Noether's theorem

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Abstract

We derive and explain Noether's theorem in the case of particle mechanics, i.e. systems described by sets of discrete variables. Some general comments on Lagrangian methods are also provided.

Noether's theorem asserts that whenever a physical system has a symmetry (in a sense to be described) then there is a conserved quantity (i.e. one whose value does not change over time). We will describe this more fully in the following. We begin with some general remarks.

1 Some thoughts about Lagrangians

A Lagrangian, for some given physical system, is a function of various coordinates and velocities and possibly time:

$$L = L(\mathbf{x},\ \mathbf{v},\ t)$$

where $\mathbf{x}$ is a way of referring to a complete set of coordinates, and $\mathbf{v}$ to their rates of change. This is often written

$$L = L(x_i,\ \dot{x}_i,\ t). \tag{1}$$

In this notation it is taken as understood that the x_i and $\dot{x}_i$ in the bracket signify that the Lagrangian can depend on *all* of the coordinates and velocities (not just one of them).

For given initial and final conditions, the *action* is defined

$$S = \int_A^B L\,\mathrm{d}t \tag{2}$$

where A refers to a complete specification of the initial state $(\mathbf{x}_A,\ \mathbf{v}_A)$ at some time t_A, and B refers to a complete specification of the final state $(\mathbf{x}_B,\ \mathbf{v}_B)$ at some time t_B. The value of the integral, and therefore the value of S, depends on the path along which the integration is performed. Thus S is a function not only of A and B but also of the path taken between them:

$$S = S(A,\ B,\ \text{path}) \tag{3}$$

In other words, S is a function of a function (the path). It is called a *functional*. Here the path is fully specified by some function $\mathbf{x}(t)$. Such a function also suffices to specify the velocities since $\mathbf{v} = \mathrm{d}\mathbf{x}/dt$.

A noteworthy fact about the Lagrangian is as follows. At any given *location and time*, specified by the values of the coordinates x_i and the time t, the Lagrangian can have many values because it depends on the velocities as well. However, if we specify a *path* being followed by the system, then at any given $(\mathbf{x},\ t)$ we will also know the velocities $\mathbf{v}$ *for that particular path.* So L can also be thought of as a function of a path $\mathbf{x}(t)$. The noteworthy fact is that once a path has been specified, L will be single-valued at each $\mathbf{x},\ t$ along that path.

1.1 Euler-Lagrange equations

For *given* start and end points A and B there are many possible paths, each giving rise to some value of the action S. It can be shown (by the calculus of variations) that a path giving a stationary value of the action satisfies

Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}. \tag{4}$$

We say equations (plural) because there is one equation for each i. But we can equally well state this as a single equation in vector quantities, thus:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \boldsymbol{\nabla} L \tag{5}$$

where we defined $\mathbf{p}$ (the canonical momentum) to be that vector whose components are

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i}. \tag{6}$$

In order to understand the Euler-Lagrange equations correctly, it is important to note what is held constant in each partial derivative, and what is meant by the use of a total derivate (not partial derivative) with respect to time on the left. This $\mathrm{d}/\mathrm{d}t$ is a derivative *along the path followed by the system*, whereas $(\partial L/\partial x_i)$ is a derivative at constant values of $\{x_{j\neq i},\ v_k, t\}$ and $(\partial L/\partial \dot{x}_i)$ is a derivative at constant values of $\{x_j,\ v_{k\neq i}, t\}$.

So far we have merely stated mathematical facts. It remains to state a law of physics. This is: *the path followed by the physical system is one for which the action has a stationary value.* It follows that the path is the one specified by the Euler-Lagrange equations. These equations are first order to time, therefore to fix a unique solution it suffices to specify the initial conditions by providing the values of $\mathbf{x}_A, \mathbf{v}_A, t_A$.

Example. Consider the case

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x) \tag{7}$$

for some function $V(x)$, where x is the position of a particle of constant mass m. We find

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

so the Euler-Lagrange equations give

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

There is a natural physical interpretation: the Lagrangian here is $L = T - V$ where T is kinetic energy, and the Euler-Lagrange equation is Newton's second law, since we can interpret $-dV/dx$ as a force.

1.1.1 More than one Lagrangian can describe the same physics

It is a familiar fact in classical mechanics that if one changes the potential everywhere by the same amount, then there is no effect on the physical behaviour because the gradient of the potential does not change and therefore the forces do not change. In the same way, if we replace L by

$$\tilde{L} = L + k \tag{8}$$

where k is a constant, then the Euler-Lagrange equations are unaffected.

More interestingly, we can also make much more convoluted changes in L and still correctly describe the system. In general the replacement

$$L \to \tilde{L} = L + \frac{\mathrm{d}f}{\mathrm{d}t} \tag{9}$$

will give the same physics, for any function $f(x_i, t)$ whatsoever! *Proof.* For any given path followed by the system between given starting state A and finishing state B,

$$\int_{t_A}^{t_B} \left(L + \frac{\mathrm{d}f}{\mathrm{d}t} \right) \mathrm{d}t = \left(\int_{t_A}^{t_B} L \, \mathrm{d}t \right) + f(t_B) - f(t_A) \tag{10}$$

The extra terms in the action depend only on the start and end points, so they do not change from one path to another between given end points. Hence the path which minimises the new action is the same path as the one which minimises the original action. QED.

Exercise. Consider a system with one coordinate, x, and one velocity, $v = \dot{x}$. Show, by substituting $\tilde{L} = L + vg(x)$ into the Euler-Lagrange equations, that the resulting motion is the same, for all functions $g(x)$.

It can seem puzzling that many different forms of Lagrangian may describe the same physical system. A way to make sense of this is to note that it is the action, not the Lagrangian, which more directly captures the physics.

2 Conservation laws for simple cases

In can happen that the Lagrangian does not depend at all one or more of the coordinates. In this case, for each coordinate x_j which does not appear in the expression for $L(\mathbf{x}, \mathbf{v}, t)$ we shall find

$$\frac{\partial L}{\partial x_j} = 0 \tag{11}$$

and therefore (from the relevant EL equation),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = 0 \tag{12}$$

so $p_j = (\partial L/\partial \dot{x}_j)$ is constant in time. We say p_j is a *conserved quantity*. Stated in words, the conclusion is

> *If L does not depend on x_i then p_i is conserved.*

Thus, for example, if the Lagrangian has no explicit spatial dependence then the linear momentum is conserved.

[There is another piece of terminology here. If L is independent of some coordinate then that coordinate is said to be a *cyclic coordinate*. This terminology adds nothing to our reasoning but it is widely adopted.]

Let's next introduce the following quantity, called the **Hamiltonian**:

$$H \equiv \left(\sum_i p_i \dot{x}_i\right) - L. \tag{13}$$

This is a definition so no proof is needed. Let's differentiate this equation with respect to time, along some path which we leave unspecified for now:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i (\dot{p}_i \dot{x}_i + p_i \ddot{x}_i) - \frac{\mathrm{d}L}{\mathrm{d}t}. \tag{14}$$

We now invoke the fundamental mathematical fact about partial derivatives of any function: a total change can be written as a sum of partial changes. Applied to the Lagrangian this mathematical fact is

$$\mathrm{d}L = \sum_i \frac{\partial L}{\partial x_i}\mathrm{d}x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\mathrm{d}\dot{x}_i + \frac{\partial L}{\partial t}\mathrm{d}t. \tag{15}$$

Therefore

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \frac{\partial L}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \sum_i \frac{\partial L}{\partial \dot{x}_i}\frac{\mathrm{d}\dot{x}_i}{\mathrm{d}t} + \frac{\partial L}{\partial t}. \tag{16}$$

$$= \sum_i \frac{\partial L}{\partial x_i}\dot{x}_i + \sum_i p_i \ddot{x}_i + \frac{\partial L}{\partial t}. \tag{17}$$

Substituting this into (14) gives

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \left(\dot{p}_i - \frac{\partial L}{\partial x_i}\right)\dot{x}_i - \frac{\partial L}{\partial t}. \tag{18}$$

Now if we consider not any path, but the one followed by the system, then it will satisfy the Euler-Lagrange equations and therefore every term in the sum here will vanish, and we deduce

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\left(\frac{\partial L}{\partial t}\right)_{\mathbf{x},\mathbf{v}} \tag{19}$$

To understand this result you need to understand clearly the difference between a total derivative and a partial derivative. On the left we have *the amount by which H changes as the system moves along the path*. On the right we have *minus a certain partial derivative of L*. The important conclusion is

> *If L has no explicit time-dependence then H is conserved.*

where the phrase 'no explicit time-dependence' means that, if you write L as a function of $\mathbf{x}, \mathbf{v}$ and t then in fact t will not appear in the expression.

3 Symmetry more generally

If the function L does not depend on a coordinate, then it is said to have a symmetry. The term 'symmetry' refers to the idea that something stays unchanged when some transformation is applied to it. The something that stays unchanged is the value of L; the transformation that is applied is a change in the coordinate.

3.1 Case where t is not affected

There are much more general types of symmetries. It can happen, for example, that

$$L(x_i, \dot{x}_i, t) = L(x_i + \eta_i,\ \dot{x}_i + \dot{\eta}_i,\ t) \tag{20}$$

for some non-trivial set of functions $\eta_i(\mathbf{x})$. In this case L is said to have a symmetry under the transformation $x_i \to x_i + \eta_i$. Here is an example:

$$L(x, y, \dot{x}, \dot{y}, t) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - \frac{1}{2} m \omega^2 \left(x^2 + y^2 \right) \tag{21}$$

with

$$\eta_x = x(\cos(\alpha) - 1) + y \sin \alpha, \tag{22}$$

$$\eta_y = -x \sin \alpha + y(\cos(\alpha) - 1), \tag{23}$$

where α is a constant (independent of x and y and t), which can take any value.

More generally, the system has a symmetry whenever

$$L(x_i + \eta_i,\ \dot{x}_i + \dot{\eta}_i,\ t) = L(x_i, \dot{x}_i, t) + \frac{\mathrm{d}f}{\mathrm{d}t} \tag{24}$$

for some function f, since, as explained in connection with eqn (9), the term $\mathrm{d}f/\mathrm{d}t$ does not influence the physical behaviour. The case where $f \neq 0$ may be called a *quasi-symmetry*.

To keep things simple we will present the reasoning in the first instance for a system with only one coordinate, and we consider the case (20) (i.e. no function f). Expand the right hand side to first order in small quantities, **under the assumption that η is itself small, and so is $\dot{\eta}$**. We find

$$L(x + \eta,\ \dot{x} + \dot{\eta},\ t) \simeq L(x,\ \dot{x},\ t) + \frac{\partial L}{\partial x} \eta + \frac{\partial L}{\partial \dot{x}} \dot{\eta} \tag{25}$$

Hence

$$\delta L = \frac{\partial L}{\partial x} \eta + \frac{\partial L}{\partial \dot{x}} \dot{\eta} \tag{26}$$

The second term here can be seen as one of the terms you would get if you took the derivative of a product. By noticing this, we deduce

$$\delta L = \frac{\partial L}{\partial x} \eta + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \eta \right) - \eta \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) \tag{27}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \eta \right) + \eta \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) \right]. \tag{28}$$

Notice that so far we have not needed to assume anything about the path $x(t)$, nor about any symmetry. We have only assumed that η and $\dot{\eta}$ are small. We now reason:

- If η is a symmetry then $\delta L = 0$
- If $x(t)$ is the path of least action then the square bracket $= 0$.

Hence when these hold, we can deduce

$$\frac{\partial L}{\partial \dot{x}} \eta \quad \text{is constant (i.e. is a conserved quantity).} \tag{29}$$

Exercise. The reader is now invited to repeat the argument for the case of a system with many coordinates. The steps are the same, the only new feature is that one gets sums of terms involving partial derivatives. The result is:

> If $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ has a symmetry described by infinitesimal quantities $\eta_i(\mathbf{x})$ then the following quantity is conserved:
>
> $$\sum_i \frac{\partial L}{\partial \dot{x}_i} \eta_i - f \tag{30}$$

where we included the possibility of function f in the more general symmetry relation (24). One is most often interested in the case $f = 0$.

Example. The transformation described by (22) and (23) can be made infinitesimal by taking small α, and then we have, at first order in α,

$$\eta_x = \alpha y, \qquad \eta_y = -\alpha x. \tag{31}$$

The conserved quantity is therefore

$$m\dot{x}\alpha y - m\dot{y}\alpha x = -\alpha m(x\dot{y} - y\dot{x}) \tag{32}$$

This can be recognized as angular momentum multiplied by $(-\alpha)$. This example could equally well have been treated by adopting polar coordinates and then one finds that the angular coordinate is cyclic.

3.1.1 Some further intuition

Figure 1 shows a visual way to gain intuition about our result (29). The thick line AB shows the path of least action between A and B. The dashed line $ACDB$ shows a nearby path. Since AB has least action, this other path has the same action (at first order in small quantities). If the Lagrangian also has a symmetry such that it is the same along AB as it is along CD, then clearly CD contributes the same amount to the action as AB, and therefore the contributions from AC and DB must cancel each other out: they are equal and opposite. It follows that the action along AC is equal to the action along BD. This is the conserved quantity.

We now wish to see why this conserved action is given by $\eta(\partial L/\partial \dot{x})$. Why isn't $(\partial L/\partial x)$ involved, for example? It is because the short sections AC and BD are horizontal. If the system were to take that path then its velocity would be infinite on the horizontal sections. We can avoid the infinity by

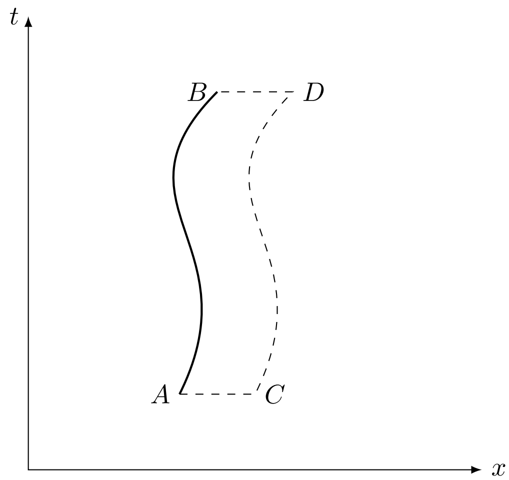

Figure 1: A path AB and a nearby path $ACDB$.

allowing a short amount of time τ to pass (placing C slightly later than A, and D slightly earlier than B, and then take a limit at the end. The difference in the action between the dashed path and the full path between times t_A and $t_A + \tau$ is then

$$(S' - S)_{AC} = \int_{t_A}^{t_A+\tau} \frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\frac{\eta}{\tau}\mathrm{d}t$$

since the velocity must be η/τ to cover the distance η in the time τ. In the limit of infinitesimal τ the second term dominates, and in this limit the integral is simply τ times the value of the integrand at t_A, so we have

$$(S' - S)_{AC} \quad = \quad \frac{\partial L}{\partial \dot{x}}\eta \tag{33}$$

(evaluated at t_A). A similar calculation applies at t_B. You can if you like regard this argument as an alternative derivation of (29).

3.2 General case

So far we treated symmetries only involving a change in x_i (and therefore also $\dot{x}_i$) but not directly involving t. Now we treat the general case. There are two types of symmetry which may arise: symmetry of the Lagrangian, and symmetry of the action.

3.2.1 Symmetry of the Lagrangian

This is the case

$$L(x_i, \dot{x}_i, t) = L(x_i + \eta_i,\ \dot{x}_i + \dot{\eta}_i,\ t + \tau) \tag{34}$$

where $\eta(\mathbf{x}, t)$ is a set of functions, as before, and $\tau(\mathbf{x}, t)$ is another function. Treating the case of a single coordinate, the analysis is just like eqns (20)–(28) except that now we have another term, coming from $(\partial L/\partial t)$. Assuming the case of small η and τ, one finds, to first order in these quantities, and after using the Euler-Lagrange equations to simplify the expression,

$$0 = \delta L = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\eta\right) + \tau \frac{\partial L}{\partial t}. \tag{35}$$

Now we employ (19), giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\eta\right) - \tau \frac{\mathrm{d}H}{\mathrm{d}t} = 0. \tag{36}$$

There is no simple way to extract a conserved quantity from this expression for a general form of the function τ. However, if τ is simply a constant then $\dot{\tau} = 0$ and then we have that the conserved quantity is

$$\frac{\partial L}{\partial \dot{x}}\eta - H\tau \tag{37}$$

3.2.2 Symmetry of the action

This is the case

$$\int_{t_A}^{t_B} L[x_i, \dot{x}_i, t]\,\mathrm{d}t = \int_{t_A+\delta t_A}^{t_B+\delta t_B} L[x_i'(t'),\ \dot{x}_i'(t'),\ t']\,\mathrm{d}t' \tag{38}$$

where

$$x_i'(t') = x_i(t) + \eta_i(\mathbf{x}, t) \tag{39}$$
$$t' = t + \tau(\mathbf{x}, t) \tag{40}$$
$$\delta t_A = \tau(\mathbf{x}_A, t_A), \qquad \delta t_B = \tau(\mathbf{x}_B, t_B) \tag{41}$$

Note that, if τ depends on time then if the symmetry of the action holds (eqn (38)) then the symmetry of the Lagrangian (34) does not (except in special cases). The reason is to do with the fact that $\delta t_B \neq \delta t_A$ so we have a different range of integration and $\mathrm{d}t' \neq \mathrm{d}t$.

Treating the case of a single coordinate, for simplicity, we have

$$\int_{t_A}^{t_B} L[x, \dot{x}, t]\,\mathrm{d}t = \int_{t_A+\delta t_A}^{t_B+\delta t_B} L[x'(t'),\ \dot{x}'(t'),\ t']\,\mathrm{d}t' \tag{42}$$

The variable t' appearing in the integral on the right is a dummy variable, so we can rename it to t in the integral, giving

$$\int_{t_A}^{t_B} L[x, \dot{x}, t] \, \mathrm{d}t = \int_{t_A+\delta t_A}^{t_B+\delta t_B} L[x'(t),\ \dot{x}'(t),\ t] \, \mathrm{d}t \tag{43}$$

Now

$$\int_{t_A+\delta t_A}^{t_B+\delta t_B} L \, \mathrm{d}t = \int_{t_A+\delta t_A}^{t_A} L \, \mathrm{d}t + \int_{t_A}^{t_B} L \, \mathrm{d}t + \int_{t_B}^{t_B+\delta t_B} L \, \mathrm{d}t \tag{44}$$

where the reader must keep aware that we are dealing with the right hand side of (43) not the left. In the limit $\delta t_A \to 0$ and $\delta t_B \to 0$ the first and last integrals can be evaluated immediately, giving $-L(t_A)\delta t_A$ and $L(t_B)\delta t_B$ respectively. Furthermore we can write

$$L(t_B)\delta t_B - L(t_A)\delta t_A = \int_{t_A}^{t_B} \frac{\mathrm{d}}{\mathrm{d}t}(L\tau)\mathrm{d}t. \tag{45}$$

Hence, by bringing these observations into (43) we have

$$\int_{t_A}^{t_B} L[x, \dot{x}, t] \, \mathrm{d}t = \int_{t_A}^{t_B} \frac{\mathrm{d}}{\mathrm{d}t}(L\tau) + L[x'(t),\, \dot{x}'(t),\, t] \, \mathrm{d}t \tag{46}$$

It is convenient now to introduce a further function ζ, defined such that

$$x'(t) = x(t) + \zeta(\mathbf{x}, t) \tag{47}$$

By using (39) and (40) we have

$$x'(t) = x(t-\tau) + \eta(x, t-\tau) = x(t) - \tau\dot{x} + \eta - \tau\eta \simeq x(t) - \tau\dot{x} + \eta \tag{48}$$

where in the last step the $\tau\eta$ term is dropped because it is second order. Hence (at first order)

$$\zeta = \eta - \tau\dot{x}. \tag{49}$$

The second term in the integrand on the right hand side of (46) is

$$\begin{aligned} L[x'(t), \dot{x}'(t), t] &= L[x(t) + \zeta,\ \dot{x}(t) + \dot{\zeta},\ t] \\ &\simeq L[x, \dot{x}, t] + \frac{\partial L}{\partial x}\zeta + \frac{\partial L}{\partial \dot{x}}\dot{\zeta} && (50) \\ &= L[x, \dot{x}, t] + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\zeta\right) + \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right]\dot{\zeta} && (51) \end{aligned}$$

By now employing the Euler-Lagrange equation, the square bracket is zero. By also arguing that the symmetry of the action is such that (46) holds for any t_A, t_B, we may infer that the integrands agree, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L\tau + \frac{\partial L}{\partial \dot{x}}\zeta\right) = 0. \tag{52}$$

Therefore the conserved quantity is

$$L\tau + \frac{\partial L}{\partial \dot{x}}\zeta = L\tau + \frac{\partial L}{\partial \dot{x}}(\eta - \tau\dot{x}) \tag{53}$$

$$= \frac{\partial L}{\partial \dot{x}}\eta - H\tau \tag{54}$$

where we recalled the definition of the Hamiltonian (13). This conserved quantity is the same as the one we found in the previous section, eqn (37), but now the result is more general because we allow any functional form for τ, not just a constant. (In practice the case τ =const. is the most common so the simpler argument of the previous section is sufficient for the most common case.)

Exercise. The reader is invited to treat the case of a system with many coordinates. The result is:

> **Noether's theorem (also called Noether's first theorem)**. If the action has a symmetry described by infinitesimal quantities $\eta_i(\mathbf{x}, t)$ and $\tau(\mathbf{x}, t)$ then the following quantity is conserved:
>
> $$\left[\sum_i \frac{\partial L}{\partial \dot{x}_i}\eta_i\right] - H\tau \tag{55}$$

It should be immediately apparent to the reader that the method of this section subsumes all the other cases. For if L is independent of a coordinate then we have a symmetry where η_i for that coordinate is a non-zero constant and for the others is zero, and then (55) informs us that the i'th momentum is conserved. Equally, the case where L has no explicit dependence on time is described by a constant non-zero τ and $\eta_i = 0$, and then we find that H is conserved.