immediately suffer collisions with the material of the resistor (nuclei and bound electrons), transferring their new-found kinetic energy to kinetic energy and field energy of the rest of the resistor, in a random form called heat. A detailed model of all these effects must end up confirming (12.13) because it is derived from the only fundamental point of interaction of field and matter.

The following beautiful argument is due to John Henry Poynting (1852-1914).

We should like to find out how u and \mathbf{N} in eq. (12.13) depend on the fields \mathbf{E} and \mathbf{B} . To this

end, we can use the Maxwell equation M4 to express \mathbf{j} in terms of the fields, giving

$$\mathbf{E} \cdot \mathbf{j} = \epsilon_0 c^2 \mathbf{E} \cdot (\nabla \wedge \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

The last term is $(\partial/\partial t)(\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E})$, so it looks as though that is at least a part of $\partial u/\partial t$. Therefore we want to turn the first term into the divergence of something.

A divergence involves $\nabla \cdot$ and a vector. The vectors we have available are \mathbf{E} , \mathbf{B} and $\mathbf{E} \wedge \mathbf{B}$. The term we are investigating involves both \mathbf{E} and \mathbf{B} so we try taking a look at $\nabla \cdot (\mathbf{E} \wedge \mathbf{B})$. The general result for the divergence of a vector product is

$$\nabla \cdot (\mathbf{E} \wedge \mathbf{B}) = \mathbf{B} \cdot (\nabla \wedge \mathbf{E}) - \mathbf{E} \cdot (\nabla \wedge \mathbf{B}). \quad (12.14)$$

The last term is just what we have. We can deduce that

$$\mathbf{E} \cdot \mathbf{j} = -\epsilon_0 c^2 \nabla \cdot (\mathbf{E} \wedge \mathbf{B}) + \epsilon_0 c^2 \mathbf{B} \cdot (\nabla \wedge \mathbf{E}) - \frac{\partial}{\partial t} (\frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E}).$$

This is a nice divergence and a time derivative, plus a part in the middle that is not in the form we want. However, Maxwell's equations will sort it out for us again, this time by using M3 to replace $\nabla \wedge E$, giving

$$\mathbf{E} \cdot \mathbf{j} = -\epsilon_0 c^2 \nabla \cdot (\mathbf{E} \wedge \mathbf{B}) - \frac{\partial}{\partial t} (\frac{1}{2} \epsilon_0 c^2 \mathbf{B} \cdot \mathbf{B} + \frac{1}{2} \epsilon_0 \mathbf{E} \cdot \mathbf{E}). \quad (12.15)$$

This beautiful result shows that we can make our energy conservation equation (12.13) apply very nicely to fields and particles together. We just need to *define*

$$\begin{aligned} u &= \frac{1}{2} \epsilon_0 c^2 B^2 + \frac{1}{2} \epsilon_0 E^2, \\ \mathbf{N} &= \epsilon_0 c^2 \mathbf{E} \wedge \mathbf{B}. \end{aligned} \quad (12.16)$$

We have not proved that (12.16) represent a unique solution: it is possible to define more complicated versions of u and \mathbf{N} , such that after differentiating one and taking the divergence of the other, one still gets (12.15), but this form is the most obvious, and it is consistent with all observations in electromagnetism. It is believed to be correct¹.

$\mathbf{N} = \epsilon_0 c^2 \mathbf{E} \wedge \mathbf{B}$ is called the Poynting vector, after its discoverer. It gives the energy flow per unit area per unit time (also called flux). For an oscillating field such as a light wave, its time average is the power per unit area, called the intensity. The Poynting vector is often written

$$\mathbf{N} = \mathbf{E} \wedge \mathbf{H}$$

¹It is not possible to use electromagnetic theory alone to distinguish this choice of u and \mathbf{N} from other choices that still satisfy (12.15). In General Relativity, however, these quantities enter into the formula for the gravitational field. A precise gravitational experiment could therefore allow a test to distinguish Poynting's choice of u , \mathbf{N} from others that could be made. However, observations to date are not sufficiently accurate to carry out such a test.

where \mathbf{H} is a field closely related to \mathbf{B} , being given in free space by $\mathbf{H} = \mathbf{B}/\mu_0 = \epsilon_0 c^2 \mathbf{B}$.

12.2.1 Examples of energy density and energy flow

Now we shall explore the physical meaning of u and \mathbf{N} by considering some examples.

Consider a stationary spherical ball of charge. We suppose the ball has a uniform charge density ρ . There is no movement, so no magnetic field. Suppose we had to construct such a ball: we would have to arrange to bring up some charge from a long way away, and push it onto the ball. At any given moment, part way through this construction process, the ball has radius r and therefore total charge

$$q(r) = (4/3)\pi r^3 \rho.$$

The work required to bring up the next little piece dq of charge from infinity to the edge of the ball is

$$dW = - \int_{\infty}^r f dr' = \frac{q(r)dq}{4\pi\epsilon_0 r}$$

where f is the Coulomb force. Let us write $Q = (4/3)\pi a^3 \rho$ for the total charge on the ball when it reaches its final size a , then

$$q(r) = (r/a)^3 Q, \quad \frac{dq}{dr} = 3r^2 Q/a^3.$$

This allows us to perform the integral for W , obtaining

$$W = \int_0^a \frac{Q(r^3/a^3)(3r^2/a^3)Q}{4\pi\epsilon_0 r} dr = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 a}.$$

Now let's calculate the energy stored in the fields, according to the energy density eq. (12.16)i.

Outside the ball the electric field is the same as the field due to a point charge: $E = Q/(4\pi\epsilon_0 r^2)$, radially outwards. Inside the ball the field at any given r is also the same as the field due to a point charge, but the total charge in question is now that contained inside the radius r , i.e. $q(r)$. This leads to a field radially outwards again, but increasing linearly with radius: $E = rQ/(4\pi\epsilon_0 a^3)$. The total field energy given by (12.16) is

$$\begin{aligned} U &= \int u dV = \frac{\epsilon_0}{2} \left\{ \int_0^a \left(\frac{rQ}{4\pi\epsilon_0 a^3} \right)^2 dV + \int_a^{\infty} \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 dV \right\} \\ &= \frac{Q^2}{4\pi\epsilon_0 a} \left(\frac{1}{10} + \frac{1}{2} \right) = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 a} \end{aligned} \quad (12.17)$$

When is it ok to use ‘potential energy’? Having strictly rejected the notion of ‘potential energy’ as a fundamental property, we can re-introduce it as a calculational device that can be convenient in some circumstances. The idea can be adopted when we have a situation where the field does not ‘leak’ energy by some process we are not taking into account. Potential energy is like money in the bank. If I have one million gold sovereigns in the bank, then as I walk around my pockets are not weighed down by one million gold sovereigns, but I am confident that, should I ask the bank for a gold sovereign, I will receive one (and my account will be diminished by 1)—except that the bank may get into financial trouble and disperse ‘my’ money. In the case of electrostatics, the field acts as a totally trustworthy ‘bank’ as far as charged particles are concerned, so that a particle can have confidence that by moving away it could pick up the kinetic energy it ‘expects’ on the basis of a potential energy calculation. We used this idea, for example, to get some insight into the (an)harmonic oscillator in section 4.2.5. The concept begins to fail, however, in dynamic problems when the fields can move energy around by wave motion. Then we have to abandon the idea of potential energy, and trust eqs (12.16).

(where the volume element is $dV = (4\pi r^2)dr$). This is a standard exercise in elementary electromagnetism, but we displayed it in full in order to comment on the result and raise some more subtle issues later on. The amount of work done against the Coulomb repulsion of the charges is found to equal the amount of energy stored in the *whole* field, both inside and outside the ball. So who ‘owns’ the energy? When one first learns electrostatics, one is usually invited to say that the work done in bringing one charge near to another charge can be described in terms of ‘potential energy’ of the charge. The work is done, but the charges are not moving at the end, so where did the energy go? In order to preserve energy conservation, this idea of ‘potential energy’ was brought in. We now see that this was misleading. The charges do not possess any energy beyond their rest energy and kinetic energy. The energy someone provided by doing the work has gone into the *field*. ‘Potential energy’ is misleading especially in relativity theory, because it is non-local and it does not contribute to the inertia of a particle.

It seems odd at first that the energy is not contained in the ball. That is where it might appear that we put it, but in fact we did not: the forces pushed on the charges throughout their journey from far far away, and they did their work *locally*, putting energy into the electromagnetic field at each place. Again, Special Relativity insists on local conservation if energy is to be conserved at all. Only a small part of the total energy ends up inside the ball. Indeed, if we had constructed instead a thin spherical shell of charge then one could arrange that only a negligible fraction of the total energy was inside the shell.

Next we investigate the Poynting vector \mathbf{N} .

Figure 12.2 shows the Poynting vector in the vicinity of a moving sphere of charge (with no fields present other than its own). We shall call this sphere a ‘particle’. Beware, however, of a difficulty in the case of point-like charges, that we shall discuss in section 12.3. Since the \mathbf{E} field is radial and \mathbf{B} circles around, \mathbf{N} is everywhere directed tangential to the surface of

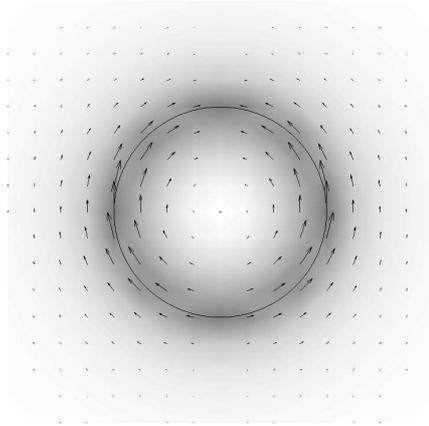


Figure 12.2: The energy density u and Poynting vector \mathbf{N} in the vicinity of a uniform sphere of charge in uniform motion, with no fields present other than its own. The shading indicates u , the arrows indicate \mathbf{N} (by their length and direction).

a sphere around the particle. The vector is directed from behind the charge to in front of it, representing the movement of field energy as the fields fade away behind the charge and build up in front of it. If the charge is not accelerating then there is no net influx or outflux of energy towards or away from it. The vanishing field behind the charge provides just enough energy for the increasing field in front to build itself up . . . until it fades in turn and passes the energy on.

Figure 12.3 shows the case of a charged sphere moving at constant velocity in a uniform applied electric field. Again we shall call it a ‘particle’. Now there is a net influx of energy into any sphere surrounding the particle. This makes sense because the applied field is doing work on the particle. To have a specific model, imagine that we have a uniform charged sphere, moving in a neutral viscous medium. It has reached its terminal velocity in the medium so moves at constant \mathbf{v} , and the net result is that the applied electric field \mathbf{E}_0 does work on the charged sphere, which in turn puts energy into the viscous medium, at the rate $\mathbf{f} \cdot \mathbf{v} = qE_0v$, where q is the total charge carried by the sphere.

Now let’s calculate the energy flow in the field. We don’t care whether we are discussing a point charge or a uniform ball of charge, since we shall only be calculating the fields outside such a ball, but we are assuming the ball is not perturbed by the applied field (i.e. it remains spherical and uniform). We shall calculate the Poynting vector at points on the surface of a sphere \mathcal{R} with radius r centered on the origin, and then integrate over this surface. N.B. the surface of integration is fixed in space: it does not move along with the charge. However, it is convenient to perform the calculation of \mathbf{N} at the moment when the charge arrives at the center of \mathcal{R} . At that moment, the electric field outside the charged ball is

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_q = E_0\hat{\mathbf{z}} + \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

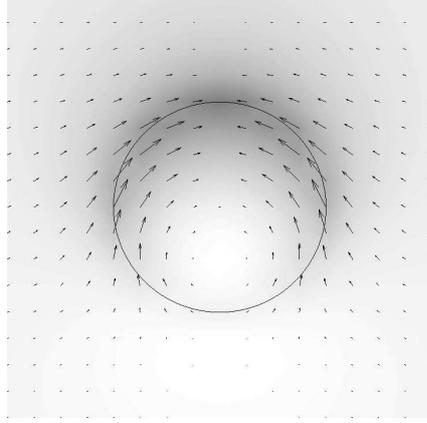


Figure 12.3: The energy density u and Poynting vector \mathbf{N} in the vicinity of a charged sphere in uniform motion, in a region where there is an applied uniform static electric field in the vertical direction. The Poynting vector now shows a net flux of electromagnetic energy towards the sphere. Since the sphere is not accelerating there must be other forces on it, and it expends the received electromagnetic energy by doing work against those other forces.

where, for simplicity, we treat the case of a slowly moving charge, $v \ll c$. The magnetic field is

$$\mathbf{B} = \frac{\mathbf{v} \wedge \mathbf{E}_q}{c^2} = \frac{qv \sin \theta}{4\pi\epsilon_0 c^2 r^2} \hat{\phi}$$

(where $\hat{\phi}$ is a unit vector in the azimuthal direction). The Poynting vector has two contributions:

$$\mathbf{N} = \epsilon_0 c^2 \mathbf{E} \wedge \mathbf{B} = \epsilon_0 c^2 (\mathbf{E}_0 \wedge \mathbf{B} + \mathbf{E}_q \wedge \mathbf{B}).$$

The second of these (the contribution from the charge's own fields) is directed around the sphere of integration, neither in nor out. The first (the applied electric field combining with the charge's magnetic field) is directed towards the z axis. This is the only term that will contribute to the surface integral. It has size

$$\frac{E_0 q v \sin \theta}{4\pi r^2}$$

so the net flux in through the surface of \mathcal{R} is

$$\int_{\mathcal{R}} \mathbf{N} \cdot (-\mathbf{n}) d\mathcal{S} = \frac{qvE_0}{4\pi r^2} \int_0^{2\pi} \int_0^\pi \sin^2 \theta r^2 \sin \theta d\theta d\phi$$

where \mathbf{n} is the unit vector normal to the surface and we used $\mathbf{N} \cdot (-\mathbf{n}) = S \sin \theta$. Thus there are three factors of $\sin \theta$: one from $\mathbf{E}_0 \wedge \mathbf{B}$, one from $\mathbf{N} \cdot \mathbf{n}$ and one from the surface element $d\mathcal{S}$. The integral is easily done using $\sin^3 \theta = \sin \theta (1 - \cos^2 \theta)$ and we obtain

$$-\int_{\mathcal{R}} \mathbf{N} \cdot \mathbf{n} d\mathcal{S} = \frac{2}{3} q E_0 v. \quad (12.18)$$

Thus the net energy flow in through our chosen surface is proportional to qvE_0 , the work done on the charge, but there seems to be a mistake: a factor $2/3$ when we expected 1. There is no mistake. It is simply that we haven't finished yet. We need to think about the field energy density u as well. Is it constant or increasing or decreasing, inside region \mathcal{R} ? At first one might imagine that we have a symmetry, so that $\int u dV$ would be constant at the moment when the particle reaches the center of \mathcal{R} , but the situation is not symmetrical forward and back. In front of the particle the fields \mathbf{E}_0 and \mathbf{E}_q are in the same direction, they reinforce one another to make a big E^2 . Behind the particle the fields \mathbf{E}_0 and \mathbf{E}_q are opposed: they tend to cancel one another out, leaving a small E^2 . So we can picture the energy density $u = \epsilon_0 E^2/2$ as 'heaped up' in front of the particle (see figure 12.3). In a given small time interval dt the particle travels a distance $dz = vdt$. If the particle is just passing through the centre of \mathcal{R} then in the next travel distance dz the distribution of energy density will shift, such that a relatively large 'chunk' of energy is lost from the upper hemisphere of \mathcal{R} , while the lower hemisphere gains a smaller amount. N.B. we are not here talking about a transport of energy (we already calculated that) but a rise or fall of energy owing to non-zero values of du/dt . The net effect is calculated in the exercises: it is found to be just $dU = -(1/3)qE_0 dz$, so

$$\frac{d}{dt} \int_{\mathcal{R}} u dV = -\frac{1}{3} q E_0 v. \quad (12.19)$$

The net effect, then, is that the energy influx of $(2/3)qE_0v$ plus a power $(1/3)qE_0v$ liberated from the field inside \mathcal{R} combine to provide the qE_0v which ends up being transferred to the viscous medium (or, more generally, to whatever further object or system the charge is pushing on).

It is noteworthy that whereas the applied force here acts along the direction of travel, the energy flows towards the charge from the *sides*, at right angles to the motion. This is connected to the fact that in this example the momentum of the charged object is not changing.

Figure ?? shows the case of two opposite charges approaching one another at constant speed. Since the charges mutually attract, then their velocity must be being maintained constant by another system which opposes their motion, so they are doing work *on* the other system. This shows that the total field energy due to this pair of particles, integrated over all space, must be diminishing. What energy there is, however, is being concentrated more and more in the region between the particles, where the field strength is increasing. Near either charge the situation is just like in figure 12.3.

Other 'canonical' examples of Poynting's vector are the capacitor, the resistor, and the plane electromagnetic wave. In a plane wave (section 6.4) \mathbf{N} points in the direction of the wave-vector \mathbf{k} , which makes sense. Its size at any moment is $\epsilon_0 c E^2$ (since the fields are perpendicular and $E = cB$). If E oscillates as $E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$ then N oscillates as $N_0 \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t)$. Thus its direction is fixed but its size oscillates between zero and $\epsilon_0 c E_0^2$. The *intensity* I is defined as the power per unit area, averaged over a cycle, i.e. $I = \langle N \rangle = \epsilon_0 c E_0^2/2$. For such a wave the electric and magnetic contributions to the energy density are equal. Their total is $u = \epsilon_0 E^2$, which also oscillates. Its spatial average is $\epsilon_0 E_0^2/2$ and one can see that the intensity is c times

this. The summary is

$$I = \langle N \rangle = \langle u \rangle c = \frac{1}{2} \epsilon_0 E_0^2 c.$$

Next consider a cylindrical resistor of length d and radius a . If a current I flows and the voltage between the ends is V then the power dissipated in the resistor is VI . The magnetic field at the surface is $B = \mu_0 I / 2\pi a = I / (2\pi \epsilon_0 c^2 a)$, directed in loops around the resistor in a right-handed sense with respect to the current. Inside the resistor there is an electric field in the direction of the current flow, of size $E = V/d$. Therefore the size of the Poynting vector is

$$N = \frac{VI}{2\pi ad}$$

and its direction is radially inwards, i.e. pointing straight in through the curved surface. The area of that surface is $2\pi ad$ (circumference times length) so the total energy flow **in** to the resistor is VI per unit time—exactly matching what we know is dissipated there. This exact balance satisfies us that energy is conserved, but the sign needs a moment's thought. Surely the power VI is leaving the resistor, not coming in? The answer is that there is a conversion of energy going on: electromagnetic field energy enters the resistor and is used up accelerating the charges that carry the current. These charges in turn collide with the material of the resistor, heating it, turning their kinetic energy into heat. This heat subsequently leaves the resistor. Therefore the sign of the flow of field energy is correct: into the resistor.

The location of this flow can seem bizarre at first, however.

The battery is pushing on the charges, which are moving up the wire, so one might think the work is being done right there in the wire. One would expect that that is where the energy is being transported too: down the wire, from the battery to the capacitor. But Poynting says it is not: it is coming in from the sides! The example of a pair of charges (figure ??) should have prepared you for this. In fact, a moment's reflection should convince you that close to zero work is being done in the wire, because the electric field (and therefore the force) is close to zero there. The work is done in the battery, which draws on energy stored in the fields of its molecules (also called 'chemical energy') to pull apart electrons and positive ions (the reverse process of figure ??). This 'pumps' energy out of the sides of the battery into the surrounding field. The energy is transported through the field and eventually comes in through the sides of the resistor.

A similar argument can be made for a parallel plate capacitor being charged at a constant rate. The field between the plates grows, and the energy it needs arrives by coming in through the curved surface in free space at the edge of the plates, not along the wires.

It turns out that we often don't need to keep track of these energy movements: we can just trust the fields to take care of it without our needing to know the details. However, if we want to hold on to the principle of energy conservation then Poynting's vector gives a clear and thorough (and correct!) treatment.

12.2.2 Field momentum

The Poynting vector describes the flux of *energy*. We would like to know also about momentum. Does an electromagnetic field carry momentum? The only Lorentz-covariant answer is *yes*. We already presented in section 4.6 the fact that there is a very general relationship between energy flux and momentum density, eq. (4.121). Therefore we should like to claim that the momentum per unit volume carried by an electromagnetic field is given by

$$\mathbf{g} = \mathbf{N}/c^2. \quad (12.20)$$

This turns out to be correct, but in the course of looking into it we shall begin to uncover the limits of classical electromagnetism. We shall explore those more fully in section 12.3 and chapter 19.

First let us give some evidence for our claim that we can apply the formula (12.20) to electromagnetic fields. Consider for example the phenomenon of radiation pressure. A plane wave incident on a (non-transparent) material body exerts a force on the body. To see how this comes about, consider the motion of a charged particle such as an electron on the surface of such a body. The electric field of the incident wave drives the electron in the *transverse* direction. For example if the wave is propagating in the z direction and is linearly polarised with its electric field along x , then the electron is pushed in the x direction. This does not give rise to a force in the direction of propagation of the wave. However, the non-zero x component of velocity causes the electron to feel also the magnetic force $q\mathbf{v} \wedge \mathbf{B}$, and this is in the z direction, and causes the radiation pressure. The electron of charge q , velocity \mathbf{v} absorbs energy from the wave at the rate $q\mathbf{E} \cdot \mathbf{v}$. For simplicity let us suppose the motion of the electron is always in the x -direction (the force in the z -direction being opposed by equal and opposite forces from the rest of the body). Then the rate at which energy is being transferred from the field to the body via the electron is qEv , and the Lorentz force component in the z direction is qvB . Therefore the energy and the impulse delivered during some time interval t are²

$$\text{energy} = \int qvE dt, \quad \text{momentum} = \int qvB dt.$$

Since for a plane wave the field strengths are related by $E = cB$, we find the ratio of the energy delivered and momentum delivered is c , the same as the ratio of energy and momentum for particles of zero rest mass. It follows that (12.20) can be asserted for electromagnetic plane waves, and therefore (by using Fourier analysis) for electromagnetic waves more generally.

²If the electron moves freely apart from the forces arising from the wave, then during each cycle of oscillation it undergoes a driven motion but does not on average absorb any energy; a body with only such particles in it would be transparent. If the electron experiences forces from the rest of the body which tend to damp its motion, then the average of $q\mathbf{E} \cdot \mathbf{v}$ over a cycle is non-zero.

The ‘4/3 problem’

Next, let us consider another example in which energy is transported by a field. Suppose there exists an electromagnetic field that presents itself as a static electric field in some reference frame. The field possesses energy, the integral of its energy density u over all space. Now consider the situation from the point of view of a reference frame moving with respect to the first. Is the energy content of the field moving in the new frame? The answer is surely ‘yes’. Whatever charges gave rise to the field in the first frame are now in motion. The energy of the field must be moving at precisely the same velocity as the charges, because we can imagine deconstructing the charge distribution at some later time and reclaiming the energy stored in the field. For example, think of a capacitor sent on an interstellar journey. It assuredly takes its field energy with it!

Now let’s calculate the momentum content of such a ‘moving static field’. For the sake of simplicity we will consider a spherically symmetric static electric field, and we will assume the new reference frame moves at low velocity $v \ll c$ relative to the frame containing the static field. Then the electric field E in the new frame is equal to that in the first frame (up to order v^2/c^2) and the magnetic field is $\mathbf{B} = \mathbf{v} \wedge \mathbf{E}/c^2$. The momentum density is

$$\mathbf{g} = \mathbf{N}/c^2 = \epsilon_0 \mathbf{E} \wedge \mathbf{B} = \frac{\epsilon_0}{c^2} \mathbf{E} \wedge \mathbf{v} \wedge \mathbf{E}.$$

Let \mathbf{v} be along the z -axis and let θ be the angle between \mathbf{r} and this axis, then \mathbf{g} has size $g = (\epsilon_0/c^2)E^2v \sin\theta$ and its z -component is $g_z = g \sin\theta$. When we integrate over all space to get the total momentum in the field, only the z component survives, and therefore we have

$$\begin{aligned} p &= \int_0^\pi \int_0^\infty \frac{\epsilon_0 E^2 v}{c^2} \sin^2\theta \, r^2 \sin\theta \, dr \, d\theta \, 2\pi \\ &= \frac{4v}{3c^2} \int_0^\infty \frac{1}{2} \epsilon_0 E^2 4\pi r^2 \, dr \end{aligned} \quad (12.21)$$

Now, for the low velocity under consideration the magnetic field is weak and contributes negligibly to the energy density compared to the electric field. Therefore we can recognise the integral on the right hand side of eq. (12.21) as the total energy content \mathcal{E} of the field (here we write \mathcal{E} for energy to avoid confusion with the electric field strength). We conclude that

$$\frac{\mathbf{p}}{\mathcal{E}} \stackrel{!}{=} \frac{4}{3} \frac{\mathbf{v}}{c^2}. \quad (12.22)$$

This result violates the relation $\mathbf{p} = \mathcal{E}\mathbf{v}/c^2$, which is the universal relationship between energy and momentum for bodies moving at any speed. This “4/3 problem” troubled early workers such as Lorentz. It implies, for example, that the total energy and momentum of this field cannot be considered as a 4-vector.

There is nothing wrong with our calculations of the energy and momentum in the field. Both are correct. The ‘problem’ is merely that we can’t consider this energy and momentum to be parts of a 4-momentum. The reason is that the field we have considered is not an isolated system. It is in continual interaction with the charges which act as its source. The relation $\mathbf{p} = \mathcal{E}\mathbf{v}/c^2$ applies only to particles or to extended objects that can be considered as isolated entities, free of external influences. We did not encounter this problem for electromagnetic waves because they have a special property: they can be *source-free*. That is to say, although the disturbance which gives rise to electromagnetic waves is usually a charged object in motion, once the source ceases to accelerate the emitted radiation continues to propagate, such that there can exist a source-free volume of space completely containing the electromagnetic radiation field. Such a field *can* be considered to be an isolated system possessing an energy and momentum of its own. Therefore it is legitimate to regard a light pulse as a single entity with a well-defined energy-momentum 4-vector.

The non-4-vector nature of \mathcal{E}, \mathbf{p} for a static field is also an illustration of the issue we discussed in connection with figure 4.9 and eq. (4.80): one cannot assume that adding up 4-vectors evaluated at different points in space will necessarily give a 4-vector total. It requires something like a conservation law to come into play, to guarantee that the total will give the same 4-vector no matter which time slice is used to calculate it. In the present case we are adding up (i.e. integrating over volume) the energy and momentum of each small region of field, which combine to form a well-defined 4-vector at each point in space for *any* field, but the sum is a 4-vector only if the field is an isolated system, not exchanging energy and momentum with anything else. A static field is not exchanging *energy* with other things, but it is in a state of continuous interaction with its sources, pulling on them. We can think of this, roughly, as a process of continuous elastic collision. If the sources are not accelerating then it must be that some other force is constraining them, and the net result is an interaction between the electromagnetic field and the other force-providing entity, mediated via the charges. A more complete understanding will emerge after we have grappled with this idea in more general terms. That is the subject of the next section.

12.2.3 Stress-energy tensor

Our aim now is to construct the 4-vector equivalent of Poynting’s argument that gave us the Poynting vector and field energy density. We already have a complete account of energy, so we shall not learn anything more about that, but now we shall learn about momentum and its conservation.

First let’s consider for a moment what sort of quantities we can expect to be dealing with. We may guess that u and \mathbf{N} should come together to form a 4-vector $\mathbf{N} \stackrel{?}{=} (uc, \mathbf{N})$. This will turn out to be not quite right, but it is on the right track. The argument involves the rate of transfer of energy from the field to the particles, so we should expect it to involve the rate of transfer of momentum from the field to the particles also. ‘Rate of transfer of momentum’ equates to force. This will be brought in as soon as we use the 4-force to describe the rate of change of

both the energy and momentum together. Our aim, then, is to equate the rate of transfer (per unit volume) of energy-momentum *to* the particles to a quantity which must represent the rate of transfer (per unit volume) of energy-momentum *out of* the field. Let's call the first of these W (in the expectation that it is a 4-vector). The zeroth component of W should be $(\mathbf{E} \cdot \mathbf{j})/c$ and it should emerge that it satisfies

$$cW^0 = \mathbf{E} \cdot \mathbf{j} = -\partial_\lambda \text{“}N^\lambda\text{”} = -\text{“}\square \cdot \mathbf{N}\text{”} \quad (12.23)$$

(eq. (12.13)). The inverted commas are there to signal that we have an abuse of notation: if \mathbf{N} really were a 4-vector then its divergence must yield a Lorentz scalar. However on the left we don't have such a scalar: we have one component of a 4-vector. It follows that \mathbf{N} must really be part of a 2nd rank tensor.

The spatial part of W should equate to a rate of flow of momentum out of a small volume of field. We can expect it to be given by some sort of divergence. Since none of us find divergence quite as intuitive a notion as flow across a surface, let's invoke the divergence theorem in our imagination for a moment, then we can see that the quantity we shall need is essentially about flow of momentum across a surface. Eventually we will make this a closed surface, but consider for a moment a horizontal flat surface. Take the z axis as vertical and let the surface be the xy plane. Clearly field momentum can flow across this surface: for example, think of a pulse of light emitted by an atom below the surface, some of which is focussed into a parallel beam propagating vertically upwards. This pulse will give a momentum kick to any atom above the plane that absorbs it. It must be that some momentum was transferred from the first atom to the electromagnetic field, which carried it upwards across the horizontal surface, and delivered it to the second atom. Such a pulse does not have to propagate exactly vertically, it could propagate in any direction. Therefore an amount of *horizontal* momentum p_x can be carried across the plane, as well as vertical momentum. In general, x , y and z momentum can all be transported or 'flow' in the z direction.

Similar statements could be made about flow in the x direction or the y direction.

Now construct a small closed box. The total amount of x -momentum flowing out of this box will have contributions from all the sides of the box: some x -momentum could be flowing out of the top, some out of the bottom, and some out of all the sides. You can see that the net transfer of x -momentum, per unit volume, must have the form of a divergence: we are expecting the form

$$W^x = -\partial_\lambda (\text{something}^x)^\lambda$$

We can make similar statements concerning W^y and W^z and it follows that the 'something' must be a 2nd-rank tensor (or possibly some more complicated object):

Transfer of 4-momentum per unit volume from fields to matter

$W^b = -\partial_\lambda T^{\lambda b} \quad [W = -\square \cdot T \quad (12.24)$

I have highlighted this equation because it will turn out to be correct. \mathbb{T}^{ab} is called the *stress-energy tensor* of the electromagnetic field. The word ‘stress’ here refers to force per unit area (which equates to speed times momentum per unit volume). It was introduced by Maxwell and Minkowski because of a close analogy between this tensor and the stress tensor describing internal stresses in a solid. It is as though the field can pull and push on itself. However, if you don’t like that idea then you can always think of it as energy flow and momentum flow.

\mathbb{T}^{ab} has the physical dimensions of force per unit area or (equivalently) energy per unit volume.

We can already see that the first row of \mathbb{T}^{ab} should be equal to $(u, \mathbf{N}/c)$ (equation (12.23)). Now we will show how the rest is obtained from the field equations. We shall do this in two ways. Both give useful insight. The first method uses 3-vectors and Maxwell’s equations to look at just the momentum flow; this results in a suggestion as to how the stress-energy tensor might be formed. The second method uses Lorentz covariant language (4-vectors and 4-tensors) throughout, and therefore proves that the resulting object is a 4-tensor (i.e. it is guaranteed to transform in the right way). This also offers practice in 4-tensor manipulation.

Method 1: 3-vector approach

We examine $\partial\mathbf{g}/\partial t$. This should tell us about the rate of change of momentum, and therefore about the force. N.B. although \mathbf{g} is related to the Poynting vector \mathbf{N} , it is best to temporarily forget that relation here. In the conservation argument, the momentum density \mathbf{g} plays the role, for momentum, which was played by *energy density* u in the Poynting argument. The quantity handling the flow of momentum (i.e. the job done for energy by \mathbf{N}) is the *tensor* \mathbb{T} .

Conservation of momentum will be achieved if the force on the particles in a small volume dV is equal and opposite to $dV\partial\mathbf{g}/\partial t$ plus another term which signifies the rate at which the field is carrying momentum away. Using Maxwell’s equations M3, M4 we have

$$\frac{\partial\mathbf{g}}{\partial t} = \epsilon_0 \left(\mathbf{E} \wedge \frac{\partial\mathbf{B}}{\partial t} + \frac{\partial\mathbf{E}}{\partial t} \wedge \mathbf{B} \right) = -\mathbf{j} \wedge \mathbf{B} + \epsilon_0 \left((\nabla \wedge \mathbf{E}) \wedge \mathbf{E} + c^2 (\nabla \wedge \mathbf{B}) \wedge \mathbf{B} \right).$$

The first term is the magnetic part of the Lorentz force per unit volume (recall $\mathbf{j}dV = q\mathbf{v}$), with a minus sign as expected. The rest must be either to do with the electric part of the force, or with momentum flow. The electric part of the force per unit volume is $\rho\mathbf{E}$, which in terms of the fields alone is $\epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E}$ (using M1). Adding this on, in order to obtain the total force, we have

$$\rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = -\frac{\partial\mathbf{g}}{\partial t} + \epsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} + (\nabla \wedge \mathbf{E}) \wedge \mathbf{E} + c^2 (\nabla \wedge \mathbf{B}) \wedge \mathbf{B} \right]. \quad (12.25)$$

The term in the square bracket can be written, we hope, as minus the divergence of something. It can, but this argument does not offer an automatic way to see it. Take a look at eq. (12.32)

and you will find the answer is

$$\epsilon_0 [\dots]_i = - \left(\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z} \right)$$

where

$$\sigma_{ij} = \frac{1}{2} \epsilon_0 (E^2 + c^2 B^2) \delta_{ij} - \epsilon_0 (E_i E_j + c^2 B_i B_j).$$

You are invited to check this by performing the differentiation. You will find that the B^2 term is needed to give part of $(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}$; it also contributes a $(\nabla \cdot \mathbf{B})\mathbf{B}$ term, but this vanishes by M2.

Method 2: 4-vector calculation

Now for the manifestly covariant approach. First we use the 4-force equation (12.2) to learn how the field transfers 4-momentum to the matter. We need to recall the argument for the form $\mathbf{E} \cdot \mathbf{j}$ which we gave before eq. (12.13): on the right hand side of the force equation the charge q is replaced by $\rho_0 dV$ and we use $\mathbf{J} \equiv \rho_0 \mathbf{U}$, then divide out the dV , to obtain

$$\begin{aligned} W^a &= \mathbb{F}^{a\mu} J_\mu & (= (\mathbf{E} \cdot \mathbf{j}/c, \rho \mathbf{E} + \mathbf{j} \wedge \mathbf{B})). \\ [W &= \mathbb{F} \cdot \mathbf{J}] \end{aligned} \quad (12.26)$$

The 4-vector $\mathbb{F}^{a\mu} J_\mu$ is called the *Lorentz force density*. It has dimensions of force per unit volume, as you can see from the presence of ρ instead of q in (12.26).

Next, just as we used a Maxwell equation to express \mathbf{j} in terms of the fields in Poynting's argument, now we use the first field equation (12.10) to express \mathbf{J} in terms of \mathbb{F} :

$$\begin{aligned} W^a &= -\mathbb{F}^{a\mu} (\epsilon_0 c^2) \partial_\lambda \mathbb{F}_\mu^\lambda = -\epsilon_0 c^2 \mathbb{F}^{a\mu} \partial^\lambda \mathbb{F}_{\lambda\mu} \\ [W &= -\epsilon_0 c^2 \mathbb{F} \cdot (\square \cdot \mathbb{F})] \end{aligned} \quad (12.27)$$

The second form helps to see clearly what we have: it has the structure “ $a\partial a$ ” so it should be possible to relate it to “ $\partial(aa)$ ”. This is the equivalent of step (12.14) in Poynting's argument, and eq. (9.43) (the product rule) contains the result we need:

$$\mathbb{F}^{a\mu} \partial^\lambda \mathbb{F}_{\lambda\mu} = \partial^\lambda (\mathbb{F}^{a\mu} \mathbb{F}_{\lambda\mu}) - \mathbb{F}_{\lambda\mu} (\partial^\lambda \mathbb{F}^{a\mu}). \quad (12.28)$$

The first term in (12.28) is a divergence of a tensor, the very thing we are looking for, so next we concentrate on the second term:

$$\mathbb{F}_{\lambda\mu} (\partial^\lambda \mathbb{F}^{a\mu}).$$

Just as in Poynting's argument, we need to bring in the other field equation, this time the homogeneous one (the one without a source term). That equation, (12.11) has things like " $\partial^c \mathbb{F}^{ab}$ " in it. Clearly what we need to do is contract it with \mathbb{F}_{cb} :

$$\begin{aligned} \mathbb{F}_{cb} (\partial^c \mathbb{F}^{ab} + \partial^a \mathbb{F}^{bc} + \partial^b \mathbb{F}^{ca}) &= 0 \\ \Rightarrow \quad \mathbb{F}_{cb} \partial^c \mathbb{F}^{ab} &= -\mathbb{F}_{cb} \partial^a \mathbb{F}^{bc} - \mathbb{F}_{cb} \partial^b \mathbb{F}^{ca}, \\ \text{i.e.} \quad \mathbb{F}_{\lambda\mu} (\partial^\lambda \mathbb{F}^{a\mu}) &= -\mathbb{F}_{\lambda\mu} \partial^a \mathbb{F}^{\mu\lambda} - \mathbb{F}_{\lambda\mu} \partial^\mu \mathbb{F}^{\lambda a}, \end{aligned} \quad (12.29)$$

where all we did was to carry two terms to the right hand side, and then relabel dummy indices to make them look more like the thing we want.

Now, practising the advice given in section 9.4 to 'look for scalars', we spot that the first term on the right hand side is almost a scalar. It is an example of eq. (9.46), except one pair of indices is the wrong way round. However, since \mathbb{F} is antisymmetric, we can swap them and introduce a minus sign, so we have $+\partial^a D$ where $D = \mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu} / 2$ (c.f. eq. (??)).

With the hint that a transpose might be handy, now take a look at the last term in (12.29) and transpose *both* occurrences of \mathbb{F} . This makes it look just like the left hand side, except the dummy indices are labelled differently. That doesn't matter, so we have found that

$$\begin{aligned} \mathbb{F}_{\lambda\mu} (\partial^\lambda \mathbb{F}^{a\mu}) &= \partial^a D - \mathbb{F}_{\lambda\mu} (\partial^\lambda \mathbb{F}^{a\mu}) \\ \Rightarrow \quad \mathbb{F}_{\lambda\mu} (\partial^\lambda \mathbb{F}^{a\mu}) &= \frac{1}{2} \partial^a D. \end{aligned}$$

Substituting this into (12.28), and returning to (12.27), we have

$$\begin{aligned} W^a &= -\epsilon_0 c^2 \left(\partial^\lambda (\mathbb{F}^{a\mu} \mathbb{F}_{\lambda\mu}) - \frac{1}{2} \partial^a D \right), \\ \Rightarrow \quad W^b &= -\epsilon_0 c^2 \left(\partial_\lambda (\mathbb{F}^{\lambda\mu} \mathbb{F}_\mu^b) - \frac{1}{2} \partial^b D \right). \end{aligned} \quad (12.30)$$

(by reversing the order of the product then using the see-saw rule twice and changing from a to b). We would like to set this equal to $-\partial_\lambda \mathbb{T}^{\lambda b}$, so we want to convert the ∂^b in the second term to ∂_λ . This is easily done by

$$\partial^b = g^{\lambda b} \partial_\lambda$$

Using finally the antisymmetry of \mathbb{F} , we have

Stress-energy tensor^a

$$\mathbb{T}^{ab} = \epsilon_0 c^2 \left(-\mathbb{F}^{a\mu} \mathbb{F}_\mu{}^b - \frac{1}{2} g^{ab} D \right), \quad (12.31)$$

where $D = \frac{1}{2} \mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}$.

[i.e. $\mathbb{T} = \epsilon_0 c^2 \left(-\mathbb{F} \cdot \mathbb{F} - \frac{1}{2} g D \right) \cdot$]

^aSee exercise ?? for the result using a metric of different signature.

Substituting for \mathbb{F} from eq. (12.5), we find

$$\mathbb{T}^{ab} = \left(\begin{array}{c|c} u & \mathbf{N}/c \\ \hline \mathbf{N}/c & \sigma_{ij} \end{array} \right)$$

where

$$\begin{aligned} \text{energy density } u &= \frac{1}{2} \epsilon_0 (E^2 + c^2 B^2) \\ \text{Poynting vector } \mathbf{N} &= \epsilon_0 c^2 \mathbf{E} \wedge \mathbf{B} \\ \text{3-stress tensor } \sigma_{ij} &= u \delta_{ij} - \epsilon_0 (E_i E_j + c^2 B_i B_j) \\ &= -\sigma_{ij}^M \end{aligned} \quad (12.32)$$

The “Maxwell stress tensor” σ_{ij}^M is often used in the literature, and its standard definition is such that it is the negative of \mathbb{T}^{ij} . By using $\sigma_{ij} = -\sigma_{ij}^M$ we preserve a greater uniformity in the equations describing conservation of energy and momentum below.

\mathbb{T} is fully symmetric. The symmetry of the space-space part is not surprising; the symmetry of the time-space part merits a comment. Suppose \mathbf{N} is a 4-vector direction, then $\mathbb{T} \cdot \mathbf{N}$ quantifies the flow of energy and momentum in that direction. The first *row* of \mathbb{T} is used to calculate the flow of *energy*; the elements of the first *column* are used, together with σ_{ij} , to calculate the flow of *momentum*. That the first row is equal to the first column is an example of the equality of energy flux and momentum density that we noted in section 4.6.

You can now confirm that the time component of the relation $\mathbf{W} = -\square \cdot \mathbb{T}$ is indeed eq. (12.13) as expected, which represents energy conservation.

Using (12.26) and examining the x -component of $\square \cdot \mathbb{T}$, we find

$$(\rho \mathbf{E} + \mathbf{j} \wedge \mathbf{B})_x = -\frac{1}{c^2} \frac{\partial N_x}{\partial t} - \nabla_j \sigma_{jx}.$$

To interpret the equation it may be helpful to integrate over a small volume to make the terms more familiar. If the volume is taken small enough that all the charge q in it moves at the same

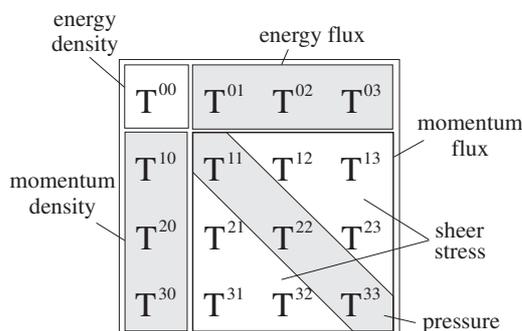


Figure 12.4: A summary of the physical interpretation of the stress-energy tensor. The terms ‘pressure’ and ‘stress’ here refer to contributions to the momentum flux; in the case of a flowing fluid they equate to what is ordinarily called ‘pressure’ and ‘stress’ in the rest frame, but not necessarily in other frames—see next chapter.

velocity \mathbf{v} , then we have

$$q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})_x = - \int \frac{\partial g_x}{\partial t} + \nabla \cdot \boldsymbol{\sigma}_x dV \tag{12.33}$$

where we wrote $\boldsymbol{\sigma}_x$ for σ_{jx} , this is the flux of x -momentum, and we used (12.20) to convert the Poynting vector into a momentum density. Eq. (12.33) can be ‘read’ as a statement of Newton’s 3rd law for the interaction of charge and field. On the left is the force on the charge, on the right is the force on (i.e. rate of injection of momentum into) the field. The equation states that these forces (‘action’ and ‘reaction’ if you like) are equal and opposite. The rate of injection of momentum into the field appears in two parts: $\partial g_x / \partial t$ is the rate at which the momentum of the local field is growing; $\nabla \cdot \boldsymbol{\sigma}_x$ is the rate at which momentum is being supplied to the rest of the field by flowing out of the region under consideration. This confirms the rough sketch we made before embarking on the calculation of \mathbb{T} .

Gathering the energy and all three momentum components together, we now have the overall conclusion:

Conservation of 4-momentum of both matter and field together

$$\left(\mathbf{E} \cdot \mathbf{j} / c, \quad \rho \mathbf{E} + \mathbf{j} \wedge \mathbf{B} \right) = - \left(\frac{1}{c} \frac{\partial}{\partial t}, \quad \nabla \cdot \right) \left(\begin{array}{c|c} u & \mathbf{N} / c \\ \hline \mathbf{N} / c & \boldsymbol{\sigma} \end{array} \right) \tag{12.34}$$

This result is at the heart of all energy-momentum conservation in electromagnetism.

It is sometimes stated that Newton’s third law (on action and reaction) breaks down in Special Relativity. It certainly does not, and eq. (12.34) is the proof for the case of electromagnetic interactions. However it is true to say that Newton’s third law should not be taken to be a

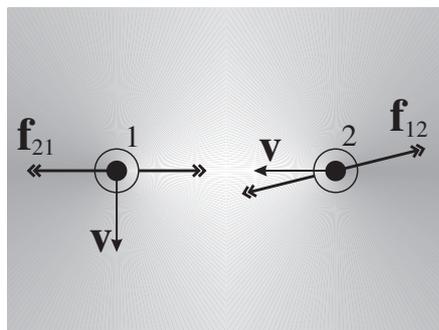


Figure 12.5: A repeat of figure 12.1, but with the effects on the field shown. Each black dot represents a particle, the attached arrows show the velocity of and force on the particle. Each circle represents a small volume of field, the attached arrows show the rate of injection of momentum from the sources into that volume of field. Conservation of momentum is achieved locally. The stress and momentum density throughout the rest of the field is not shown; it satisfies a continuity equation for each component.

statement about forces at separate locations (e.g. on particles with finite separation); it must be applied locally. What was missing in figure (12.1) was a pair of arrows showing the rate of change of momentum in the field. We can now provide those arrows—see figure 12.5.

The intuition that 4-momentum should be conserved has been fully born out by the theory. Indeed, the requirement to conserve energy in all reference frames *implies* the conservation of momentum, by the zero component lemma. We have discovered that in order to make sense of these great conservation principles it is necessary to credit an electromagnetic field not only with energy and momentum, but also with pressure and stress.

Simple examples of stress and pressure

The notion of a stress tensor is already present in classical mechanics, in the analysis of solids and fluids. Whether there or in field theory, it can be hard to get a good physical intuition of it. Some examples are given in table 12.1.

The part of the stress acting normal to the boundary of an object, in its own rest frame, is the *pressure*, and this is the easiest part to understand. Consider an ideal gas, for example. It exerts a pressure on the walls of the chamber confining it, but no other forces, leading to the tensor shown in table 12.1. ‘Dust’ is defined to be a type of gas in which the particles don’t move so exert no pressure. For a parallel plate capacitor aligned along x there is a uniform electric field E in the x direction, and no magnetic field. The force on either plate is $f = QE/2$ where the charge on the plate $Q = \epsilon_0 AE$, so $f = \epsilon_0 E^2 A/2$. Now look at the 3-stress tensor. It

Dust	$\begin{pmatrix} \rho_0 c^2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$	$T^{ab} = \rho_0 U^a U^b$
Ideal fluid	$\begin{pmatrix} \rho_0 c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix},$	$T^{ab} = (\rho_0 + p/c^2)U^a U^b + pg^{ab}$
Capacitor	$\frac{1}{2}\epsilon_0 E^2 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	
Solenoid	$\frac{1}{2}\epsilon_0 c^2 B^2 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	
Plane wave	$\epsilon_0 E_0^2 \cos^2(kx - \omega t) \begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = \epsilon_0 c^2 (E^2/\omega^2) K^a K^b$	
Point charge	$\frac{q^2}{16\pi^2\epsilon_0 r^6} \begin{pmatrix} \frac{r^2}{2} & 0 & 0 & 0 \\ 0 & \frac{r^2}{2} - x^2 & -xy & -xz \\ 0 & -yx & \frac{r^2}{2} - y^2 & -yz \\ 0 & -zx & -zy & \frac{r^2}{2} - z^2 \end{pmatrix}$	

Table 12.1: Example stress-energy tensors. Entries left blank are zero. The examples are necessarily given in some suitably chosen reference frame. For the first two cases a covariant equation for the tensor in question is also supplied; for the electromagnetic examples this is provided by eq. 12.31. The capacitor, solenoid and plane wave are aligned along the x axis. ‘Point charge’ refers to the Coulomb field of a point charge at the origin. All but the last example are stress-free in the chosen frame. Note that for electromagnetic fields, $\mathbb{T}_\lambda^\lambda = 0$ (easily proved from eq. (12.32)).

is

$$\sigma_{ij} = \frac{1}{2}\epsilon_0 E^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (12.35)$$

A negative pressure is a *tension*, it means that in the x direction the field is pulling its boundary (i.e. the charges on the plate) in towards it (the field). This is the attraction between the capacitor plates that we normally describe as the mutual attraction of opposite charges. The positive xx and yy terms tell us something further: there is an outwards pressure at right angles to the field direction. In general, in the absence of magnetic fields there is always a tension along the electric field lines, providing the ‘mechanism’ by which opposite charges attract, and there is a pressure at right angles to the field lines, tending to push them apart, providing the ‘mechanism’ by which like charges repel.

In a solenoid with a magnetic field B along the x direction, the 3-stress tensor is the same as (12.35) but with E replaced by cB . There is an outward pressure $B^2/2\mu_0$ on the walls of the solenoid, and a tension along the axis.

An electromagnetic wave in free space exerts a force in the direction of travel, and no transverse force. We can always align the x axis with the direction of propagation and write down both \mathbb{T} and the 4-wave vector $\mathbf{K} = (k, k, 0, 0)$. By spotting that for this case

$$\mathbb{T}^{ab} = \epsilon_0 c^2 \frac{E_0^2}{\omega^2} \cos^2(\mathbf{X}_\mu \mathbf{K}^\mu) \mathbf{K}^a \mathbf{K}^b$$

we deduce that this is the general relationship. By the quotient rule, this implies that E_0^2/ω^2 is a scalar, and therefore for a given wave examined in two arbitrarily related reference frames, the energy density, momentum flux, pressure, electric field and frequency are related by

$$\frac{u'}{u} = \frac{g'}{g} = \frac{p'}{p} = \frac{E_0'^2}{E_0^2} = \frac{\omega'^2}{\omega^2}.$$

This is the result we established by less sophisticated methods in chapter 3 (eq. (??)).

The point charge (Coulomb field) exhibits a negative pressure and stresses directed towards the origin. What this means is that if you arrange a ‘boundary wall’ in some region, such as to leave the field on one side of the boundary unaffected but reduced to zero on the other side, then in this case the boundary will be pulled in the direction of the side where the field remains non-zero. For example, place a uniform spherical shell of total charge $-q$ around the point charge q , at some finite radius. Then the field inside the shell is unchanged and the field outside is reduced to zero (easily proved using Gauss’ theorem). The stress tensor tells us that the spherical shell will experience forces pulling it in towards the origin—which of course we know to be true from the attraction of opposite charges.

What is striking about all this is that the electromagnetic field is behaving like a substantial thing, like a lump of jelly that we could push or pull and be pushed and pulled by in return. It is

no wonder that so much time and energy was devoted to the aether model of electromagnetism in the 19th century. This time was not wasted: it forced physicists such as Maxwell, Lorentz and Minkowski to discern and expound these properties. They make the field seem very much like a mechanical entity. Now we have come full circle, and one could say that we do have an ‘aether’ after all, but *the field itself is the ‘aether’*. It is an aether with properties that could not be grasped before the advent of Special Relativity, such as the ability to propagate signals that you can’t catch up with.

Having started in chapter 6 with the idea that electromagnetic fields can seem intangible, it is time to reconsider. Far from being ‘hard to see and touch’ the electromagnetic field is just about the only thing we ever see or touch! The retinas of our eyes respond to incoming light waves; the nerve receptors in our fingers respond to the pressure that results when we push the electron clouds of our skin molecules up against the fields supplied by the electron clouds of other objects. The chemical reactions that stimulate our taste buds are a dance of electrons in response to fields in further molecules. Even sound, a pressure wave, relies on electromagnetic fields to allow the air molecules to pass the pressure on as they collide.

12.2.4 Resolution of the “4/3 problem” and the origin of mass

[Section omitted in lecture-note version.]

12.3 Self-force and radiation reaction

[Section omitted in lecture-note version.]

[Section omitted in lecture-note version.]