Electromagnetic waves in plasmas and conductors

Andrew M. Steane
Exeter College and Department of Atomic and Laser Physics, University of Oxford.

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Abstract

This note describes electromagnetic waves in plasmas and conductors, at second year undergraduate level. The treatment of waves in a dielectric medium also follows (it is simpler).

Contents

1 Preliminaries

Our starting point is Maxwell's equations:[1]

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \text{M1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{M2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{M3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{M4}$$

where ρ is the total charge density at any given point, $\mathbf{j}$ is the total current density and $\mathbf{E}, \mathbf{B}$ are the electric and magnetic fields. In classical electromagnetism (which is the setting of the whole our discussion) these equations always hold everywhere.

In order to complete the theory of classical electromagnetism, we also need the Lorentz force equation, telling how the fields influence the motion of charged particles:

$$\mathbf{f} = q(\mathbf{E} = \mathbf{v} \times \mathbf{B}). \tag{1}$$

Current and current density. The quantity ρ is a charge per unit **volume**; the quantity $\mathbf{j}$ is a current per unit **area**. A current density is not quite the same as a current (it has different physical dimensions for example) but obviously they are closely related. It is common practice in field theory to use the short name 'current' when referring to $\mathbf{j}$. We shall adopt this practice. The physicist reader is expected to understand that this is a short-hand and $\mathbf{j}$ is a current per unit area in fact.

[1] We adopt SI units throughout.

There is a related set of equations which we will not need, but we shall mention them briefly at the end:

$$\nabla \cdot \mathbf{D} = \rho_{\rm f} \qquad \text{M1a}$$
$$\nabla \cdot \bar{\mathbf{B}} = 0 \qquad \text{M2a}$$
$$\nabla \times \bar{\mathbf{E}} = -\frac{\partial \bar{\mathbf{B}}}{\partial t} \qquad \text{M3a}$$
$$\nabla \times \mathbf{H} = \mathbf{j}_{\rm c} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{M4a}$$

where

$$\mathbf{D} = \epsilon_0 \bar{\mathbf{E}} + \bar{\mathbf{P}} \tag{2}$$
$$\mathbf{H} = \frac{\bar{\mathbf{B}}}{\mu_0} - \bar{\mathbf{M}}. \tag{3}$$

In these equations, $\mathbf{P}$ is polarization (defined as electric dipole moment per unit volume) and $\mathbf{M}$ is magnetization (defined as magnetic dipole moment per unit volume), $\rho_{\rm f}$ is free charge density and $\mathbf{j}_{\rm c}$ is conduction current density, and the bar indicates a spatial average over a region large compared to the atomic structure of the local material medium such as a solid or a gas, but small compared to the wavelength of any electromagnetic waves under discussion. It practice one often takes this spatial average for granted and then the bar is not used; we shall adopt this practice.

The derivation of (M1a)–(M4a) from (M1)–(M4) is not included here. The essence of the derivation is to work out how $\nabla \cdot \mathbf{P}$ relates to the charge, and how $\nabla \times \mathbf{M}$ relates to the current. We have

$$\rho = \rho_{\rm f} + \rho_{\rm b}, \qquad \mathbf{j} = \mathbf{j}_{\rm c} + \mathbf{j}_{\rm b} \tag{4}$$

where $\rho_{\rm b}$ and $\mathbf{j}_{\rm b}$ (the 'bound' charge and current) are those parts of the charge and current that are associated with dipoles as opposed to charges that can move over longer distances.

The relationship between electric field and polarization is sometimes expressed in terms of a dimensionless quantity called *susceptibility*, and similarly for the relationship between magnetization of magnetic field. The standard definitions are (in SI units)

$$\mathbf{P} = \epsilon_0 \chi_{\rm e} \mathbf{E}, \qquad \mathbf{M} = \chi_{\rm m} \mathbf{H}. \tag{5}$$

where $\chi_{\rm e}$ is the electric susceptibility and $\chi_{\rm m}$ is the magnetic susceptibility. Note: these susceptibilities are introduced merely for convenience in gaining physical insight and simplifying some mathematical expressions. They do not introduce any physical concept or quantity that is not already expressible in terms of $\mathbf{E}, \mathbf{B}, \mathbf{P}, \mathbf{M}$.

A final piece of notation is the notion of *relative permittivity* ϵ_r and *relative permeability* μ_r. These are not always applicable, but they are applicable when the polarization and magnetization are in the same direction as the relevant fields, such that $\mathbf{D}$ is in the same direction as $\mathbf{E}$ and $\mathbf{H}$ is in the same direction as $\mathbf{B}$ (note, we are dropping the bars now). In this situation we define ϵ_r and μ_r through

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}, \qquad \mathbf{B} = \mu_0 \mu_r \mathbf{H}. \tag{6}$$

One then finds

$$\epsilon_r = 1 + \chi_{\mathrm{e}}, \qquad \mu_r = 1 + \chi_{\mathrm{m}}. \tag{7}$$

One may also invoke tensor versions of these quantities in order to treat anisotropic media.

1.1 Mathematical results

The following are useful mathematical results which the reader is encouraged to first prove (e.g. by writing out components) and then commit to memory: If

$$\mathbf{F} = \mathbf{F}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{8}$$

where $\mathbf{F}$ is any field, and $\mathbf{F}_0$, $\mathbf{k}$ and ω are all constants (with no dependence on either position or time), then

$$\frac{\partial \mathbf{F}}{\partial t} = -i\omega\mathbf{F} \tag{9}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F} = i\mathbf{k}\cdot\mathbf{F} \tag{10}$$

$$\boldsymbol{\nabla}\times\mathbf{F} = i\mathbf{k}\times\mathbf{F} \tag{11}$$

$$\nabla^2\mathbf{F} = -k^2\mathbf{F}. \tag{12}$$

We will also use that, for any vector field $\mathbf{A}$,

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{13}$$

(this can be memorized by committing to memory the phrase 'curl curl equals grad div minus del-squared').

The reader should note that whereas $\boldsymbol{\nabla}\cdot\mathbf{A}$ yields a scalar field, the quantity $\nabla^2\mathbf{A}$ is a vector field whose components are

$$\begin{pmatrix} \nabla^2 A_x \\ \nabla^2 A_y \\ \nabla^2 A_z \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \\ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \\ \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \end{pmatrix}. \tag{14}$$

Written out in full like this, it seems a bit of a mouthful, but in practice we often don't need to resort to component form.

2 Wave equation

Let's take the curl of (M4), and immediately use (13) on the left hand side:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \nabla^2\mathbf{B} = \mu_0\boldsymbol{\nabla}\times\mathbf{j} + \epsilon_0\mu_0\boldsymbol{\nabla}\times\frac{\partial\mathbf{E}}{\partial t}. \tag{15}$$

The div $\mathbf{B}$ term is zero by (M2), and we can reverse the order of partial differentiation in the final term, giving $(\partial/\partial t)(\boldsymbol{\nabla}\times\mathbf{E})$ which is $-\partial^2\mathbf{B}/\partial t^2$ using (M3). Hence we have

$$\boxed{\epsilon_0\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} = \mu_0\boldsymbol{\nabla}\times\mathbf{j}.} \tag{16}$$

We will comment on this in a moment. Before doing so, let's take a similar approach to (M3). We take the curl, and reverse the order of partial differentiation on the right, yielding

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\mathbf{B} \tag{17}$$

so, by bringing in (M1) and (M4):

$$\boxed{\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla^2\mathbf{E} = -\boldsymbol{\nabla}(\rho/\epsilon_0) - \mu_0\frac{\partial\mathbf{j}}{\partial t}.} \tag{18}$$

Equations (16) and (18) are important for several reasons:

1. They always apply: we have made no special assumptions about the conditions (amounts of charge and current etc.)
2. They have the form of *wave equations* in which $\mathbf{B}$ or $\mathbf{E}$ is the field under discussion, and the right hand side serves as source.
3. In the special case of a current that is either uniform or zero, (16) is the ordinary (i.e. source-free) wave equation for $\mathbf{B}$.
4. In the special case of a charge that is uniform or zero and a current that is constant in time (or zero), (18) is the ordinary (i.e. source-free) wave equation for $\mathbf{E}$.

The source-free wave equation has plane wave solutions. These are solutions of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{19}$$

$$\mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{20}$$

where $\mathbf{E}_0, \mathbf{B}_0, \mathbf{k}$ and ω are constants. $\mathbf{E}_0$ is the amplitude of the wave of electric field, vk is its wave-vector and ω is its angular frequency. $\mathbf{B}_0$ is the amplitude of the wave of magnetic field, vk is its wave-vector and ω is its angular frequency. By substituting the solutions into the wave equation, one finds that the phase velocity (ω/k) is given by

$$\frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0\mu_0}}. \tag{21}$$

This is the speed of light in free space, c.

In the rest of this note we discuss two situations where there are sources (currents and/or charges) related to the waves in such a way that plane-wave solutions can still be found.

3 The ohmic conductor

A conductor such as a metal is said to be *ohmic* when it gives rise to behaviour satisfying Ohm's law: $V = IR$. If we apply this law to a small region of such a material we shall find that the current density is proportional to the electric field:

$$\mathbf{j} = \sigma \mathbf{E} \tag{22}$$

The proportionality constant σ is called the *conductivity*. (Proof: consider a small cylinder of length L and cross-section A, then when an electric field $\mathbf{E}$ is in the directed along the cylinder, the resistance is $L/(\sigma A)$ and the voltage is $V = EL$, so Ohm's law gives $I = (EL)(\sigma A/L) = E\sigma A$ so the current density is $j = I/A = \sigma E$.)

We shall now consider electromagnetic waves in the case where

$$\mathbf{j} = \sigma \mathbf{E}, \qquad \boldsymbol{\nabla}\rho = 0. \tag{23}$$

In this case (18) gives

$$\epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t}. \tag{24}$$

We now propose a trial solution in the form of a plane wave (19) and we make free use of the mathematical results (9)–(12), thus obtaining

$$-\epsilon_0\mu_0\omega^2 \mathbf{E} + k^2 \mathbf{E} = i\omega\mu_0\sigma\mathbf{E}. \tag{25}$$

Notice that owing to the simple behaviour of planes waves, the field itself divides out of the equation, leaving just a relationship between k and ω. This is extremely useful and therefore important! Gathering terms related to ω on the right, and using $c^2 = 1/\epsilon_0\mu_0$, we have

Dispersion relation for a conductor

$$k^2 = \frac{\omega^2}{c^2} + i\omega\mu_0\sigma \tag{26}$$

In wave theory in general, the term *dispersion relation* refers to the relationship between k and ω. It is one of the most useful pieces of information in order to understand what the waves are doing. The above dispersion relation applies whenever the situation is an ohmic medium ($\mathbf{j} = \sigma\mathbf{E}$) with no or uniform charge ($\boldsymbol{\nabla}\rho = 0$), and we assumed that the conductivity σ is constant in time. We did not need to assume it is uniform in space, and it can have any dependence on ω.

3.1 Good conductor

The interaction between electromagnetic waves and metals can be treated to good approximation by taking σ to be a real number, as long as the wavelength is large compared to the interatomic spacing and the frequency lies below that of atomic resonances. By this method we can treat radio waves accurately and even visible light roughly, but not ultra-violet radiation or X rays.

For simplicity we first examine two limiting cases, where the dispersion relation simplifies:

$$\text{insulator} \qquad \omega\mu_0\sigma \ll \omega^2/c^2, \qquad k = \omega/c \tag{27}$$

$$\text{good conductor} \qquad \omega\mu_0\sigma \gg \omega^2/c^2, \qquad k = \sqrt{i\omega\mu_0\sigma} \tag{28}$$

These are called 'poor' and 'good' conductor because the condition is on the conductivity: the 'good' conductor limit is

$$\sigma \gg \frac{\omega}{\mu_0 c^2} = \epsilon_0\omega. \tag{29}$$

From the simplified dispersion relations (27), (28) we deduce that the poor conductor leads to a behaviour just like waves in free space (but see comments after (48)), and the good conductor gives

$$k = \alpha(1+i) \tag{30}$$

where

$$\alpha = \sqrt{\omega\mu_0\sigma/2} \tag{31}$$

where we used that $\sqrt{i} = (1+i)/\sqrt{2}$. (Notice that we are here allowing k to be a complex number, not just a real number; you should confirm that all the equations and methods we have employed do not require any assumption that k is real.) Substituting this solution into the plane wave (19) and taking the z axis in the direction of $\mathbf{k}$, we have

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\alpha(1+i)z-\omega t)} \\ &= \mathbf{E}_0 e^{-\alpha z} e^{i(\alpha z-\omega t)} \end{aligned} \tag{32}$$

The physical electric field will be given by the real part of this solution:

e.m. wave in a good conductor

$$\mathbf{E}_{\text{phys}} = \mathbf{E}_0\, e^{-\alpha z} \cos(\alpha z - \omega t) \tag{33}$$

where we assumed $\mathbf{E}_0$ is real. The result is plotted in Fig. 1.

We find that as it 'tries' to propagate into the conductor, the wave is rapidly suppressed. Its amplitude decays on the length scale $1/\alpha$; this is called the

skin depth

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu_0\sigma}} \tag{34}$$

This is the formula in the case of a good conductor; a more general formula is derived in the next section.

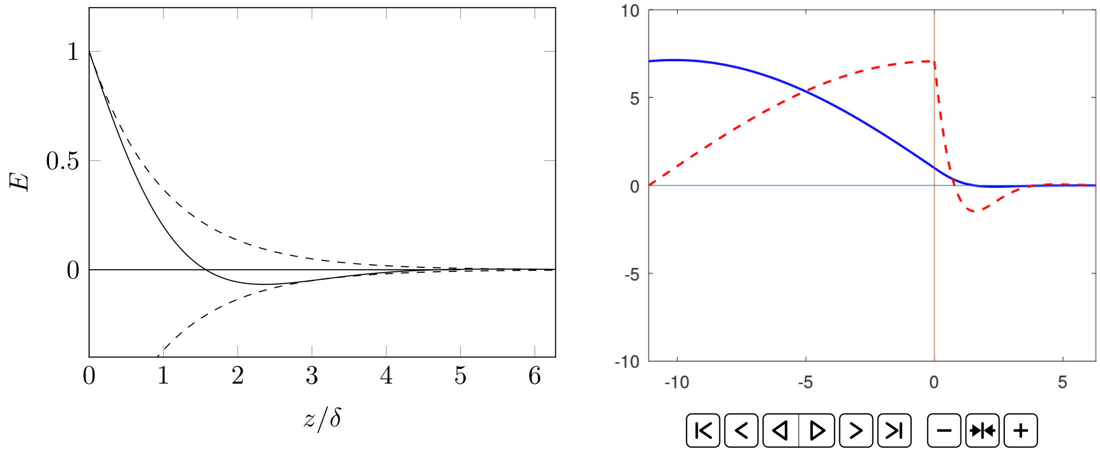

Figure 1: Left: the function $E = e^{-z/\delta}\cos(z/\delta)$, describing a wave propagating into a good conductor. The dashed curves show $\pm e^{-z/\delta}$. The wave amplitude decreases by a factor $\exp(-\pi/2) \simeq 0.208$ each quarter-wavelength. Right: the animation shows a wave incident from the left and mostly reflected, for the case $\sigma = 100\epsilon_0\omega$. Blue full line = E_x, red dashed line = B_y.

Now let's recall that our conductor is ohmic, so the behaviour of the electric field also tells us the behaviour of the current, through $\mathbf{j} = \sigma\mathbf{E}$. In a typical case the electric field is directed along a wire, and therefore so is $\mathbf{j}$. But we see that if δ is small compared to the radius of the wire then $\mathbf{j}$ is confined to a region close to the surface of the wire. Since δ falls with frequency (assuming σ does not depend strongly on frequency) this is liable to happen at higher frequencies, and it leads to an increase in resistance since the current is confined to only part of the cross-section of the wire, rather than all of it. This is why wires are often made multi-stranded, in order to increase the surface area and thus reduce the resistance. The skin effect is an important consideration in r.f. circuit design.

Example.
The conductivity of aluminium is $3.8 \times 10^7\ (\Omega\mathrm{m})^{-1}$ and may be taken to be independent of frequency. Find the resistance of an aluminium wire of length $\ell = 1\,\mathrm{m}$ and radius $a = 0.2\,\mathrm{mm}$, at 50 Hz and at 50 MHz.
Solution.
First we find the ratio $\sigma/\epsilon_0\omega$. At 50 Hz it is about 10^{16} and at 50 MHz it is about 10^{10}. Therefore in both cases we may employ the good conductor approximation. Next we obtain the skin depth using (34). At 50 Hz it is $1.2\,cm$ and at 50 MHz it is $11.5\,\mu$m. Therefore at 50 Hz the wire conducts throughout its width and the resistance is $R = \ell/(\pi r^2\sigma) = \underline{0.21\,\Omega}$. At 50 MHz, on the other hand, the skin depth is small so the current only flows in the region near the surface. Since $\delta \ll a$ we can ignore the effect of the curvature of the

surface, treating it as a plane of width $2\pi a$ (and length ℓ). The current is

$$\begin{aligned} I &= 2\pi a \int_0^a \sigma j \mathrm{dz} \\ &\simeq 2\pi a \int_0^\infty \sigma E_0 e^{-\alpha z} e^{i(\alpha z - \omega t)} \mathrm{dz} \\ &= 2\pi a \frac{\sigma E_0}{2\alpha}(1+i)e^{-i\omega t}. \end{aligned} \tag{35}$$

Taking the voltage to be ℓE_0 we find the complex impedance $Z = V/I$ is

$$Z = \frac{\ell}{\sigma 2\pi a \delta}(1-i) \tag{36}$$

where we used $\delta = 1/\alpha$. The impedance therefore has a resistive and an inductive part.[2] They are in parallel physically, but the result is equivalent to an arrangement in series with resistive part $R = \ell/(\sigma 2\pi a \delta) = \underline{1.8\,\Omega}$.

A final remark. The alert reader will realise that we started with a quadratic dispersion relation but we only discussed one solution. For a quadratic equation there should be two solutions, and indeed there are. The full story is

$$k = \pm\alpha(1+i). \tag{37}$$

The second solution yields

$$\mathbf{E} = \mathbf{E}_0\, e^{\alpha z} e^{i(-\alpha z - \omega t)}. \tag{38}$$

This is a wave propagating in the negative z direction (which you can deduce from the phase, i.e. the argument of the complex exponential), decaying as it goes, just like the other solution (because the $\exp(\alpha z)$ factor gets smaller as z goes towards $-\infty$).

3.2 Any conductivity

Next we treat a general conductor, retaining all terms in the dispersion relation (26). The convenient way to handle the mathematics is to express k in terms of its real and imaginary parts:

$$k = \alpha + i\beta \tag{39}$$

Then we have

$$k^2 = \alpha^2 - \beta^2 + 2i\alpha\beta. \tag{40}$$

[2] When the voltage goes as $\exp(-i\omega t)$ a capacitor has impedance $i/\omega C$ and an inductor $-i\omega L$.

Substituting this into the dispersion relation gives the simultaneous equations

$$\alpha^2 - \beta^2 = \frac{\omega^2}{c^2} \tag{41}$$

$$2\alpha\beta = \omega\mu_0\sigma \tag{42}$$

Taking the second equation ($\beta = \omega\mu_0\sigma/2\alpha$) and substituting it into the first one obtains

$$\alpha^4 - \alpha^2\omega^2/c^2 - (\omega\mu_0\sigma/2)^2 = 0. \tag{43}$$

This is a quadratic equation for α^2 whose solution is

$$\alpha^2 = \frac{\omega^2}{2c^2}\left(1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon_0\omega}\right)^2}\right) \tag{44}$$

where we must take the + sign because α^2 must be positive (we have assumed α to be real by definition). Substituting this into (41) gives

$$\beta^2 = \frac{\omega^2}{2c^2}\left(-1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon_0\omega}\right)^2}\right). \tag{45}$$

Also, substituting α into (42) yields

$$\beta = \frac{\mu_0\sigma c}{\sqrt{2}\sqrt{1 + \sqrt{1 + s^2}}} \tag{46}$$

where $s = \sigma/\epsilon_0\omega$. This version is more useful for obtaining the skin depth, as we now show.

The plane wave form is

$$\mathbf{E} = \mathbf{E}_0\, e^{i(kz-\omega t)} = \mathbf{E}_0\, e^{-\beta z} e^{i(\alpha z - \omega t)} \tag{47}$$

so we deduce that the general formula for the skin depth is

skin depth

$$\delta = \frac{1}{\beta} = \frac{\sqrt{2}}{\mu_0\sigma c}\sqrt{1 + \sqrt{1 + (\sigma/\epsilon_0\omega)^2}} \tag{48}$$

The good conductor limit of this formula reproduces our previous result (34). The poor conductor limit ($\sigma \ll \epsilon_0\omega$) gives

$$\delta \to \frac{2}{\mu_0\sigma c}. \tag{49}$$

This seems to negate our previous observation from (27) that the situation is 'just like in free space'. That observation was not entirely wrong, however, since as $\sigma \to 0$ we get $\delta \to \infty$. The interesting

freq.	aluminium	copper	silver	sea water	drinking water	deionised pure water
σ ($\Omega^{-1}\text{m}^{-1}$)	3.78×10^7	5.97×10^7	6.29×10^7	5	0.01	5.6×10^{-6}
50 Hz	0.0116	0.0092	0.0090	32	710	30200
1 kHz	0.0026	0.0021	0.0020	7	160	6800
1 MHz	82×10^{-6}	65×10^{-6}	63×10^{-6}	0.23	5	960
1 GHz	2.6×10^{-6}	2.1×10^{-6}	2.0×10^{-6}	0.007	0.5	960
6×10^{14} Hz	3.3×10^{-9}	2.7×10^{-9}	2.6×10^{-9}			

Table 1: Skin depth in metres for various materials and frequencies. The values for water are rough, because the conductivity is a function of temperature, frequency and concentration of impurities. The last row applies to visible light of wavelength 500 nm.

further information to be seen in (49) is that even before σ reaches zero, the skin depth becomes independent of frequency in poor conductors. This is observed at high frequencies in especially poor conductors such as semi-conductors.

Some example values of skin depth are displayed in table 1. For metals the depth is about 1 cm at the mains frequency of 50 Hz, and a few microns at 1 GHz. For visible light the skin depth in a metal is a few nanometres, which is a few tens of atomic spacings. This implies that the continuous model of the material remains reasonably reliable, but at optical frequencies the conductivity of metals depends on frequency and becomes complex.

The conductivity of water depends on frequency above a few Hz and also on temperature; the table shows the order of magnitude of the values. One implication is that radio communication with submarines is difficult. (The behaviour of water at optical frequencies is not adequately modelled by simple conductor theory. In fact water transmits quite well at optical frequencies, and considerably less well at both infra-red and ultra-violet.)

3.3 The magnetic field in a conductor

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So far we discussed the electric field. To treat the magnetic field, recall the wave equation (16) and adopt the plane wave (20) as a trial solution. Using also $\mathbf{j} = \sigma \mathbf{E}$ on the right hand side, one finds

$$-\epsilon_0 \mu_0 \omega^2 \mathbf{B} + k^2 \mathbf{B} = i \mu_0 \sigma \mathbf{k} \times \mathbf{E}. \tag{50}$$

The fields also satisfy (M3) which gives

$$i\mathbf{k} \times \mathbf{E} = i\omega \mathbf{B} \tag{51}$$

(obviously the factor i divides out of this equation but I left it in at this stage in order to make it clear how the result was obtained.) After substituting this into (50) we obtain the same dispersion relation as before, (26), therefore at any given ω the two fields have the same k (as we should expect).

Eqn. (51) shows that **B** is perpendicular to both **k** and **E** and we already know **k** is perpendicular to **E**. It follows that the (possibly complex) amplitudes of the fields are related by

$$B_0 = \frac{k}{\omega} E_0. \tag{52}$$

We shall now adopt the good conductor approximation (the reader will be able to go back and deduce the more general case should they wish to). In the good conductor limit we have $k = (1+i)\alpha$ (this is (51)) where α is given by (31), so we find

$$B_0 = \frac{1+i}{\sqrt{2}} \sqrt{\frac{\mu_0 \sigma}{\omega}} E_0 = e^{i\pi/4} \sqrt{\frac{\mu_0 \sigma}{\omega}} E_0. \tag{53}$$

Two observations follow: first, the magnetic field is out of phase with the electric field by $\pi/4$ (**B** lags **E**); secondly: the magnetic field is 'large'. That is, the ratio B_0/E_0 is much larger than it is for waves in free space. To express this, examine the dimensionless ratio

$$\frac{c|B_0|}{|E_0|} = \sqrt{\frac{\sigma}{\epsilon_0 \omega}} \gg 1\,. \tag{54}$$

(In free space the value would be 1.)

3.4 Reflection from a good conductor

We will find the coefficient of reflectivity for electromagnetic waves incident on a good conductor. For simplicity we just treat the case of normal incidence on a flat boundary. The calculation is most easily done by treating the fields **E** and **H**. This is because the fields are transverse to the wave vector, so they are directed *along* the surface of the boundary in the case of normal incidence, and it is $\mathbf{E}_\parallel$ and $\mathbf{H}_\parallel$ (as opposed to **D** and **B**) that are continuous at the boundary in the absence of surface charge or current.

In view of the fact that we have been discussing the current **j**, the reader may wonder whether there is a surface current. For the moment we shall simply assert that there is not; we shall return to this point at the end.

Under the assumed conditions, then, the standard reflection calculation applies and one finds that the amplitude reflection coefficient, for waves passing from medium 1 to medium 2

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{55}$$

for the electric field amplitude, where Z_1, Z_2 are the wave impedances. For a plane electromagnetic wave of amplitude E_0 propagating in the positive z direction the impedance is defined through

$$E_0 = Z H_0 \tag{56}$$

In free space we have $E = cB$ and $B = \mu_0 H$, hence one finds $Z = \mu_0 c$. Thus

$$Z_1 = \mu_0 c = \sqrt{\mu_0/\epsilon_0}. \tag{57}$$

In a good conductor (53) gives

$$Z_2 = \sqrt{\mu_0 \omega/\sigma}\, e^{-i\pi/4} \tag{58}$$

(we took the permeability $\mu = \mu_0$; for remarks on this see section 3.6). Note that $|Z_2/Z_1| = \sqrt{\epsilon_0 \omega/\sigma} \ll 1$ (consistent with the observation that 'the magnetic field is large' as we said above.)

After substituting Z_2/Z_1 into (55) we find

$$r = \frac{-1+s}{1+s} \qquad \text{where} \;\; s = e^{-i\pi/4}\sqrt{\epsilon_0\omega/\sigma} \tag{59}$$

Using that $|s| \ll 1$ we can simplify this to

$$r = -1 + 2s + O(s^2). \tag{60}$$

The -1 indicates there is a phase change on reflection. The intensity reflection coefficient is

$$\left|r^2\right| \simeq (-1+2s)(-1+2s^*) = 1 - 2(s+s^*) \simeq 1 - \sqrt{8\epsilon_0\omega/\sigma}. \tag{61}$$

For example, for silver at optical frequencies one finds $|r^2| \simeq 0.93$. This is consistent with the measured value. More generally, metals make good reflectors. Together with its small chemical reactivity, this explains why silver is commonly used as a coating on glass to make mirrors.

But is there a surface current or isn't there? Now let's return to the point about surface current which was postponed above. The more general version of the continuity equation for $\mathbf{H}$ is

$$\mathbf{H}_{\parallel,2} - \mathbf{H}_{\parallel,1} = \mathbf{K} \tag{62}$$

where $\mathbf{K}$ is the current per unit length flowing along the surface (such that the two $\mathbf{H}$ fields make a loop around $\mathbf{K}$). When there is a volume current $\mathbf{j}$ the surface current in a thickness δz is $\mathbf{j}\delta z$ and this tends to zero in the limit $\delta z \to 0$. This is why we correctly took $\mathbf{K} = 0$ in the above theory of reflection at a conducting surface. It is all about what we mean by the symbol $\mathbf{H}_2$. We used that symbol to refer to the magnetic field at a distance within the surface of just one or a few atomic spacings, a distance small compared to the skin depth. We accounted for the current $\mathbf{j} = \sigma\mathbf{E}$ by including it in the calculation of the fields.

3.5 Permittivity of a conductor?

Our discussion of the conductor invoked Maxwell's equations (M1)–(M4) and we only ever needed to mention the fields $\mathbf{E}$ and $\mathbf{B}$ and the current $\mathbf{j}$. There was no mention of either polarization $\mathbf{P}$ or magnetization $\mathbf{M}$ and therefore the conductor is correctly treated by taking

$$\epsilon_r = 1, \qquad \mu_r = 1. \tag{63}$$

That is not to say you cannot also have a medium which both conducts and is also polarizable or magnetizable, but here we are just dealing with a non-dielectric conductor.

However you often see quoted a different result:

$$\epsilon_r = 1 + \frac{i\sigma}{\epsilon_0 \omega}. \tag{64}$$

This is a very different answer! It is complex and has a magnitude large compared to 1 for a good conductor! So what is going on?

The two results represent two ways of interpreting the motion of the charges. In the first way we say the oscillating charges are making a current, which we have accounted for correctly. It is the whole current and there is nothing further to say; in particular there is no polarization, so $\mathbf{P} = 0$ and therefore $\epsilon_r = 1$.

In the second approach, we treat the very same motion of the charges, but we interpret it as an oscillating polarization. Suppose that particles of charge q are oscillating with displacement $\mathbf{x} = \mathbf{x}_0 e^{-i\omega t}$. This will produce a current density $\mathbf{j} = nq\dot{\mathbf{x}} = -i\omega nq\mathbf{x}$ where n is the number density of the charges. And if for each moving particle there is an equal charge $-q$ that does not move, then the displacement of any given particle gives rise to an electric dipole moment $q\mathbf{x}$, and therefore the overall polarization (electric dipole moment per unit volume) is

$$\mathbf{P} = nq\mathbf{x} = i\mathbf{j}/\omega. \tag{65}$$

Using now $\mathbf{j} = \sigma\mathbf{E}$ we find

$$\mathbf{P} = nq\mathbf{x} = i\sigma\mathbf{E}/\omega \tag{66}$$

which implies the susceptibility is $\chi_{\rm e} = i\sigma/\epsilon_0\omega$ and eqn (64) follows. In this second approach we must take $\mathbf{j}_c = 0$ because we already accounted for the motion of the charges by including it in $\mathbf{P}$.

The lesson is *you can use either of (63) or (64) as long as you know what you are doing and define* $\mathbf{P}$ *in a way consistent with your choice.*

3.6 Field energy and Poynting vector

Electromagnetic waves in a conductor provide a nice example of the way energy conservation is treated in classical electromagnetism. There are three fundamental concepts:

field energy density	$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$	(67)
Poynting vector (field energy flux)	$\mathbf{S} = \mathbf{E}\times\mathbf{H}$	(68)
power density	$p = \mathbf{E}\cdot\mathbf{j}$	(69)

For comments on the related quantities $(\epsilon_0 E^2 + B^2/\mu_0)/2$ and $\mathbf{E}\times\mathbf{B}/\mu_0$ see the note *Energy in electromagnetism*. For the ordinary conductor there is no magnetisation so $\mu_r = 1$, and if we treat the current as entirely (not polarization) current then $\epsilon_r = 1$ as explained in section 3.5.

The power density p is the rate at which energy is transferred from the fields to the charges, per unit volume.[3] The conservation of energy is expressed by[4]

Conservation of energy

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{E} \cdot \mathbf{j} = 0. \tag{70}$$

To apply these ideas to waves, the first step is to beware of the complex notation! The use of complex numbers to describe physical quantities is useful when we are concerned with *linear* combinations such as a sum of fields, but less so then we are concerned with non-linear terms such as E^2 or EB. This is because, for any complex numbers z_1 and z_2,

$$\mathrm{Re}\,(z_1)\,\mathrm{Re}\,(z_1) \neq \mathrm{Re}\,(z_1 z_2) \tag{71}$$

It follows that if $\mathbf{E}$ and $\mathbf{H}$ are complex then the Poynting vector is *not* $\mathbf{E} \times \mathbf{H}$. Rather, it is

$$\mathbf{S} = \mathrm{Re}\,(\mathbf{E}) \times \mathrm{Re}\,(\mathbf{H}) \tag{72}$$

Is this a contradiction of (68)? No, but it is a hidden change of notation. What happened is that in (68) I quoted the standard form of the definition where it is understood that all the quantities in play are real-number-valued, but when dealing with waves we often use symbols such as $\mathbf{E}$ and $\mathbf{H}$ to refer to complex numbers whose real part yield the physical fields, so in that case we must use (72). Similar considerations apply to power and energy density.

The equation for energy conservation (70) applies at all times and places and correctly tracks energy movements in detail. In the case of wave motion typically all the quantities $u, \mathbf{S}$ and p oscillate at the wave frequency, and we can in principle calculate the detailed movements of energy within each cycle and each wavelength (for an example see section 4.5). However we are often mainly interested in the average over time, and that is all we shall consider here. The following mathematical 'trick' will be useful. In case of oscillating quantities of the form $z = z_0 e^{-i\omega t + \phi}$ (such as our plane waves (19),(20)) the time average (here indicated by the notation $\langle \cdots \rangle$) can be found from

$$\langle \mathbf{S} \rangle = \langle \mathrm{Re}\,(\mathbf{E}) \times \mathrm{Re}\,(\mathbf{H}) \rangle = \frac{1}{2}\mathrm{Re}\,(\mathbf{E}^* \times \mathbf{H})\,. \tag{73}$$

$$\langle u \rangle = \frac{1}{4}\mathrm{Re}\,((\mathbf{E}^* \cdot \mathbf{D} + \mathbf{B}^* \cdot \mathbf{H})) \tag{74}$$

$$\langle p \rangle = \frac{1}{2}\mathrm{Re}\,(\mathbf{E}^* \cdot \mathbf{j}) \tag{75}$$

Note, this mathematical 'trick' is not self-evident and the reader should either prove it for themselves or else consult *Energy in electromagnetism*.

[3] To derive this, observe that the work done per unit time on a charge q moving at velocity $\mathbf{v}$ is $\mathbf{f} \cdot \mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}$ and therefore the rate per unit volume, if there are n such particles per unit volume, is $nq\mathbf{E} \cdot \mathbf{v} = \mathbf{E} \cdot \mathbf{j}$ by using $\mathbf{j} = nq\mathbf{v}$.

[4] Derived in the above-mentioned note, *Energy in electromagnetism*.

We shall now apply the above equations to the case of waves in a conductor, using the solution for **E** and **B** (eqs (47), (52)) which we repeat here for convenience:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{-\beta z} e^{i(\alpha z - \omega t)}, \\ \mathbf{B} &= \frac{k}{\omega}\mathbf{E} = \frac{\alpha + i\beta}{\omega}\mathbf{E}_0 e^{-\beta z} e^{i(\alpha z - \omega t)}. \end{aligned} \tag{76}$$

By substituting these into (74), and using (44), (45), we obtain

$$\langle u \rangle = \frac{1}{4}\left(1 + \sqrt{1 + (\sigma/\epsilon_0\omega)^2}\right)\epsilon_0 E_0^2 e^{-2\beta z}. \tag{77}$$

Notice that for a good conductor the field energy is dominated by the magnetic part (and in an insulator the electric and magnetic parts contribute equally).

Turning now to the Poynting vector and power density, (73),(75) and (76) give

$$\langle \mathbf{S} \rangle = \frac{\alpha}{2\mu_0\omega} E_0^2 e^{-2\beta z}\hat{\mathbf{z}}, \tag{78}$$

$$\langle p \rangle = \frac{1}{2}\sigma E_0^2 e^{-2\beta z}. \tag{79}$$

By using (42) it is now straightforward to confirm that

$$\langle \boldsymbol{\nabla} \cdot \mathbf{S} \rangle = \boldsymbol{\nabla} \cdot \langle \mathbf{S} \rangle = -\langle p \rangle. \tag{80}$$

Therefore the conservation of energy is respected (since the time average of u is constant here). A good way to grasp the physical interpretation is to write the result in the form

$$S(z + \mathrm{d}z) = S(z) - p \tag{81}$$

which asserts that "the energy flowing to the plane at $z + \mathrm{d}z$ is equal to the energy arriving at z, less the energy given up to drive the current against the resistance of the matter in the conductor." By integrating $\langle p \rangle$ between z and ∞ you can also confirm that all the field energy arriving at any given z is eventually transferred to the matter.

3.7 Anisotropic conductor

Throughout our discussion of a conductor we have taken Ohm's law in its simplest form $\mathbf{j} = \sigma\mathbf{E}$ where σ is a scalar quantity. This is the form it takes in an isotropic medium (one in the which the conductivity is the same in all directions). In an anisotropic medium such as a crystalline solid, the relationship may be

$$\mathbf{j} = \bar{\bar{\sigma}}\mathbf{E} \tag{82}$$

where the notation indicates that $\bar{\bar{\sigma}}$ is a *tensor*: that is, a quantity which can be expressed by a 3×3 matrix (and which obeys the mathematical rules for tensors). The treatment of this situation is a more advanced exercise. If the fields are directed along the principle axes of the tensor, the mathematics is not much more elaborate, but we shall not investigate this here.

4 The neutral plasma

A neutral plasma is a fluid medium composed of equal densities of positive and negative charge, such as positive ions and electrons. Such a plasma can be regarded as a fourth state (also called *phase*) of matter, in addition to solid, liquid, gas. This state is of especially high significance in astrophysics, because stars are plasmas and so is much of the interstellar medium.

In the presence of an electromagnetic wave the charges in a plasma will be driven to oscillate at the frequency of the wave, and the mathematical treatment is almost identical to that for a conductor, but with a pure-imaginary conductivity, as we now show.

Let's suppose a plane wave is propagating in a plasma. We treat a small region of the plasma, and examine the motion of a single charged particle in the first instance. It obeys Newton's second law:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{83}$$

where q is the charge and m the mass of the particle. We assume the magnetic contribution to the force is negligible in comparison to the electric. Then

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m}\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{84}$$

For transverse waves ($\mathbf{E}$ perpendicular to $\mathbf{k}$) the particle is displaced in a direction orthogonal to the wave vector and therefore experiences the same field, at any given time, no matter how far displaced it is. For longitudinal waves ($\mathbf{E}$ parallel to $\mathbf{k}$) the displacement is along the direction of $\mathbf{k}$ and this complicates matters. However, if the size of the particle's motion remains small compared to the wavelength then this complication goes away: we place the origin of coordinates near the particle and then $\mathbf{k}\cdot\mathbf{x} \ll 1$ and we have

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m}\mathbf{E}_0 e^{-i\omega t}. \tag{85}$$

This equation is straightforward; it can be integrated immediately, giving

$$\begin{aligned} \mathbf{v} &= \frac{q}{-i\omega m}\mathbf{E}_0 e^{-i\omega t} \\ &= \frac{iq}{m\omega}\mathbf{E} \end{aligned} \tag{86}$$

(and the constant of integration is zero since we assume the particle has no drift velocity in addition to this oscillating velocity).

Now let's use the above to make deductions about the plasma as a whole. As long as the plasma is not too dense, we can apply (86) to each charged particle in it. If there are n particles per unit volume then we have a net flux of electric charge given by

$$\mathbf{j} = nq\mathbf{v} = n\frac{iq^2}{m\omega}\mathbf{E}. \tag{87}$$

If there are several species of particle with different masses and charges (as is usually the case) then we must sum over them. However in a plasma of electrons and ions the current is dominated by the electrons (since their mass is about 2000 times smaller than that of the ions) so we shall ignore the other contributions. The most important fact is that (87) has the form $\mathbf{j} = \sigma \mathbf{E}$ with

Conductivity of plasma

$$\sigma = i \frac{q^2 n}{m\omega} \tag{88}$$

Now we can treat the propagation of waves in the plasma. We take as starting-point equation (18), and substitute into it a plane wave solution. This yields

$$-\epsilon_0 \mu_0 \omega^2 \mathbf{E} + k^2 \mathbf{E} = -\boldsymbol{\nabla}(\rho/\epsilon_0) - \mu_0 \frac{\partial \mathbf{j}}{\partial t} \tag{89}$$

Now use $\mathbf{j} = \sigma \mathbf{E}$ and observe that for the plane wave the charge density term can itself be written in terms of $\mathbf{k}$ and $\mathbf{E}$:

$$\boldsymbol{\nabla}(\rho/\epsilon_0) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) = -\mathbf{k}(\mathbf{k} \cdot \mathbf{E}). \tag{90}$$

Hence we obtain

$$k^2 \mathbf{E} = \frac{\omega^2}{c^2} \mathbf{E} + i\omega\mu_0\sigma \mathbf{E} + (\mathbf{k} \cdot \mathbf{E})\mathbf{k} \tag{91}$$

This is almost a dispersion relation. It would yield a dispersion relation if we could divide out a factor of $\mathbf{E}$. The overall form of the equation implies that $\mathbf{k}$ must be along the same direction as $\mathbf{E}$ unless $\mathbf{k} \cdot \mathbf{E} = 0$. In other words, there are two cases:

$$\text{transverse} \quad \mathbf{k} \cdot \mathbf{E} = 0, \qquad k^2 = \frac{\omega^2}{c^2} + i\omega\mu\sigma \tag{92}$$

$$\text{longitudinal} \quad \mathbf{k} \parallel \mathbf{E}, \qquad k^2 = \frac{\omega^2}{c^2} + i\omega\mu\sigma + k^2 \tag{93}$$

where in the second case we used that when $\mathbf{k}$ is parallel to $\mathbf{E}$ we have $\mathbf{k} \cdot \mathbf{E} = kE$ and $\hat{\mathbf{k}} = \hat{\mathbf{E}}$ so $(\mathbf{k} \cdot \mathbf{E})\mathbf{k} = (kE)k\hat{\mathbf{E}} = k^2\mathbf{E}$. The $\mathbf{E}$ factor then divides out of the whole equation. Upon substituting σ from (88) we find

plasma dispersion relation

$$\text{transverse} \quad \omega^2 = k^2 c^2 + \omega_{\mathrm{p}}^2 \tag{94}$$

$$\text{longitudinal} \quad \omega = \omega_{\mathrm{p}} \tag{95}$$

where

Plasma frequency

$$\omega_{\mathrm{p}} = \sqrt{\frac{nq^2}{\epsilon_0 m}} \tag{96}$$

In the following we shall say a bit more about these solutions. The transverse waves are like the more familiar waves in free space: they have electric and magnetic parts, they carry energy, etc. The longitudinal waves are a direct property of the plasma: they amount to the fact that the plasma has a natural oscillation frequency and can oscillate without being driven. The longitudinal waves have the same frequency irrespective of their wave vector (and therefore zero group velocity).

4.1 Transverse waves

The dispersion relation for transverse waves can usefully be displayed in the form

$$k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}} \tag{97}$$

This form makes it clear that k is real when $\omega > \omega_{\rm p}$ and k is pure imaginary when $\omega < \omega_{\rm p}$. Hence: high frequency waves propagate, low frequency waves decay. The phase velocity is

$$\frac{\omega}{k} = \frac{c}{n_{\rm r}} \tag{98}$$

(this defines the refractive index $n_{\rm r}$), so we find the refractive index is

$$n_{\rm r} = \sqrt{1 - \frac{\omega_{\rm p}^2}{\omega^2}}\,. \tag{99}$$

For the case of propagating waves the refractive index is less than 1 and the phase velocity is greater than c. This is allowed in special relativity because the individual wavefronts do not themselves convey information (for a monochromatic wave): their arrival is ‘expected’.

To obtain the group velocity it is easiest to employ (92) which gives

$$2\omega\frac{{\rm d}\omega}{{\rm d}k} = 2kc^2, \tag{100}$$

hence

$$\frac{{\rm d}\omega}{{\rm d}k}\frac{\omega}{k} = c^2 \tag{101}$$

which is

$$v_{\rm gr}v_{\rm ph} = c^2. \tag{102}$$

Hence $v_{\rm gr} < c$ when $v_{\rm ph} > c$, and

$$v_{\rm gr} = n_{\rm r}c = \left(1 - \omega_{\rm p}^2/\omega^2\right)^{1/2}\, c. \tag{103}$$

The transverse waves have $\mathbf{k}\cdot\mathbf{E} = 0$ and therefore $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, which implies $\rho = 0$. This shows that they do not disturb the neutrality of the plasma: in these waves the charges move from side to side without ‘bunching up’ (c.f. Fig. 2). Using (M3) we have also that

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B} \tag{104}$$

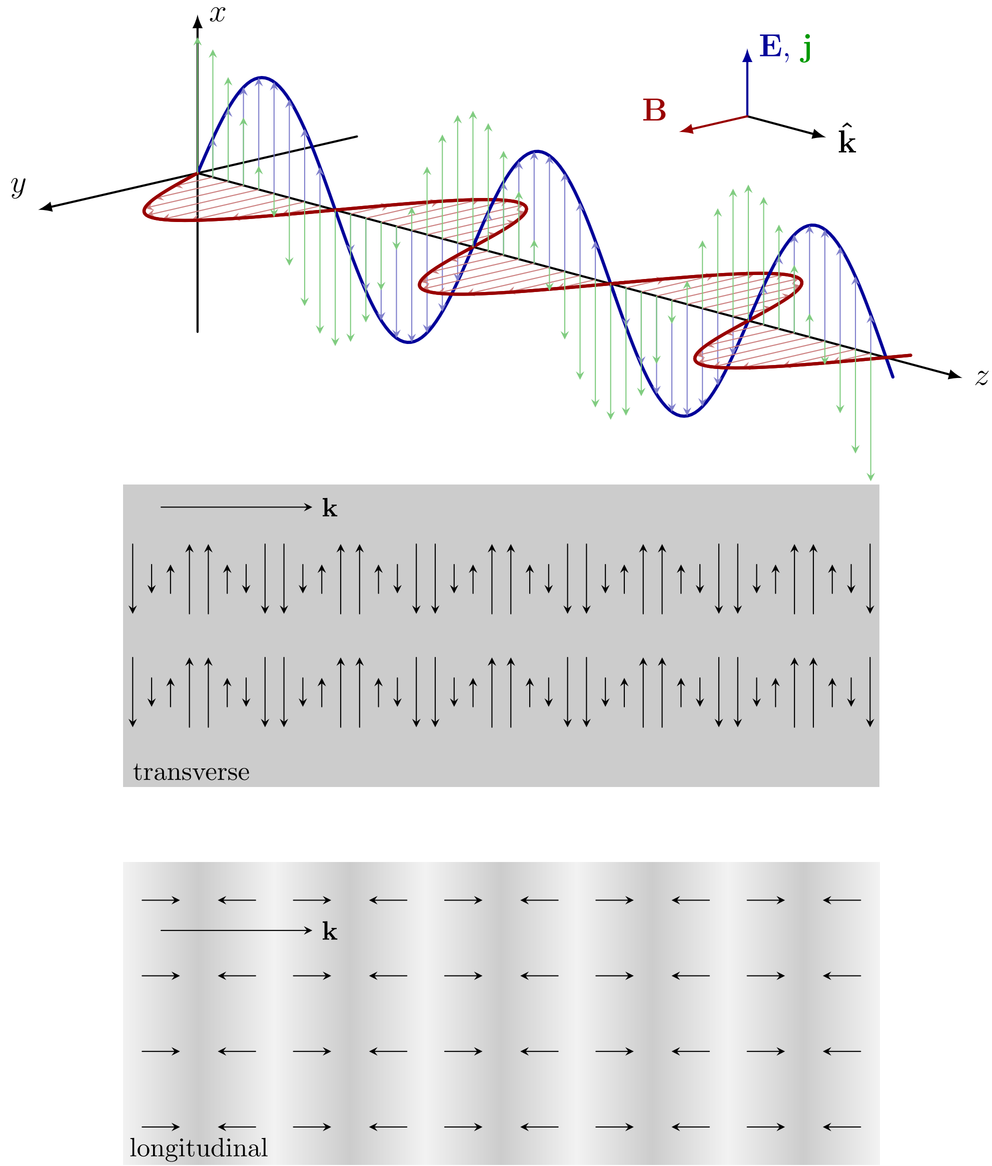
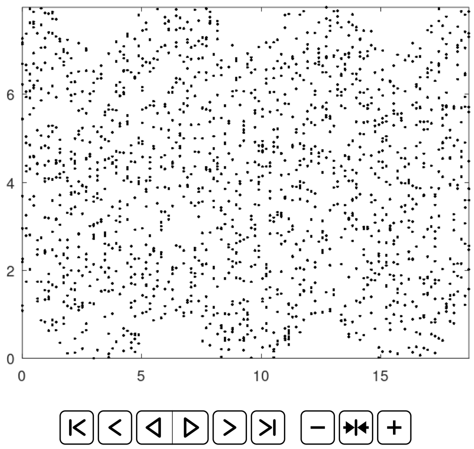
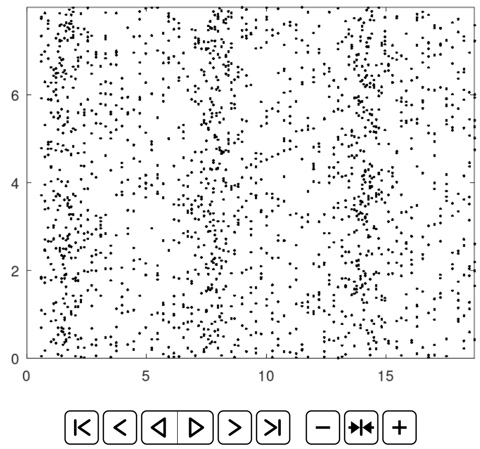


Figure 2: Waves in a plasma. The top diagram shows a plot of the **E** and **B** fields and the current **j** (at some instant of time) for a transverse wave propagating in the z direction and linearly polarized along x. The middle and bottom diagrams show the plasma with a transverse and longitudinal wave, respectively. Shading represents the charge density, arrows indicate the current density. In all cases the electric field is along the same direction as **j**, oscillating $\pi/2$ radians out of phase with it.





so we deduce there must be a magnetic wave accompanying the electric wave, with amplitude $kE/\omega = n_{\rm r}E/c$. It is transverse too (says (M2)).

The solutions at high frequency behave very much like ordinary electromagnetic waves in free space, in that they are transverse electromagnetic waves which propagate over long distances without decaying. What do the solutions at low frequency mean? They do not propagate at all, since when k is pure imaginary we have $k = i\alpha$ for real α, and the form is

$$\mathbf{E} = \mathbf{E}_0 e^{-\alpha z} e^{-i\omega t} \tag{105}$$

(for a z axis aligned with $\mathbf{k}$). This type of behaviour is called an *evanescent wave*. A good physical insight is obtained by examining the impedance Z, defined by

$$\mathbf{E}_0 = Z\mathbf{H}_0 = Z\mathbf{B}_0/\mu_0. \tag{106}$$

By combining this with (104) we find

$$Z = \mu_0 \omega / k. \tag{107}$$

Thus when k is pure imaginary, so is Z. Now consider waves incident on a boundary between two media, one with real Z, the other with imaginary Z. At normal incidence the intensity reflection coefficient is

$$R = \left| \frac{Z_2 - Z_1}{Z_2 + Z_1} \right|^2 \tag{108}$$

$$= \frac{(Z_2 - Z_1)}{(Z_2 + Z_1)} \frac{(Z_2^* - Z_1^*)}{(Z_2^* + Z_1^*)} \tag{109}$$

But if $Z_1 = Z_1^*$ and $Z_2 = -Z_2^*$ (that is, Z_1 is real and Z_2 is imaginary) then this yields $R = 1$. In other words, there is perfect reflection. The lesson is that *low-frequency electromagnetic waves, incident from some other medium, are reflected by a plasma.*

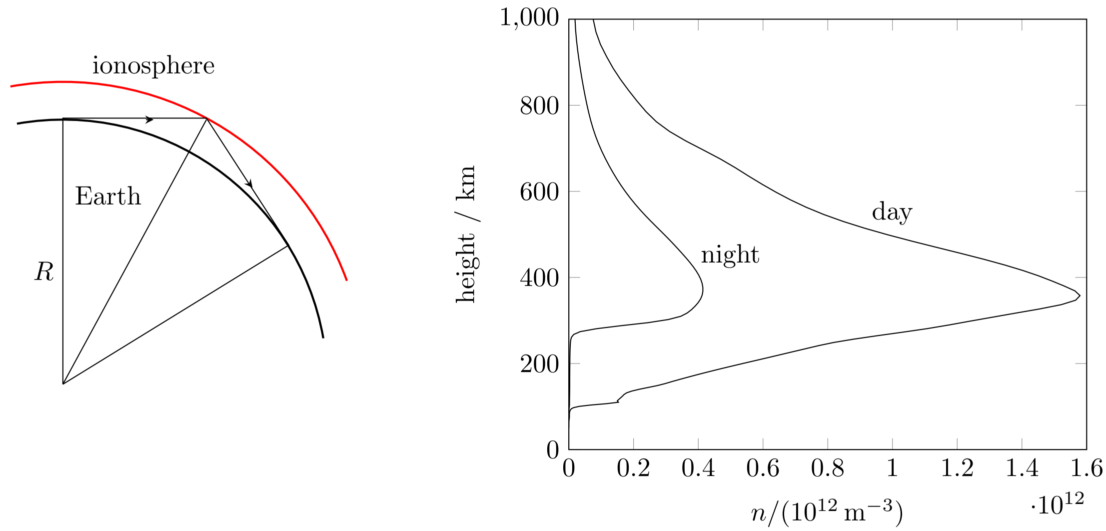


Figure 3: Left: long-distance radio communication using reflection from the ionosphere. A signal sent horizontally will hit the layer at height 400 km at an angle of 70° to the normal, and therefore will be reflected if its frequency is below about $3\omega_\mathrm{p}$. Right: the electron density as a function of height (there are considerable variations with weather, season and especially sunspot activity). The plasma frequency for $n = 1.5 \times 10^{12}\,\mathrm{m}^{-3}$ is $(\omega_\mathrm{p}/2\pi) = 11\,\mathrm{MHz}$.

In the case $\omega > \omega_\mathrm{p}$, when waves can in principle propagate in the plasma, one has $n < 1$ and therefore there will also be total reflection for waves incident at angles to the normal above a critical angle which can be determined from Snell's law:

$$n_i \sin\theta_i = n_\mathrm{r} \sin\theta_r \,. \tag{110}$$

$\sin\theta_r$ cannot exceed 1 so this equation has no solution (i.e. no refracted wave, therefore complete reflection) when $\sin\theta_i > (n_\mathrm{r}/n_i)$.

There is a plasma over our heads, in the region of Earth's atmosphere called the *ionosphere*—see Fig. 3. Radio signals can be bounced off the ionosphere, and this is used for long-distance communication without the need for wires or satellites. An easy way to remember the fact that it is the high frequencies that can be transmitted is to recall that when we look up at night, we can see the stars. Their visible radiation can propagate through the ionosphere, whereas radio signals (of frequency below about 20 MHz) can not.

Another system that can be treated by plasma theory to good approximation is a metal at very high frequencies (the X ray region of the spectrum). At the highest frequencies the response of the metal is not ohmic; one can ignore the scattering processes associated with ohmic conduction and just assume the electrons respond directly to the fields by undamped driven oscillations, just as for a plasma.

4.2 Longitudinal waves

The longitudinal waves are also known as plasma oscillations or Langmuir waves. They are not like electromagnetic waves in various respects. For one thing, they are not magnetic, since when $\mathbf{k}$ is parallel to $\mathbf{E}$ we have $\mathbf{k} \times \mathbf{E} = 0$ so (M3) informs us that $\partial \mathbf{B}/\partial t = 0$: the magnetic field (if there is one) is not oscillating. For this reason these waves are sometimes called 'electrostatic waves' (a name I do not like because they are not static). The phase velocity ω/k can have any value and the group velocity is zero. The Poynting vector is also zero: there is no propagation of energy (there is just exchange of between the current and the field at any given location.)

The longitudinal waves are associated with non-zero values of the charge density. In other words, they disturb the charges in the plasma in such a way that the net charge density is positive in some places and negative in others (c.f. Fig. 2). This case did not arise in our treatment of conductors because there we assumed $\boldsymbol{\nabla}\rho = 0$ throughout. If we now allow for the possibility of non-zero charge variations in a conductor we shall obtain (91)–(93) but for a medium in which σ is real-valued. The only solution in the longitudinal case is then $\omega = 0$; in other words longitudinal waves do not arise in an ohmic conductor.

4.3 Further remarks

Our treatment of a plasma assumed that the particles in the plasma were not moving significantly apart from the oscillations in response to the oscillating field. We also ignored the magnetic contribution to the force. In practice the motion of the plasma will introduce pressure variations, and these will add to the restoring forces. The approximation of neglecting the pressure forces is called the 'cold plasma' approximation. For a 'hot' plasma, where pressure gradients are non-negligible, the transverse waves are unchanged and the longitudinal waves have a dispersion relation

$$\omega^2 = \omega_{\rm p}^2 + 3k^2 v_{\rm th}^2 \tag{111}$$

where the thermal velocity $v_{\rm th}$ is defined as

$$v_{\rm th} = \sqrt{k_{\rm B} T/m} \tag{112}$$

The plasma is 'hot' when $k v_{\rm th}$ is not negligible compared with $\omega_{\rm p}$.

For the transverse waves there is a magnetic field but we ignored its contribution to the force. Let's confirm that this is legitimate. The ratio of magnetic to electric force is at most vB/E for a given charged particle moving at speed v and (104) gives $B = n_{\rm r} E/c$ so we find that the ratio of magnetic to electric force is at most $n_{\rm r} v/c$. This is small compared to 1 for non-relativistic motion.

If there is an applied static magnetic field whose size is sufficient to produce non-negligible forces then the situation is significantly more complex. In this case there are several types of wave, having different dispersion relations. Plasmas in strong fields occur in tokamak fusion reactors. Their full description is a highly complex, and highly significant, area of physics in its own right.

4.4 Postscript: treatment via polarization and permittivity

We treated the plasma using the concept of conductivity, and in consequence we never needed the concepts of polarization, susceptibility and relative permittivity. An alternative approach is to note that the charged particles form dipoles when they move. There is an equal charge density of either sign in the undisturbed plasma, so if a given electron is displaced by $\mathbf{x}$, with the nearby positive ions unmoved, then an electric dipole of size $q\mathbf{x}$ is produced. The polarization (= dipole moment per unit volume) of the plasma is therefore

$$\mathbf{P} = nq\mathbf{x}\,. \tag{113}$$

Starting from the equation of motion (85) we integrate twice, obtaining $\mathbf{x} = -q/(m\omega^2)\mathbf{E}$ (with constants of integration equal to zero) and therefore

$$\mathbf{P} = -\frac{nq^2}{m\omega^2}\mathbf{E}. \tag{114}$$

Hence the susceptibility is $\chi_{\rm e} = -nq^2/(\epsilon_0 m\omega^2)$ and the relative permittivity is

$$\epsilon_r = 1 - \frac{q^2 n}{\epsilon_0 m\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega^2}. \tag{115}$$

Meanwhile there is no magnetization. This is because the currents in the plasma are in straight lines: they do not form loops and hence they do not make magnetic dipoles. Therefore $\mu_r = 1$.

We can now obtain the refractive index from

$$v_{\rm ph} = \frac{c}{n_{\rm r}} = \frac{1}{\sqrt{\epsilon_0\epsilon_r\mu_0\mu_r}} \tag{116}$$

so

$$n_{\rm r} = \sqrt{\epsilon_r\mu_r} = \sqrt{1 - \omega_{\rm p}^2/\omega^2} \tag{117}$$

as before.

This method via polarization is equally good as the one via conductivity, but one should be careful not to muddle them. When using conductivity the current $\mathbf{j} = \sigma\mathbf{E}$ is the *total* current: the quantity that appears in (M4). It is **not** the quantity $\mathbf{j}_{\rm c}$ that appears in (M4a). Indeed we never invoked any of (M1a)–(M4a) when we did the calculation the first way.

In the treatment via polarization, by mentioning quantities such as relative permittivity we are implicitly invoking the equations (M1a)–(M4a). The whole of the current in the plasma is now ascribed to oscillations of dipoles so we must take $\mathbf{j}_{\rm c} = 0$ in (M4a).

When $\omega = \omega_{\rm p}$ we have $\epsilon_r = 0$. In the wave equation for $\mathbf{E}$ it is this vanishing of the permittivity that makes it possible for there to be longitudinal as well as transverse wave solutions. A zero permittivity does not occur in dielectrics or ohmic conductors, so they don't exhibit longitudinal waves.

4.5 Energy flow in a plasma

We will treat energy flow in a plasma in the case of transverse waves.

In the case $\omega > \omega_{\rm p}$ the field energy density is

$$\begin{aligned} u &= \frac{1}{2}\left(\epsilon_0(\mathrm{Re}\,(\mathbf{E}))^2 + \frac{1}{\mu_0}(\mathrm{Re}\,(\mathbf{B}))^2\right) \\ &= \frac{\epsilon_0}{2}(2-\omega_{\rm p}^2/\omega^2)E_0^2\cos^2(kz-\omega t) \end{aligned} \tag{118}$$

where we used $B = kE/\omega = (1-\omega_{\rm p}^2/\omega^2)^{1/2}E/c$. The Poynting vector is

$$\begin{aligned} \mathbf{S} &= \mathrm{Re}\,(\mathbf{E}) \times \mathrm{Re}\,(\mathbf{H}) \\ &= \epsilon_0 c\sqrt{1-\omega_{\rm p}^2/\omega^2}\,E_0^2\cos^2(kz-\omega t)\hat{\mathbf{z}} \end{aligned} \tag{119}$$

The power density is

$$\begin{aligned} p &= \mathrm{Re}\,(\mathbf{E}) \cdot \mathrm{Re}\,(\mathbf{j}) \\ &= -\frac{\omega_{\rm p}^2\epsilon_0}{\omega}E_0^2\cos(kz-\omega t)\sin(kz-\omega t). \end{aligned} \tag{120}$$

Therefore

$$\begin{aligned} \frac{\partial u}{\partial t} &= \epsilon_0\omega(2-\omega_{\rm p}^2/\omega^2)E_0^2\cos(kz-\omega t)\sin(kz-\omega_t) \\ \boldsymbol{\nabla}\cdot\mathbf{S} &= -2\epsilon_0\omega(1-\omega_{\rm p}^2/\omega^2)E_0^2\cos(kz-\omega t)\sin(kz-\omega_t) \end{aligned} \tag{121}$$

and one finds $(\partial u/\partial t)+\boldsymbol{\nabla}\cdot\mathbf{S}+p=0$ as required for energy conversation (recall the continuity equation (70)).

Let's consider now the time averages of the above quantities. For the power density the time average is $\langle p\rangle = 0$. This shows that whereas there is an oscillation in which energy passes between field and matter, there is no net energy transfer after averaging over a cycle, with the result that the waves propagate without losing energy. This is confirmed by

$$\langle \mathbf{S}\rangle = \frac{1}{2\mu_0 c}\sqrt{1-\omega_{\rm p}^2/\omega^2}\,E_0^2\,\hat{\mathbf{z}} \tag{122}$$

which is non-zero and independent of z.

The above applies to propagating waves, with $\omega > \omega_{\rm p}$. Next we treat the case of evanescent waves, with $\omega < \omega_{\rm p}$. We have $B = kE/\omega$ (from (M3) as usual) which gives

$$B = i(\alpha/\omega)E_0e^{-\alpha z}e^{-i\omega t} \tag{123}$$

where

$$\alpha = (\omega/c)\sqrt{\omega_{\rm p}^2/\omega^2-1}. \tag{124}$$

Therefore

$$u = \frac{1}{2}\epsilon_0 E_0^2 e^{-2\alpha z}\left(\cos^2(\omega t) + (\alpha c/\omega)^2 \sin^2(\omega t)\right), \tag{125}$$

$$\mathbf{S} = \frac{1}{2}\epsilon_0 \frac{\alpha c^2}{\omega} E_0^2 e^{-2\alpha z}\sin(2\omega t), \tag{126}$$

$$p = \frac{1}{2}\epsilon_0 \frac{\omega_{\rm p}^2}{\omega} E_0^2 e^{-2\alpha z}\sin(2\omega t). \tag{127}$$

In this case we find $\langle p \rangle = 0$ as before, but now $\langle \mathbf{S} \rangle = 0$ as well. The evanescent waves do not transport energy on average; in this respect they are like standing waves. This is connected to the fact that the expressions for $\mathbf{E}$ and $\mathbf{B}$ factorize into a part dependent on position and a part dependent on time.

4.5.1 Energy treatment via polarization

In the discussion of energy presented in (118)–(127) we have treated the current in the plasma in terms of conductivity. But we can also treat it in terms of polarization, as presented in section 4.4. In this case the physical behaviour is unchanged but it is expressed in terms of different quantities.

We have $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ with, $\epsilon_r = 1 - \omega_{\rm p}^2/\omega^2$ (eqn (115)). Hence

$$\begin{aligned} u &= (1/2)\left(\mathrm{Re}\,(\mathbf{E})\cdot \mathrm{Re}\,(\mathbf{D}) + \mathrm{Re}\,(\mathbf{B})\cdot \mathrm{Re}\,(\mathbf{H})\right) \\ &= \epsilon_0 \epsilon_r \mathrm{Re}\,(\mathbf{E})^2 \\ &= \epsilon_0 \epsilon_r E_0^2 \cos^2(kz - \omega t). \end{aligned} \tag{128}$$

Both magnetic and electric energy have been included; they contribute equally.

The Poynting vector is unchanged.

For the power density, since we have attributed the whole of the current to the oscillating polarization, we must take the conduction current to be $\mathbf{j}_c = 0$ and therefore the power density is

$$p = 0. \tag{129}$$

We now appear to have a discrepancy between (118) and (128), and also between (120) and (129). The reason is that the quantity called u in (118) only includes electric and magnetic field energy, whereas the quantity called u in (128) includes those and also polarization energy as well. Also the quantity called p in (129) includes all transfer of energy from field to matter, whereas the quantity called p in (129) only includes the part not involving polarization or magnetization. One finds

$$\frac{\partial u_{\rm 2nd}}{\partial t} = \frac{\partial u_{\rm 1st}}{\partial t} + p_{\rm 1st} \tag{130}$$

where the subscripts 1st and 2nd refer to the first and second methods of calculation. It follows that everything is consistent, but one must keeps ones wits about one when invoking either method of calculation.

5 Waves in a dielectric

For a pure dielectric medium—that is, one with no free charge and zero conductivity so no conduction current—the equations (M1a)–(M4a) simplify to

$$\nabla \cdot \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{131}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{132}$$

In the case of a linear isotropic medium, where $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, these further simplify to

$$\nabla \times \mathbf{E} = -\frac{1}{\mu}\frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{133}$$

where $\epsilon = \epsilon_0 \epsilon_r$ is the permittivity and $\mu = \mu_0 \mu_r$ is the permeability. By taking the curl of either of these equations one finds a wave equation in which the wave speed (phase velocity) is

$$v_{\rm ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \epsilon_r \mu_0 \mu_r}} = \frac{c}{\sqrt{\epsilon_r \mu_r}}. \tag{134}$$

From (M1a) and (M2a) one deduces that the waves are transverse, and from (M3a) one deduces that the amplitudes are related by $B = (k/\omega)E$ as usual.