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Tracking the radiation reaction energy when charged bodies accelerate

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We consider radiation reaction and energy conservation in classical electromagnetism. We first treat the well-known problem of energy accounting during radiation from a uniformly accelerating particle. This gives rise to the following paradox: when the self-force vanishes, the system providing the applied force does only enough work to give the particle its kinetic energy—so where does the energy that is eventually radiated away come from? We answer this question using a modern treatment of radiation reaction and self-force, as it appears in the expression due to Eliezer and Ford and O’Connell. We clarify the influence of the Schott force, and we find that the radiated power is $2q^2\mathbf{a}_0 \cdot \mathbf{f}_0/(3mc^3)$, which differs from Larmor’s formula. Finally, we present a simple and highly visual argument that enables one to track the radiated energy without the need to appeal to the far field in the distant future (the “wave zone”). © 2015 American Association of Physics Teachers.

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I. INTRODUCTION

This paper discusses the physics of an accelerating electric charge, with particular regard to the emitted radiation and the radiation reaction force. In the first part of the paper, we present some issues concerning radiation reaction that often confuse students. As an appealing way to introduce the subject, we start with a paradox that has been treated before but perhaps is less well known than it might usefully be. A well-framed paradox, and its resolution, offers a helpful way to capture a physical idea, and is often easily memorable for students. In the second part of the paper, we invoke some simple and easily visualized arguments to gain further understanding of the radiated power.

The paradox concerns the radiation of a charge undergoing uniform acceleration. When students first learn about radiation reaction, it often leads to confusion because the self-force vanishes, which appears to imply either that there is no radiated energy, or, if there is, then energy conservation has broken down (as we elaborate below). Indeed, this situation caused confusion in the professional physics community for a long time, even though the essential insight was already given by Schott in 1915.¹ Since 1960, and perhaps earlier, there has not been good reason to doubt that, when observed by an inertial observer, a uniformly accelerating charge radiates,^{2–5} but the right way to describe self-force and radiation reaction remained unclear until at least 1991.^{6–9} The work done by the self-force was described by Ford and O’Connell,^{10,11} who thus obtained the formula for the radiated power. It is slightly different from Larmor’s formula, owing to the fact that the radiating charge cannot be truly point-like. Here, we describe both this and the work done by the Schott term either differently or somewhat more fully than was done previously.^{12–15} Our treatment is informed by a discussion of Rohrlich, who performed the equivalent analysis based on a slightly different formula for the self-force.¹⁶ We find that the radiated energy per unit time taken to emit it is

$$\frac{2q^2}{3mc^3} \dot{v}_\lambda f^\lambda, \quad (1)$$

where \dot{v}_λ is the four-acceleration, f^λ is the applied four-force, q and m are the charge and mass, and Gaussian electromagnetic units are adopted (to obtain the result in SI units,

replace q^2 by $q^2/4\pi\epsilon_0$). This expression differs slightly from the formula given by Ford and O’Connell, but the difference is insignificant, as we discuss in Sec. II B; however, the difference from Larmor’s formula is significant. Equation (1) was previously noted in Ref. 12 and, at low velocities, in Ref. 14.

In the second half of the paper, we introduce a convenient method to understand the radiant energy in the field, which avoids the need to consider the details of the field either close to or far from the particle. We obtain the radiated power from the field energy and show that it agrees with the conclusions obtained from the self-force.

The two parts of the paper have in common that they concern energy movements in the electromagnetic field, and they offer ways to calculate the power in the electromagnetic radiation without the need to invoke the “wave zone.” For any given event at which a charge is accelerating, the wave zone is the exterior of a large spherical surface centered on the event’s position in a given frame, at a time in the distant future such that the surface lies on the future light cone of the event. This surface is usually invoked (either explicitly or implicitly) in derivations of Larmor’s formula for the emitted power,^{17,18} but it is useful to note that the formula can be obtained without appealing to that abstraction.

II. AN ENERGY PARADOX

We will consider a charged object whose spatial size is small compared to most other relevant distances in the problem, so we will refer to it as a “particle,” though it should not be assumed that this “particle” is truly point-like, only small.^{6,8,19} The precise shape of the object is not important, but if we take it to be roughly spherical then its radius R must exceed $\sim q^2/mc^2$, where q is its charge and m its observed mass, and we assume the motion under discussion has an acceleration small compared to c^2/R .

Consider two such particles. The first moves permanently at constant velocity v in the positive x -direction, whereas the second undergoes a period of acceleration. To be specific, the second particle moves initially with speed v in the opposite direction to the first, then it undergoes constant proper acceleration in the positive x -direction, until its velocity matches that of the first, after which it moves inertially (Fig.

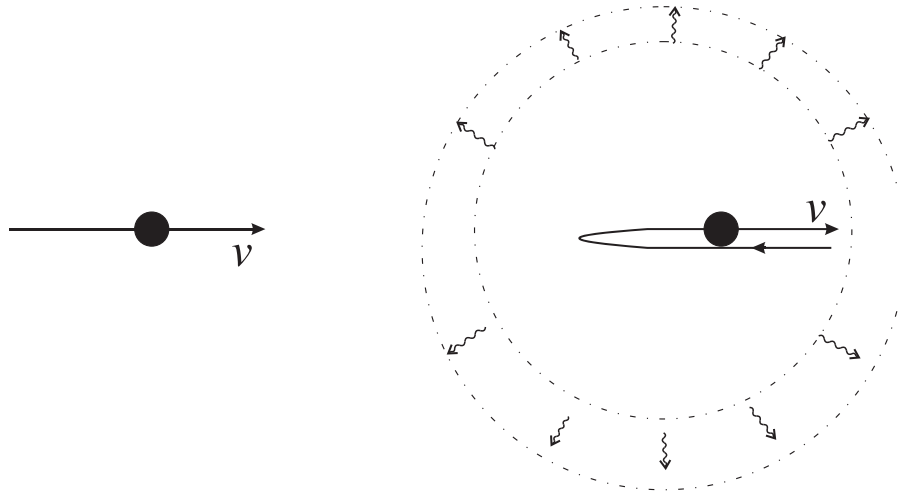


Fig. 1. A pair of small charged objects with the same final state of motion; one has accelerated, one has not.

1). These two motions represent two different scenarios that we will compare.

In these two scenarios, the initial and final kinetic energies of the two particles are the same. Also, the initial electromagnetic fields are the same in the two cases, up to a translation and a reversal of orientation in the magnetic part. The final electromagnetic field is not the same, because in the second case there is electromagnetic radiation, while in the first case there is not. However, in the second case, as the radiation propagates outward the field becomes identical to that in the first case throughout a larger and larger region of space, and the radiated pulse conserves its own energy as it propagates. Therefore, the total energy in the final electromagnetic field in the second case is greater than that in the first, by the energy W in the radiated pulse. It also follows that the net change in field energy between initial and final conditions in the second case is equal to W .

The question is, where has the energy W come from?

Consider the work done by the applied force \mathbf{f} . (To be clear, throughout this paper the symbol \mathbf{f} (and f) without label refers to the external force that is applied to the particle in question and is not caused by any field sourced by the particle). The exact relativistic equation of motion is

$$\mathbf{f} + \mathbf{f}_{\text{self}} = \frac{d}{dt}(\gamma m \mathbf{v}), \quad (2)$$

where \mathbf{f}_{self} is the self-force that is associated with momentum movements in the field sourced by the charge, such as radiation reaction, and m is the observed rest mass (which includes a contribution from electromagnetic energy and binding energy). Motion at constant proper acceleration has the special property that the self-force vanishes: $\mathbf{f}_{\text{self}} = 0$ (see Eq. (3) and Sec. IIB). But if \mathbf{f}_{self} vanishes in Eq. (2) then the equation is the same as that describing the motion of an uncharged particle, and, in particular, the total work done by the external force, in the motion under consideration, is precisely zero (since there is no overall change in the particle's kinetic energy). In other words, the work done by \mathbf{f} is just sufficient to provide the observed kinetic energy change of the particle—zero in total—and no more. Therefore, it would appear that the external force has not supplied the radiated energy W . So where has the radiated energy come from?

One can see that the radiated energy has not come from the bound field of the charge in question, because the final bound field eventually becomes identical to the initial bound field, apart from a translation and a sign change in the magnetic part. Once again, then, what physical system has supplied the energy that ends up in the radiation?

In the scenario under consideration, the energy is distributed over an extended system (the electromagnetic field), whereas energy conservation is enforced locally, so perhaps the problem is that we have added up the contributions in the wrong way, or over the wrong hyper-surface in spacetime? Or could it be something to do with the nonzero spatial extent of the object and a failure to construct its momentum in the right way?

Readers who are unfamiliar with this paradox are invited to come to their own conclusions before reading on.

A. Resolution of the paradox

The paradox is closely connected to the long-studied question of whether a uniformly accelerated charge radiates at all.²⁻⁴ Relative to an inertial observer it certainly does, but subtleties arise when one considers the observations of a uniformly accelerated observer.^{20,21} Here, we restrict attention to inertial observers, and then the resolution is simple. The above presentation of the paradox has neglected to consider the two brief periods when the motion does not have constant proper acceleration, at the beginning and end of the period of hyperbolic motion. Even though those periods are brief, it turns out that they contribute non-negligibly because during these times the external force provides all of the energy that is eventually radiated away, as we now show.

For the sake of simplicity, consider the case of low velocities (the non-relativistic limit), which retains all the important features of the paradox. In this limit, the spatial part of the self-force is [c.f. Eq. (9)]⁶⁻⁸

$$\mathbf{f}_{\text{self}} = \tau_q \dot{\mathbf{f}}, \quad (3)$$

where $\tau_q = 2q^2/3mc^3$, so the equation of motion is

$$\mathbf{f} = m\dot{\mathbf{v}} - \tau_q \dot{\mathbf{f}}. \quad (4)$$

If the initial and final speed is u then the acceleration during the hyperbolic motion is $a = 2u/T$, where T is its duration.

Let δt be the duration of the brief period when the applied force changes from zero to ma and assume that it also takes this same time δt for the force to change from ma to zero at the end. Then during the first such period we have $\mathbf{f} \simeq ma/\delta t$ and during the second we have $\mathbf{f} \simeq -ma/\delta t$ (we shall make a more precise statement in Sec. II B). The work done by the external force during each period is approximately $\mathbf{f} \cdot \mathbf{v} \delta t$. Using Eq. (4), this has a part $m\dot{\mathbf{v}} \cdot \mathbf{v} \delta t$, which goes into changing the kinetic energy of the particle, and a part

$$\pm \tau_q \frac{ma}{\delta t} \cdot \mathbf{v} \delta t, \quad (5)$$

which contributes energy to the electromagnetic field around the particle. In this equation, \mathbf{a} is a constant but \mathbf{v} is not, and in fact it has opposite sign in the two contributions, so that they are both equal to

$$\tau_q ma u = \tau_q ma^2 \frac{T}{2}, \quad (6)$$

because the initial and final speed is $u = aT/2$. In other words, in both the initial and the final periods of changing applied force, \mathbf{f} is in the opposite direction to \mathbf{v} , so the external force in Eq. (4) has to do some extra positive work, putting energy *into* the field, to the total amount

$$2 \times \tau_q ma^2 \frac{T}{2} = P_L T, \quad (7)$$

where

$$P_L = \frac{2q^2}{3c^3} \dot{v}_\lambda \dot{v}^\lambda = \frac{2q^2}{3c^3} a^2 \quad (8)$$

is Larmor's formula for the radiated power. We conclude that the external force does, in total, just the required amount of work to supply all the radiated energy, and therefore there is no energy-conservation paradox here. An exact treatment is given in the next section.

The surprise is that the external force provides all this energy in two brief periods at the start and end of the accelerated motion. Does this mean that the particle is not radiating in between these periods? Not at all. The particle radiates whenever it accelerates. The energy accounting during accelerated motion has to consider exchange of energy between the bound, or co-moving, field of the charged particle and the radiated field. During the motion at constant acceleration, energy is continuously moving from the former to the latter, as was first noted by Schott,¹ and as we show explicitly below. Although this basic picture has been available since Schott's work in 1915, it is surprising how often it has been unclear in the literature; see Rowland¹⁵ for further comments.

Figure 2 summarizes the argument by showing the results of an example exact calculation. We start from a given assumed $\mathbf{f}(t)$ and then obtain its derivative and hence $\dot{\mathbf{v}}$ and $\mathbf{v}(t)$. From this, it is easy to extract the work done.

Note that for this example, and more generally whenever the force does not change too abruptly, the acceleration as a function of time (dashed curve in the top graph in Fig. 2) is almost the same as the applied force per unit mass evaluated at a slightly later time (solid curve). This is *not* an example

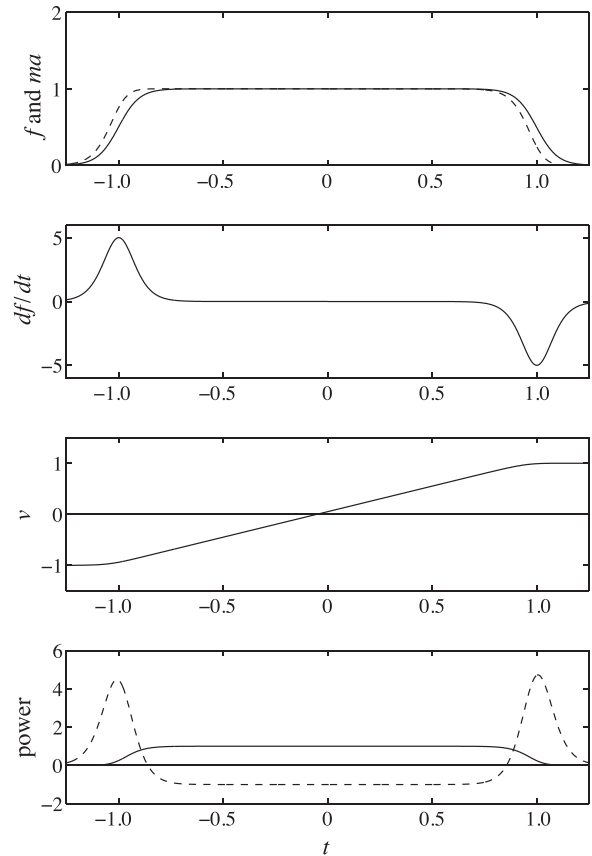


Fig. 2. A summary of the forces and powers involved in the example motion considered in the text, each plotted as a function of time. The top graph shows the applied force (solid line) and resulting ma (dashed line). The next graph shows df/dt ; the self-force is proportional to this. The third graph shows the particle speed $v(t)$. The last graph shows two contributions to the power delivered to the electromagnetic field: the radiated power (solid line) and the power delivered to the bound field (dashed line). When f is changing most of the power goes to the bound field. When f is constant but nonzero the two contributions are equal and opposite. The total area under the dashed curve is zero. (The essential insight here is not new, but the author has not been able to find a presentation like the above in prior work.)

of “pre-acceleration”; the equation of motion is strictly causal; the acceleration at any time is given by the total force $\mathbf{f} + \tau_q \dot{\mathbf{f}}$ evaluated at that same time, without regard to what may happen at later times.

B. Self-force is not just radiation reaction

The situation of constant applied force, which leads to zero self-force, is easy to misunderstand because of the common practice of calling the self-force by the name “radiation reaction.” This is a poor choice of terminology that has misled the physics community for a century. As Rohrlich rightly emphasizes,^{5,16} it is a misnomer because in fact the self-force has three parts. First there is an “inertial” part describing the supply of 4-momentum to the bound field, which has been absorbed into the definition of the mass of the particle in our discussion. Next there are the two terms in the following expression for the self-4-force:

$$f_{\text{self}}^\mu = \tau_q [f^{\dot{\mu}} - (\dot{v}_\lambda f^\lambda) v^\mu / c^2], \quad (9)$$

where the dot signifies $d/d\tau$ (differentiation with respect to proper time along the worldline). Readers unfamiliar with

this form of the self-force equation (because, perhaps, they learned the Lorentz-Abraham-Dirac approach) are referred to Refs. 7, 8, 22, and 23. It is the equation first proposed by Eliezer and obtained by Ford and O’Connell; it is closely related to but slightly different from the equation proposed by Landau and Lifshitz.^{8,24} The first term in Eq. (9) is the Schott term, which accounts for redistribution of 4-momentum within the bound field. The second term describes the supply of energy and momentum to the radiated field. It would be logical to reserve the phrase “radiation reaction” for the second term alone, but it is commonly applied to both. In this paper, we will use the unambiguous phrase “self-force” when discussing both terms together. The radiated field always transports energy away from the source, but the bound field may act either to accelerate or decelerate the source, depending on the recent history of the motion. In physical terms, to push a charged particle is to push something that is permanently attached to a “springy” medium.

Equation (9) comes from a treatment that is relativistically consistent but not guaranteed to be exact, because the self-force in general depends on the shape and internal motion of the accelerating body. However, for a body of given total charge and not exhibiting extreme behaviors such as internal resonance, the corrections to the equation are of higher order in powers of τ_q , so for small entities such as electrons, Eq. (9) is very accurate. The non-relativistic form (3) follows by substituting dt for $d\tau$ and neglecting the second term in comparison with the first, which does *not* amount to neglecting the radiation. This subtle point is discussed in Ref. 16.

One way to prove that the self-4-force vanishes for motion at constant proper acceleration (hyperbolic motion) is to search for the motion in which the self-4-force vanishes. When $f_{\text{self}}^\mu = 0$, the equation of motion reads $f^\mu = m\dot{v}^\mu$ (assuming constant rest mass m) so the bracket on the right-hand side of Eq. (9) is $(m\ddot{v}^\mu - ma_0^2 v^\mu/c^2)$, where a_0 is the proper acceleration, obtained from $a_0^2 = \dot{v}_\lambda \dot{v}^\lambda$ using a metric signature $(-1, 1, 1, 1)$. But the condition $c^2 \ddot{v}^\mu = a_0^2 v^\mu$ implies hyperbolic motion,¹⁸ so we find $f_{\text{self}}^\mu = 0$ if and only if the motion is hyperbolic, and we also see that this result arises by virtue of equal and opposite contributions from two effects. In the case of hyperbolic motion, during the initial short period during which f increases from zero to some non-zero value, the external force does more work than is needed to supply the energy eventually required by the bound field. In the subsequent hyperbolic motion, according to Eq. (4), the applied force does less total work than is needed to supply both the radiated energy and the kinetic energy of the particle; this is because during such motion the bound field near the particle also does work on the particle. While the particle slows, the system providing the force has work done on it by the particle, and the bound field holds the particle back a little, tending to maintain its kinetic energy. As the speed passes through zero this process continues, but now the system providing the force does work on the particle, and the bound field “helps” by pulling the particle along a little, doing work on it. At this stage, an energy deficit is building up: the bound field has less energy than it will eventually require. This deficit is filled by the applied force during the second period when it changes. In the second such short period, \mathbf{f} is again opposed to \mathbf{v} so again the external force does more work than is needed to supply either kinetic energy or radiated energy; the energy passes to the bound field and stays there.

We now provide a quantitative statement of the above ideas by calculating the rate of doing work by the applied force, in both the general (any v) and low-velocity ($v \ll c$) cases. Our discussion of the various contributions matches that of Rohrlich,¹⁶ except that we use a different expression for the self-force. Previously several authors have treated the radiation power and the Schott power implied by Eq. (9); our discussion slightly extends or modifies the prior ones.^{10,12–14}

Assuming that the rest mass is constant, the relativistic equation of motion is

$$f^\mu + \tau_q [\dot{f}^\mu - (\dot{v}_\lambda f^\lambda) v^\mu / c^2] = m\dot{v}^\mu. \quad (10)$$

The rate of doing work is given by the zeroth component of this 4-force:

$$\frac{dW}{dt} = f^0 c = m\dot{v}^0 - \tau_q c \dot{f}^0 + \tau_q \dot{v}_\lambda f^\lambda \gamma, \quad (11)$$

where γ is the Lorentz factor. The three terms on the right-hand side are the rate of change of kinetic energy, the Schott power, and the radiated power. The Schott power takes the form of a total derivative, so the net work done by the Schott term between any two events where the 4-force has no net change is zero. The radiation term gives, for the radiated power per unit time taken to emit it,

$$P_R = \frac{dW_R}{dt} = \frac{2q^2}{3mc^3} \dot{v}_\lambda f^\lambda, \quad (12)$$

where we used $dt/d\tau = \gamma$, and we note that the resulting expression is Lorentz-invariant. By evaluating this expression in the instantaneous rest frame, one finds that $P_R = \tau_q \mathbf{a}_0 \cdot \mathbf{f}_0$, where \mathbf{a}_0 is the proper acceleration and \mathbf{f}_0 is the applied force in the instantaneous rest frame (in the absence of self-force the latter would be equal to \mathbf{a}_0/m , but here that is not the case). Equation (12) is not quite the same as Larmor’s expression (8), because the latter does not take the nonzero size of the accelerating body into account. We will elaborate on this point below and in Sec. III.

In the low-velocity limit, Eq. (4) gives the rate of doing work:

$$\mathbf{f} \cdot \mathbf{v} = m\dot{\mathbf{v}} \cdot \mathbf{v} - \tau_q \dot{\mathbf{f}} \cdot \mathbf{v}. \quad (13)$$

The first term on the right-hand side is the rate of change of kinetic energy of the particle. To clarify the physical interpretation of the second term, use

$$\dot{\mathbf{f}} \cdot \mathbf{v} = \frac{d}{dt} (\mathbf{f} \cdot \mathbf{v}) - \mathbf{f} \cdot \dot{\mathbf{v}}, \quad (14)$$

so we have

$$\mathbf{f} \cdot \mathbf{v} = \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) - \tau_q \frac{d}{dt} (\mathbf{f} \cdot \mathbf{v}) + P, \quad (15)$$

where

$$P = \frac{2q^2}{3c^3} \frac{\dot{\mathbf{v}} \cdot \mathbf{f}}{m}. \quad (16)$$

The three contributions to the rate of doing work in Eq. (15) correspond to the three appearing in the more general

expression (11). The radiated power agrees exactly with the expression (12) when P is evaluated in the instantaneous rest frame.

Ford and O’Connell¹⁰ also considered this question based on the same starting point (4), but they arrived at a different result for the radiated power,

$$P_{\text{FO}} = \frac{2q^2}{3c^3} \left(\frac{f}{m} \right)^2, \quad (17)$$

and a more detailed subsequent treatment came to the same conclusion.¹³ In order to understand this discrepancy, express $\dot{\mathbf{v}}$ in terms of the force using Eq. (4): $m\dot{\mathbf{v}} \cdot \mathbf{f} = \mathbf{f} \cdot \mathbf{f} + \tau_q \dot{\mathbf{f}} \cdot \mathbf{f}$; substituting into Eq. (16) then gives

$$P = P_{\text{FO}} + \frac{\tau_q^2}{2m} \frac{d}{dt} (f^2). \quad (18)$$

Hence, the two expressions differ only when the size of the applied force is changing, and, furthermore, they predict the same total radiated power between any two events at which the applied force has the same size. It follows from this that the choice between P and P_{FO} is largely a matter of convention, concerning how to apportion the energy between the Schott field and the radiation while f is changing. We are here making one choice, in agreement with two previous authors,^{12,14} while Heras and O’Connell made the other.¹³ Also, even when f is changing, the two expressions differ only at the next higher order in τ_q , where our original expression (9) is not guaranteed to be accurate, so one should not over-interpret this small difference. (This point was also noted by Rohrlich.¹²) Rowland¹⁵ offers essentially the same physical interpretation of the energy movements, but his detailed calculation is limited to the case of constant proper acceleration, where P_R , P_{FO} , and P_L all agree. Therefore, his result (Eq. (15) of Ref. 15) should not be assumed to apply to general motion, and in fact it is imprecise in general because it adopts the Larmor radiation formula.

For the case of a constant force ($\dot{\mathbf{f}} = 0$), the Schott power evaluates to $-\tau_q \mathbf{f} \cdot \dot{\mathbf{v}} = -\tau_q f^2/m$, and then Eq. (15) gives

$$\mathbf{f} \cdot \mathbf{v} = \frac{d}{dt} \left(\frac{1}{2} m v^2 \right) + \frac{2q^2}{3c^3} \left(\frac{f^2}{m^2} - \frac{f^2}{m^2} \right). \quad (19)$$

Here, we explicitly exhibit both the radiated power and the power leaving the bound field for this case. This helps one to see clearly that in the presence of an applied force, the radiation is happening throughout the motion, not just when the force is changing.

The overall conclusion is that energy conservation is maintained, and the external force does indeed supply the energy required by both the bound field and the radiated field. The inertial contribution to the energy of the bound field has been absorbed into the definition of m , and we have exhibited the other part (the Schott term) explicitly.

III. FINDING RADIATED ENERGY WITHOUT RECOURSE TO THE WAVE ZONE

We now turn to the direct calculation of radiated energy by examining the field around a particle that has accelerated.

The standard derivations of Larmor’s formula (8) for the power radiated by an accelerating point charge require an

assumption that only the part of the field associated with acceleration leads to radiation, and that one may legitimately calculate the energy associated with this part of the field alone, and call it radiated energy. One way to justify this assumption is to take the limit $r \rightarrow \infty$, where r is the distance from the source event to the field event. For a given source event, the field events in such a calculation are located on an infinitely large spherical surface in the infinite future—the “wave zone” or “radiation zone.” Sometimes, the consideration of this limit is problematic. In the radiation zone, the radiated field carries almost all the energy, when integrated over all directions, but in some directions, it vanishes completely where the bound field does not, and even where it is strong it does not dominate in all respects. For example, its divergence is everywhere equal and opposite to that of the bound field. In any case, it is interesting to ask whether one can avoid an appeal to the radiation zone. It should, after all, be possible to learn about something happening in the here and now without recourse to the far distance and the infinite future.

In a classic paper, Teitelboim²⁵ addressed this issue, among others, and gave much insight into the energy and momentum movements in the fields sourced by a charged particle undergoing arbitrary motion. Subsequent work has further elucidated particular cases or has extended the ideas, for example, to non-flat spacetimes. In the present discussion, we wish to give an argument that, owing to its visual nature and great simplicity, might be useful as a teaching aid. The aim of the argument is to get some general insight into the movement of energy in an electromagnetic field and to derive Larmor’s formula without invoking the wave zone.

A. Point particle

First, we consider a point particle. Strictly, a point charge is impossible for finite q and m , but it is a convenient case to consider because the fields are simple to calculate.

Consider a charged particle that moves initially at some constant velocity and finally at some constant velocity (not necessarily the same), relative to a given inertial frame. For any such motion, there exists an inertial frame relative to which the initial and final velocities are equal and opposite. Adopt this frame, oriented so that the initial and final velocities are $\mp \mathbf{u}$ in the x -direction and suppose that the part of the worldline for which the motion is arbitrary (but always timelike) extends between events (t_1, \mathbf{x}_1) and (t_2, \mathbf{x}_2) . At any time $t > t_2$, divide all of space into three regions. Region 1 is the exterior of a sphere of radius $c(t - t_1)$ centered at \mathbf{x}_1 . Region 2 is the interior of a sphere of radius $c(t - t_2)$ centered at \mathbf{x}_2 . Since the worldline is timelike, these regions do not overlap. Define region 3 as the region between them. These regions are shown, for a particular case, in Fig. 3(a). By reasoning about this figure, we will make an important observation about the energy movements in the field.

In region 1, the electromagnetic field is that of a charge uniformly moving at the initial velocity; in region 2 the field is that of a charge uniformly moving at the final velocity; in region 3 the field is more complicated, having both radiative and bound parts. The electric field in region 2 extends radially outwards from the present position of the particle. The electric field in region 1 extends radially outwards from the *projected position* given by

$$\mathbf{x}_p = \mathbf{x}_1 - \mathbf{u}(t - t_1). \quad (20)$$

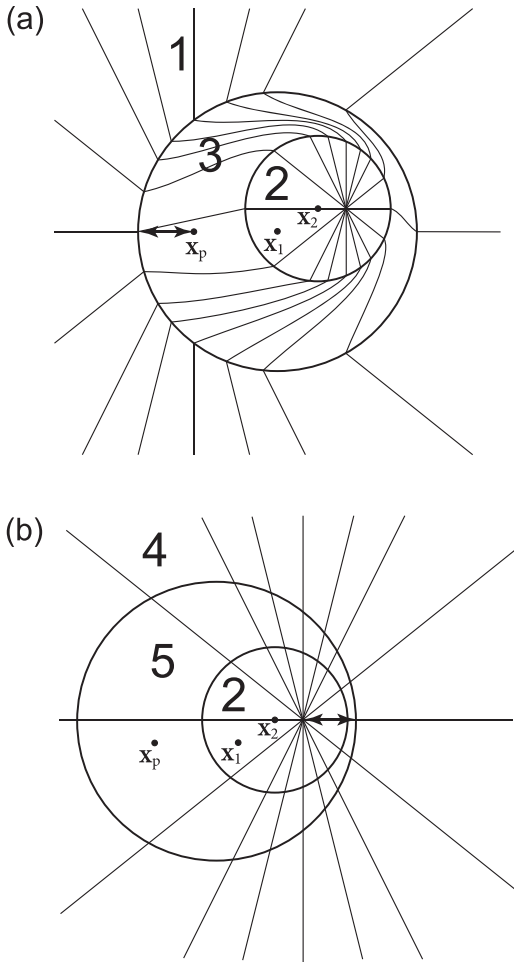


Fig. 3. The top diagram shows the electric field around a particle that has recently accelerated and now undergoes uniform motion to the right at a velocity equal and opposite to the velocity it had before the acceleration began at t_1 . The bottom diagram shows the electric field of a particle that has the same final state of motion but has not accelerated. The circles indicate various regions defined in the text. The bold double-ended arrow has the same length and direction in the two diagrams. This ensures that the electromagnetic field in region 1 is everywhere the same as that in region 4, apart from a translation and reflection. It follows that the energy in region 3 exceeds that in region 5 by an amount that is independent of the moment in time chosen to draw the diagram (see text).

This is the position the particle would now have (at time t) had it continued permanently in its initial state of motion with velocity $-\mathbf{u}$. Since $u < c$, the projected position is located inside the sphere enclosed by region 1.

So far we have simply taken a general look at the form of the electromagnetic field. The only assumption has been that the particle moves initially and finally at constant velocity, and for convenience we have adopted the reference frame in which those velocities are equal and opposite.

Let U_i be the energy contained in the electromagnetic field in region i . By conservation of energy, we have for all times $t > t_2$,

$$U_1 + U_2 + U_3 = \text{constant}. \quad (21)$$

There is a divergent contribution to U_2 coming from the fields close to the particle, but we will show how to regularize this.

Next, consider the case of a particle that has never accelerated but has always moved in the final state of motion of the particle just described. In other words, this “reference”

particle has constant velocity \mathbf{u} and after t_2 it is located at the same place as the original particle; its electric field is illustrated in Fig. 3(b). Let U_i^H be the energy in the electromagnetic field of the reference particle in region i (the H here stands for “Heaviside”; we shall call the field of a uniformly moving point charge a “Heaviside field” in honor of O. Heaviside, who was the first to calculate it.²⁶ This field is, roughly speaking, a “moving Coulomb field,” but we have in mind an exact treatment including Lorentz contraction). Since the fields of the actual particle and the reference particle are identical in region 2, clearly $U_2^H = U_2$, but we cannot make any such simple statement about U_1 or U_3 .

The crucial part of the argument is to identify yet another region of interest. This region, region 4, is the exterior of a sphere of radius $c(t - t_1)$ centered at

$$\mathbf{x}_t - \mathbf{u}(t - t_1) = \mathbf{x}_2 - \mathbf{u}(t_2 - t_1), \quad (22)$$

where $\mathbf{x}_t = \mathbf{x}_2 + \mathbf{u}(t - t_2)$ is the present position of the reference particle (this is also the location of the original particle after t_2). This means the center of the spherical surface defining region 4 is displaced from the present location of the reference particle by the same amount (but in the opposite direction) that the center of the spherical surface defining region 1 is displaced from the projected position of the actual particle. There are two useful implications. First, because $u < c$, region 4 does not overlap region 2. Second, the electromagnetic field of the reference particle in region 4 is the same as the electromagnetic field of the actual particle in region 1, except for a displacement and a reflection in a plane normal to the x -axis. An example of this correspondence can be seen by examining the pattern of the field lines in Fig. 3. Such a displacement and reflection does not change the energy content of the field, and therefore

$$U_1 = U_4^H. \quad (23)$$

Finally, define region 5 as that between region 2 and region 4. (Note that regions 3 and 5 share a common inner boundary but have different outer boundaries.) Since the reference particle is in a state of steady motion in which no energy is being supplied, we must have

$$U_4^H + U_2^H + U_5^H = \text{constant}. \quad (24)$$

Subtracting this from Eq. (21) gives

$$U_1 + U_2 + U_3 - (U_4^H + U_2^H + U_5^H) = \text{constant}. \quad (25)$$

Now, using $U_2^H = U_2$ and Eq. (23), we obtain

$$U_3 - U_5^H = \text{constant}. \quad (26)$$

This simple result tells us something interesting about the electromagnetic field sourced by a particle undergoing arbitrary motion: the energy content of that field differs from what it would need to be to construct the Heaviside field of a particle in the same final state of motion by an amount that does not change with time. As time goes on, all the light spheres we have identified grow, and the energy contents of all the regions change. But regions 3 and 5 have the interesting property we have identified, which is

$$U_3(t) = U_5^H(t) + U_R, \quad (27)$$

for some U_R that is independent of time. Since as time goes on, the field around the actual particle becomes more and more like a Heaviside field, we can identify U_R as an energy that has become detached from the particle. It is the radiated energy.

This argument allows us to make the standard observations about the source of electromagnetic radiation (for inertial observers in the absence of gravity), namely that accelerated motion always results in radiated energy, non-accelerated motion never does, and the radiated energy moves outward from the source at the speed of light. We can also obtain Larmor's formula, as follows.

The fields of a particle in an arbitrary state of motion are, in Gaussian units,

$$\mathbf{E} = \frac{qr^3}{(\mathbf{r} \cdot \mathbf{r}_0)^3 c^2} \left[(c^2 - v^2)\mathbf{r}_0 + \mathbf{r} \times (\mathbf{r}_0 \times \mathbf{a}) \right], \quad (28)$$

$$\mathbf{B} = \mathbf{n} \times \mathbf{E}, \quad (29)$$

where \mathbf{r} is the vector from the source event to the field event, $\mathbf{r}_0 = \mathbf{r} - \mathbf{v}r/c$, $\mathbf{n} = \mathbf{r}/r$, and \mathbf{v} and \mathbf{a} are the particle's velocity and acceleration at the source event. For a given source event, in the instantaneous rest frame these expressions simplify to

$$\mathbf{E} = q \left[\frac{\mathbf{n}}{r^2} + \frac{\mathbf{n} \times (\mathbf{n} \times \mathbf{a})}{c^2 r} \right], \quad \mathbf{B} = \frac{-q}{c^2 r} \mathbf{n} \times \mathbf{a}. \quad (30)$$

The energy density in the field is

$$u = \frac{1}{8\pi} (E^2 + B^2) = \frac{q^2}{8\pi} \left(\frac{1}{r^4} + \frac{2a^2 \sin^2 \theta}{c^4 r^2} \right), \quad (31)$$

where θ is the angle between \mathbf{n} and \mathbf{a} (in SI units one would have $q^2/4\pi\epsilon_0$ instead of q^2 in the last expression).

Now consider the argument of Fig. 3 applied to a particle that accelerates only during a short interval δt . In this case, region 3 is, to lowest-order approximation, a spherical shell of thickness $\delta r = c\delta t$. Integrating Eq. (31), we obtain for the total field energy in such a shell,

$$\delta U = \int_0^\pi \int_0^{2\pi} \int_0^{\delta r} u r^2 \sin \theta d\theta d\phi \delta r = q^2 \left(\frac{1}{2r^2} + \frac{2a^2}{3c^4} \right) \delta r. \quad (32)$$

This δU is an example of the energy we have called U_3 in the argument above. To get the radiated energy, we subtract from it U_5^H , the energy in the Heaviside field in the appropriate shell. This energy is equal to the first term in Eq. (32); hence we find

$$U_R = \frac{2}{3} \frac{q^2 a^2}{c^4} \delta r. \quad (33)$$

The time taken to emit this energy is the time taken for a light sphere to grow from radius r to $r + \delta r$, so we find that the energy radiated, per unit time taken to emit it, is as given by Larmor's formula, Eq. (8). This derivation assumed that the acceleration took place only during a short burst. However, the result can be extended immediately to arbitrary motion, by arguing from causality: the fields at any given event depend only on the part of the worldline that acted as

source, so Eq. (33) gives the radiated energy for the part of the worldline we considered, no matter what the motion was at other times.

B. Extension to objects of nonzero size

The preceding treatment, like Larmor's, gives the answer for a point charge. No point-like object can have a nonzero charge, however, unless the observed mass tends to infinity, owing to the contribution from the electromagnetic field energy.^{8,19} Therefore, Larmor's formula and Eq. (33) are valid only in the limit $q \rightarrow 0$. In that limit $\tau_q \rightarrow 0$ and then Eq. (8) agrees with Eq. (16). For an entity of nonzero charge and finite mass, and therefore nonzero spatial extent, we should expect a departure from Larmor's formula, and Eq. (16) or (17) gives, to first approximation, what that departure is. It can be understood as a small modification in the energy in the radiation field, owing to the difference between the field of a small extended object and the field of a point-like object. For a small rigid body moving non-relativistically, this departure can be obtained from Eq. (10) of Ref. 11, which gives the squared electric field in the radiation zone as

$$|\mathbf{E}(\mathbf{r}, \omega)|^2 = \frac{q^2 |f_q(\mathbf{k})|^2 \omega^4}{c^4 r^2} (\alpha \tilde{\mathbf{f}})^2 \sin^2 \theta, \quad (34)$$

where $\tilde{\mathbf{f}}$ is the Fourier transform of the applied force, α is the linear response function, and f_q is the form factor of the rigid charge distribution. A suitable expression for the latter is^{6,11}

$$|f_q(\mathbf{k})|^2 = \frac{1}{1 + \omega^2 \tau_q^2}. \quad (35)$$

It follows that the radiated power is proportional to

$$\omega^4 |f_q|^2 |\alpha|^2 |\tilde{\mathbf{f}}|^2 = \frac{|\tilde{\mathbf{f}}|^2}{m^2}, \quad (36)$$

in agreement with Eq. (17), where we used the expression $\alpha = (-1 + i\omega\tau_q)/m\omega^2$ that describes the response of a free particle according to Eq. (4). The above calculation was presented at greater length in Ref. 27. This result does not necessarily offer a reason to prefer Eq. (17) over Eq. (16), because the difference between them appears at a higher order in powers of τ_q than is assumed in the approximations leading to Eq. (34).

We note that Eq. (17) can also be reproduced to this order of accuracy by replacing $\mathbf{a} \rightarrow \mathbf{a} - \tau_q \dot{\mathbf{a}}$ in the Larmor formula, but this is a mere observation, not a derivation.

The argument of Fig. 3 and Eqs. (21)–(27) remain valid for the case of an extended charged body, if we adapt it as follows. We consider the case where the body moves in such a way that its initial and final motions are inertial, as before, and we adopt the frame in which the initial and final velocities are equal and opposite. We assume that the body has the same proper size and shape in the initial and final states, and that it produces no magnetic field in its rest frame when it is not accelerating. By transforming from the rest frame, one finds the electric field (in some given frame, not the rest frame) of such a body is the same for two inertial states of motion that differ only in the sign of the velocity, and the magnetic field is equal and opposite. This allows the argument to proceed essentially as before.

The spherical surfaces are located such that regions 1 and 2 only contain fields owing to inertial motion, so all the complications owing to acceleration are contained in region 3. This can be arranged by choosing these regions in a variety of ways, but for clarity, a specific choice may be helpful. This could be, for example, spheres centered on \mathbf{x}_1 and \mathbf{x}_2 as before, where now these refer to the locations, at t_1 and t_2 , of a convenient point on the extended body, such as its centroid. The spheres lie on the future light cones of events 1 and 2; in other words, their radii are $c(t - t_1)$ and $c(t - t_2)$, respectively, at time t . One must choose t_1 early enough, and t_2 late enough, to ensure that regions 1 and 2 only contain fields owing to inertial motion. Equation (21) applies as before.

The primary aim of the argument is to show that there is a contribution to the total field energy that is independent of time and that propagates outwards from the charged body at the speed of light. A secondary aim is to give one way to locate and calculate this energy. The primary aim is achieved immediately, because regions 1 and 2 contain two parts of the field that would be produced by a similar body in permanent inertial motion (the “reference field”), apart from translations and a sign change in the magnetic part. Hence, the extra energy in the actual field, compared to the energy in the reference field, is all in region 3, and it is constant because the total energies of both fields are constant.

In order to make this argument as general as possible, in particular in order not to require reflection symmetry in the field, one must locate region 4 differently from before. This region is defined such that, at any time t , the reference field in region 4 matches the actual field in region 1, up to translation and sign changes. In order to guarantee this, region 4 is centered at $\mathbf{x}_r + \mathbf{u}(t - t_1)$; this replaces Eq. (22). This means the boundary of region 4 may move at speeds above c , and regions 2 and 4 may overlap, but this does not affect the argument. The only important point is that the difference between the total field energy and the energy of the reference field is wholly owing to an excess energy located in region 3, and this excess energy is independent of time. Since the boundaries of region 3 are moving outwards at the speed of light, so is this energy.

IV. CONCLUSION

This paper has offered contributions of two types: accurate statements about radiant energy and self-force and easily visualized or remembered ways of thinking about them. The statements correct or clarify earlier work (by a modest amount). Equation (1) is more accurate than Larmor’s equation for emitted power. The physical scenarios offer, we hope, a useful teaching method, whose ideas are captured in the three figures.

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