Example finals questions

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1998 3. The first two excited configurations of the helium atom, 1s2l, each give rise to terms labelled ^1L and ^3L. What does the notation ^1L and ^3L signify? List the terms that arise from the two configurations. Sketch the energy level diagram of the helium atom, showing these terms, the ground state, and the ionization energy. [6]

Indicate the positions of the $n = 1$ and $n = 2$ levels of hydrogen on the same scale, using the ionization limit of each atom as the common zero energy. (Ignore fine structure.) Discuss the physical origin of the similarities and differences between the energy level diagrams of the hydrogen and helium atoms. [11]

The singlet term of the 1s2p configuration lies at an energy of 3.37 eV below the helium ionization limit. The wavelength of the transition from this level to the ground state is 58.4 nm. Estimate (giving your reasoning) the wavelength of the transition between the singlet term of the 1s3p configuration and the ground state. Explain why electric dipole transitions cannot occur between the terms of the lowest excited configuration and the ground state. [8]

1998 6. A particle is confined in a one-dimensional well described by a potential $V(x)$ which is zero for $|x| < \frac{1}{2}a$ and infinite for $|x| > \frac{1}{2}a$. Use the Schrödinger equation to find the energies available to the particle. Give sketch graphs showing the wavefunctions of the three states of lowest energy. [6]

Without detailed calculation, explain how the wavefunction of the ground state is changed if the well is perturbed by a small added potential ΔV

(a) $\Delta V = V_1 x$
($V_1 > 0$, $V_1 a \ll E_0$, the ground-state energy)

(b) $\Delta V = V_2 x^2$
($V_2 > 0$, $V_2 a^2 \ll E_0$). [6]

The well is perturbed by the addition to $V(x)$ of a potential $\Delta V = V_3 \sin(\pi x/a)$, as a result of which the ground-state wavefunction acquires an admixture of other wavefunctions. Find the amount by which the first excited-state wavefunction appears in the mixture. Show that in the first order of perturbation theory this is the only wavefunction to be mixed with the ground-state wavefunction. [13]

1999 3. The spin-orbit interaction for the electron in a hydrogen atom can be represented by a term

$$\frac{\mu_0}{4\pi}\frac{e^2}{2m_{\mathrm{e}}^2}\frac{1}{r^3}\mathbf{l}\cdot\mathbf{s}$$

in the Hamiltonian, where r is the radial co-ordinate of the electron and $\mathbf{l}$ and $\mathbf{s}$ are the orbital and spin angular momentum operators respectively. Give simple arguments which justify the form of this expression. [8]

[For an electronic state of given n and l in hydrogen,

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a_0^3 n^3 l(l+1/2)(l+1)} \text{]}$$

Calculate in eV the positions of the 3p $^2\mathrm{P}_{3/2}$ and 3p $^2\mathrm{P}_{1/2}$ levels in hydrogen relative to the position of the unperturbed 3p configuration. Give an energy level diagram showing the splitting. Explain briefly whether your diagram would apply without modification to the corresponding levels of anti-hydrogen (a positron bound to an anti-proton). [7]

Calculate also the splittings in eV of the 3p configurations in

(a) hydrogen-like $_{11}$Na (a sodium nucleus with a single bound electron);

(b) muonic sodium (in which a muon moves in hydrogen-like states in the field of a sodium nucleus). [7]

The spin-orbit splitting of the 3p configuration in neutral sodium is $6.8 \times 10^{-4}\,\mathrm{eV}$. Comment on the order of magnitude of this splitting in relation to those you have calculated for hydrogen and hydrogen-like sodium. [3]

1999 5. Explain what is meant by the *Variational Principle*, and show how it can be applied to give an upper bound to the true ground state energy of a quantum mechanical system. [6]

A particle of mass m moves in one dimension, in the potential

$$\begin{aligned} V(x) &= \infty \qquad (x \leq 0) \\ V(x) &= ux \qquad (x > 0) \end{aligned}$$

where u is a constant. Make a rough sketch of the wavefunctions for this particle in the ground and first excited states. [4]

Show that for a trial wavefunction of the form

$$\begin{aligned} \psi(x) &= 0 && (x \leq 0) \\ \psi(x) &= Ax\exp(-ax) && (x > 0) \end{aligned}$$

the expectation value of the potential energy is

$$\langle V \rangle = 3u/2a$$

and that of the kinetic energy is

$$\langle T \rangle = \hbar^2 a^2/2m$$

Hence obtain an estimate of the ground state energy. [12]

Explain why the true wavefunction can be expected to fall off more rapidly than the trial wavefunction at large x. [3]

1999 6. Two particles, with masses m_1 and m_2 and positions $\mathbf{r}_1$ and $\mathbf{r}_2$, interact via the potential $V(r)$, where $r = |\mathbf{r}_1 - \mathbf{r}_2|$. The radial part of the wavefunction for relative motion obeys the equation

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] + V(r)\right\}\phi(r) = E\phi(r)$$

Discuss the physical significance of the terms in this equation and the meaning of the symbols μ, l and E. [6]

Consider an attractive interaction of the form

$$\begin{aligned} V(r) &= 0 \qquad (r < a) \\ V(r) &= V_0 \quad\; (r \geq a) \end{aligned}$$

where V_0 is a constant. Let $l = 0$ and set $\phi(r) = u(r)/r$. Show that

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2 u(r)}{\partial r^2} + V(r)u(r) = Eu(r)$$

State and justify the conditions which must be satisfied by $u(r)$ at $r = 0$ and $r = a$. [7]

For the case $V_0 = \infty$, find the ground state energy for relative motion of the system. [6]

In this state, what is the mean separation of the particles? [6]

1999 7. Write down expressions for the operators representing the position, momentum and kinetic energy of a particle moving in one dimension. [4]

The wavefunction $\psi(x,t)$ satisfies the time-dependent Schrödinger equation for a free particle of mass m, moving in one dimension. Consider a second wavefunction of the form

$$\phi(x,t) = \mathrm{e}^{\imath(ax-bt)}\psi(x-vt,t)$$

Show that $\phi(x,t)$ obeys the same time-dependent Schrödinger equation, provided the constants a, b and v are related by

$$b = \frac{\hbar a^2}{2m} \quad \text{and} \quad v = \frac{\hbar a}{m} \qquad [9]$$

Calculate the expectation values of position, momentum and energy for a particle in the state $\phi(x,t)$ in terms of those for a particle in the state $\psi(x,t)$. Show that the uncertainty in momentum is the same in both states. [8]

What physical interpretation can be given to the transformation from the state $\psi(x,t)$ to the state $\phi(x,t)$?

[4]

1999 8. Give an account of the elementary theory of measurement in quantum mechanics, using the following questions as a guide.

(a) What role is played in the theory by operators?

(b) What are the possible results of a measurement?

(c) How does one calculate the probability of obtaining a given result from a measurement?

(d) What happens if two successive measurements, possibly of different quantities, are made on a system?

Illustrate your answer by discussing a simple example, such as measurements of the components of the angular momentum of a spin-$\frac{1}{2}$ particle. [25]

2000. 4. Under certain assumptions the time independent Schrödinger equation for atomic helium can be written as

$$\left\{-\frac{\hbar^2}{2m_e}\nabla_1^2-\frac{\hbar^2}{2m_e}\nabla_2^2-\frac{Ze^2}{4\pi\epsilon_0 r_1}-\frac{Ze^2}{4\pi\epsilon_0 r_2}+\frac{e^2}{4\pi\epsilon_0 r_{12}}\right\}\psi=E\psi$$

where r_1 and r_2 are the radial coordinates of the two electrons and r_{12} is their separation.

What is the physical origin of each term in this expression? What approximations have been made in deriving the expression in this form? [8]

A solution of the form $\psi_{a,b} = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)$ can be found when the term $e^2/4\pi\epsilon_0 r_{12}$ is ignored, where a and b label the states occupied by electrons 1 and 2. By considering an exchange of these labels explain why a wavefunction of the form $\psi_{a,b}$ is not acceptable for two identical particles. Discuss the symmetry properties of the wavefunctions that describe the $1s^2$ and $1s2s$ electron configurations. Sketch an energy level diagram for helium showing the terms that arise from these configurations. [12]

The energy required to remove an electron from the ground state of atomic helium is approximately 25 eV. For the helium-like ion O^{6+} the corresponding energy is approximately 740 eV. For helium and oxygen compare these energies with those required to remove an electron from the ground states of the hydrogen-like ions He^{+} and O^{7+}. Comment on your results. [5]

2000 7. A particle of mass m is confined in a 3-dimensional infinitely deep potential well. The sides parallel to the x-, y-, z-axes each have length L. The origin of the coordinate system is the middle of the well, $L/2$ away from each side. Derive the normalized time-independent wavefunction of the ground state and an expression for the energy levels. What is the degeneracy of the first excited state? [10]

Explain why pressure acting on the walls of the well is given as $-\frac{1}{3L^2}\frac{\partial E}{\partial L}$ where E is the energy of the particle. [5]

The energy difference between the ground state and the first excited state is 140 MeV. The mass of the particle is $M_p/3$. Calculate the value of L in Fermi. [4]

In an instantaneous event the size of the well suddenly expands. The new side length is $2L$. The well is otherwise unchanged. Before the expansion the particle was in its ground state. Find the probability of observing the particle in its new ground state immediately after the expansion. [6]

2001 5. Two identical non-interacting particles occupy the same one-dimensional simple harmonic oscillator potential well. The single-particle spatial wavefunctions for the ground state and first excited state are $\psi_{\mathrm{g}}(x)$ and $\psi_{\mathrm{e}}(x)$ respectively, where

$$\psi_{\mathrm{g}}(x) = N \exp\left(-\frac{x^2}{2a^2}\right) \text{and} \psi_{\mathrm{e}}(x) = \sqrt{2}N\left(\frac{x}{a}\right)\exp\left(-\frac{x^2}{2a^2}\right)$$

Both N and a are positive real constants.

In an excited state of the two-particle system, one particle is in the ground state ψ_{g} and the other in the first excited state ψ_{e}. Write down a spatial wavefunction $\psi_{\S}(x_1, x_2)$ that is symmetric in the coordinates and also a spatial wavefunction $\psi_{\mathrm{A}}(x_1, x_2)$ that is antisymmetric in the coordinates, showing explicitly the dependence on the particle coordinates x_1 and x_2.

For each wavefunction $\psi_{\S}(x_1, x_2)$ and $\psi_{\mathrm{A}}(x_1, x_2)$ write down the probability densities $P_{\S}(x_1, x_2)$ and $P_{\mathrm{A}}(x_1, x_2)$. [10]

Evaluate the probability density for each wavefunction when $x_1 = x_2$ and give a simple argument to suggest which is the larger of $\langle (x_1 - x_2)^2 \rangle_{\S}$ and $\langle (x_1 - x_2)^2 \rangle_{\mathrm{A}}$. [6]

Explain why, for a given configuration, the singlet state of an energy level in helium is higher than the corresponding triplet state. [9]

2001 6. Explain how the general solution to the time-dependent Schrödinger equation for a system may be written in terms of the eigenfunctions and eigenvalues obtained from the corresponding time-independent Schrödinger equation. [4]

A particle of mass m moves in one dimension with potential energy $V(x) = \frac{1}{2}m\omega^2 x^2$. Verify that a normalized solution of the time-dependent Schrödinger equation for the particle is

$$\Psi(x,t) = \Psi_0(x,t)\sin\gamma + \Psi_1(x,t)\cos\gamma, \text{where}\Psi_0(x,t) = \alpha^{1/2}\bigg(\frac{1}{\pi}$$

γ is a fixed real parameter and $E_0 = \hbar\omega/2$. Obtain expressions for α and E_1. [10]

What are the possible outcomes of measurement of the energy of the particle in the state $\Psi(x,t)$? What is the probability of each outcome? [4]

Explain what is meant in quantum mechanics by the *expectation value* of an operator. Calculate the expectation value of position x, as a function of time t, for the particle in the state $\Psi(x,t)$. [7]

2001 7. Explain what is meant by *probability current density*. Show that the probability current density j can be written as

$$j(x,t) = \frac{\imath\hbar}{2m}\left(\Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial\Psi(x,t)}{\partial x}\right)$$

where $\Psi(x,t)$ is a wavefunction. [7]

A particle of mass m and energy E is incident from negative x on the potential step

$$V(x) = 0 \qquad \text{for } x \leq 0 V(x) = V_0 > 0 \quad \text{for } x > 0$$

Write down the form of the wavefunction $\psi(x)$ in the two regions, as well as the boundary conditions, for the cases $E > V_0$ and $0 < E < V_0$. Using probability current densities, derive an expression for the transmission probability for $E > V_0$. [8]

Make a sketch graph of the reflection and transmission probabilities as functions of E/V_0, paying attention to both cases: $E/V_0 < 1$ and $E/V_0 > 1$. [4]

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The diagram shows the reflection probability R and transmission probability T for a barrier potential

$V(x) = V_0 \quad \text{for } 0 < x < a V(x) = 0 \quad \text{for } x \leq 0, \;\; x \geq a$

Explain the physical origin of any features that are new, compared with the case of a potential step. [6]

2001 8. Hydrogen-like wavefunctions have the form

$$\psi_{n\ell m}(\mathbf{r}) = R_{n\ell}(r) Y_{\ell m}(\theta, \phi)$$

Which property of the Coulomb interaction allows for the above separation of variables? What do the symbols n, ℓ and m represent? For a given n, what values can ℓ and m take? [5]

The ground-state wavefunction is given by

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2\,\mathrm{e}^{-Zr/a_0} \qquad \text{and} \qquad Y_{00}(\theta, \phi) = \frac{1}{(4\pi)^{1/2}} \,,$$

where $a_0 = 4\pi\epsilon_0\hbar^2/(\mu e^2)$ and r is the distance from the nucleus. What do μ and Z represent? What is the significance of a_0? Sketch the probability density for the electron as a function of r. [5]

A hydrogen-like atom is formed by a proton and a π^- particle. Evaluate a_0 for this system.

The Coulomb interaction between a proton and a π^- is modified by the nuclear interaction between them. The nuclear interaction can be described by the Yukawa potential

$$U(r) = g\frac{\mathrm{e}^{-r/b}}{r}\,,$$

where $b = 1\,\mathrm{fm}$ and $g = 15\,\hbar c$. Using first-order perturbation theory, calculate the energy shift of the ground state. [10]

Compare the above energy shift with the ground-state energy due to the Coulomb interaction only, and comment on the applicability of first-order perturbation theory to this case. [5]

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[End of paper]