2006, 8

(i) Treat neutrino as massless, then $p_\nu = E_\nu$ (with $c=1$).

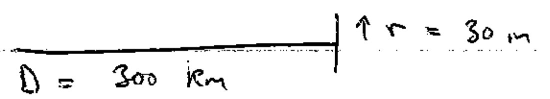
$$\begin{pmatrix} E_\nu^* \\ \underline{p}_\nu^* \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E_\nu \\ \underline{p}_\nu \end{pmatrix}$$

$$\Rightarrow E_\nu^* = \gamma(E_\nu - \beta p_\nu \cos\theta) = \underline{\gamma E_\nu (1-\beta\cos\theta)}$$

ii) $$p_x^* = p^* \cos\theta^* = \gamma(-\beta E_\nu + p_\nu \cos\theta)$$

$$\Rightarrow \cos\theta^* = \frac{\gamma E_\nu(-\beta + \cos\theta)}{\gamma E_\nu (1-\beta\cos\theta)}$$

$$= \frac{\cos\theta - \beta}{1-\beta\cos\theta}$$



$\theta \simeq r/D \ll 1$

$$\cos\theta \simeq 1 - \tfrac{1}{2}\theta^2$$

$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow 1-\beta^2 = \frac{1}{\gamma^2}$$

$$\Rightarrow \beta = (1 - 1/\gamma^2)^{\frac{1}{2}}$$

$$\simeq 1 - \tfrac{1}{2}\gamma^{-2} \quad \text{for } \gamma \gg 1$$

$$\Rightarrow \cos\theta^* \simeq \frac{1 - \frac{1}{2}\theta^2 - 1 + \frac{1}{2}\frac{1}{\gamma^2}}{1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{1}{2}\theta^2\right)}$$

$$= \frac{-\gamma^2\theta^2 + 1}{2\gamma^2 - (2\gamma^2 - 1)(1 - \theta^2/2)}$$

$$= \frac{1 - \gamma^2\theta^2}{1 + \gamma^2\theta^2 - \theta^2/2}$$

Detector receives up to angle $\theta \simeq \frac{r}{D} = \frac{30}{300\,000}$

and we have $\gamma = 100$ $\qquad = 10^{-4}$

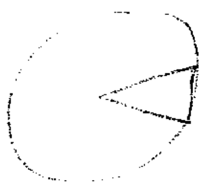
$$\Rightarrow \quad \cos\theta^* \simeq \frac{1 - (10^{-2})^2}{1 + (10^{-2})^2 - 10^{-8}/2}$$

$$\simeq (1 - 10^{-4})(1 - 10^{-4})$$

$$= 1 - 2 \times 10^{-4}$$

$$\simeq 1 - \tfrac{1}{2}\theta^{*2}$$

$$\Rightarrow \quad \theta^* = 2 \times 10^{-2}$$



collect fraction $\frac{\pi\,\theta^{*2}}{4\pi} = \frac{\theta^{*2}}{4}$

$= 10^{-4}$

2005

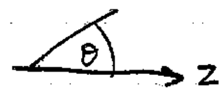
5.

$$E = \gamma m c^2$$
$$\underline{p} = \gamma m \underline{v}$$

$$\begin{pmatrix} E'/c \\ \underline{p}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$$

$$\pi \rightarrow \mu + \nu$$

$$E_\nu^* = \frac{m_\pi^2 + m_\nu^2 - m_\mu^2}{2 m_\pi} \qquad \text{as in } 2008 . 8$$



Now

$$E^*/c = \gamma \left(E/c - \beta p_z \right)$$
$$\Rightarrow E^* = \gamma E \left(1 - \beta pc \cos\theta \right)$$

but $E = pc$ for massless ν

$$\Rightarrow \qquad E_\nu^* = \gamma E_\nu (1 - \beta \cos\theta)$$
$$\Rightarrow$$
$$E_\nu = \frac{E_\nu^*}{\gamma(1-\beta\cos\theta_\nu)}$$

Max E_ν at min $1 - \beta\cos\theta_\nu$

i.e. $\theta_\nu = 0$

$$E_\nu^{max} = \frac{E_\nu^*}{\gamma(1-\beta)}$$
$$= \frac{\gamma E_\nu^*}{\gamma^2(1-\beta)} = \gamma E_\nu^* \frac{(1-\beta^2)}{1-\beta}$$
$$= \gamma E_\nu^* (1+\beta)$$

$$= \gamma \frac{(m_\pi^2 - m_\mu^2)}{2 m_\pi} (1+\beta)$$

(pion $\rightarrow \beta, \gamma$)

Now $E_\pi \gg m_\pi$ (taking $c = 1$)

$$\Rightarrow \gamma \gg 1$$
$$\Rightarrow \beta \simeq 1$$

so $E_\nu^{max} \propto \gamma(1+\beta) \propto \gamma \propto E_\pi$

Small θ:

$$E_\nu = \frac{E_\nu^* \gamma}{\gamma^2\left(1-\beta\left(1-\frac{\theta_\nu^2}{2}\right)\right)}$$

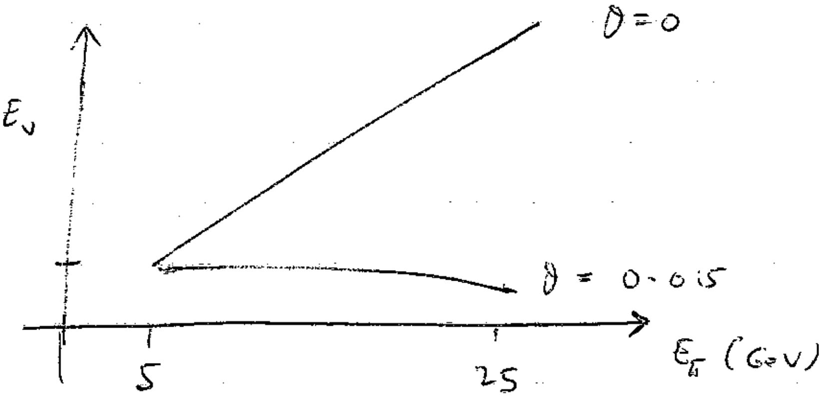
$$= \frac{2 E_\nu^* \gamma}{2\gamma^2(1-\beta) + \gamma^2\beta\theta_\nu^2}$$ with $\beta \simeq 1$

Now $\gamma^2(1-\beta) = \frac{1-\beta}{1-\beta^2} = \frac{1}{1+\beta} \simeq \frac{1}{2}$

$$\Rightarrow E_\nu \simeq \frac{2E_\nu^* \gamma}{1+\gamma^2\theta_\nu^2}$$

π : $m = 139.6$ MeV/c^2
μ : $m = 106$ MeV/c^2

5 GeV : $\gamma = \frac{5000}{140} = 36$, $\gamma\theta = 0.54$
25 GeV : $\gamma = 180$, $\gamma\theta = 2.7$



(headlight effect)

2009

8

a) $$\begin{pmatrix} E'/c \\ \underline{p}' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ \underline{x}' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ \underline{x} \end{pmatrix}$$

4-vectors:

$$P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix} , \quad X = \begin{pmatrix} ct \\ \underline{x} \end{pmatrix}$$

Square $P \cdot P \equiv P^T g P$ with metric

$$g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\Lambda^T g \Lambda = \begin{pmatrix} \cdots \end{pmatrix}$$

$$= g$$ as previously.

b) $$E_t = mc^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\simeq mc^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$= mc^2 + \frac{1}{2}mv^2$$

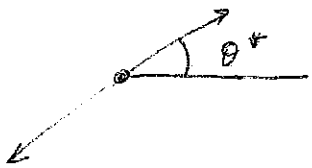
$$= E_{rest} + E_{kin}$$

Since E_{rest} is fixed, it merely shifts the zero of energy so has no dynamical consequence (for particles which retain their rest mass, i.e. do not decay).

For a massless particle we have

$$E^2 - p^2c^2 = 0$$

$$\Rightarrow \quad E = pc$$

c) 

here $*$ = rest frame

Take $c = 1$.

$$\begin{pmatrix} E_1 \\ \underline{p}_1 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & \\ \beta\gamma & \gamma & \\ & & 1 \; 1 \end{pmatrix} \begin{pmatrix} E_1' \\ \underline{p}_1' \end{pmatrix}$$

$$\Rightarrow E_1 = \gamma \left(E_1^* + \beta p_1^* \cos\theta^* \right)$$

By cons. of energy $E_1^* = m_A / 2$

$$E_1 = \gamma \frac{m_A}{2} \left(1 + \beta \cos\theta^* \right)$$

$$E_2 = \gamma \frac{m_A}{2} \left(1 \cancel{A*} - \beta \cos\theta^* \right)$$

Assume emission is isotropic in rest frame.
$\Rightarrow$ distributed as $\frac{1}{2} \sin\theta^*$

$$\Rightarrow \langle E_1 \rangle = \gamma \frac{m_A}{2} \int_0^{\pi} \frac{1}{2} \sin\theta^* + \frac{\beta}{2} \sin\theta^* \cos\theta^* \, d\theta^*$$

$$= \gamma \frac{m_A}{2} \left(\left[-\frac{1}{2} \cos\theta^* \right]_0^{\pi} + \frac{\beta}{2} \left[\frac{1}{2} \sin^2\theta^* \right]_0^{\pi} \right)$$

$$= \gamma \frac{m_A}{2}$$

similarly, $\langle E_2 \rangle = \gamma \frac{m_A}{2}$

consistent with energy conservation
since

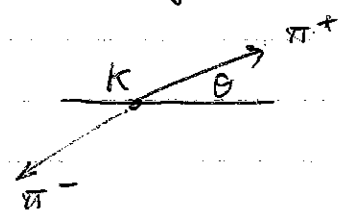
$$E_A = \gamma m_A$$

2007

8 $K \to \pi^+ \pi^-$

Equal masses $\Rightarrow$ cons. of momentum gives equal speeds $\Rightarrow$ equal γ.

K rest frame:



$\left(\overset{K}{\bullet}\!\longrightarrow \text{ in lab} \right)$

In K rest frame decay times are

$$\gamma_\pi t_+ \quad , \quad \gamma_\pi t_-$$

$\Rightarrow$ decay events are $\begin{pmatrix} \gamma_\pi t_+ \\ \gamma_\pi t_+ \beta_\pi \cos\theta \\ \gamma_\pi t_+ \beta_\pi \sin\theta \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \gamma_\pi t_- \\ -\gamma_\pi t_- \beta_\pi \cos\theta \\ -\gamma_\pi t_- \beta_\pi \sin\theta \\ 0 \end{pmatrix}$

(taking $c = 1$)

$\Rightarrow$ in lab frame (using inverse Lorentz transformation)

$$t_1 = \gamma_K \left(\gamma_\pi t_+ + \beta_K \gamma_\pi t_+ \beta_\pi \cos\theta \right)$$

$$t_2 = \gamma_K \left(\gamma_\pi t_- - \beta_K \gamma_\pi t_- \beta_\pi \cos\theta \right)$$

$$\Rightarrow t_1 - t_2 = \gamma_K \gamma_\pi \left(t_+ - t_- + \beta_K \beta_\pi \cos\theta \, (t_+ + t_-) \right)$$

$$= 0 \quad \text{when} \quad \cos\theta = \frac{-(t_+ - t_-)}{\beta_K \beta_\pi (t_+ + t_-)}$$

4-vector between the two events, in K rest frame:

$$\gamma_\pi \begin{pmatrix} (t_+ - t_-) \\ \beta_\pi \cos\theta \, (t_+ + t_-) \\ \beta_\pi \sin\theta \, (t_+ + t_-) \\ 0 \end{pmatrix}$$

Length^2 of this 4-vector is

$$\gamma_\pi^2\left(-(t_+ - t_-)^2 + \beta_\pi^2(\cos^2\theta + \sin^2\theta)(t_+ + t_-)^2\right)$$

$$= \gamma_\pi^2\left(\beta_\pi^2 (t_+ + t_-)^2 - (t_+ - t_-)^2\right)$$

= invariant.

In lab frame for simultaneous decays, this length is the distance between the events

$\Rightarrow$ distance is $c\,\gamma_\pi \sqrt{\beta_\pi^2 (t_+ + t_-)^2 - (t_+ - t_-)^2}$

not needed

Simplify

$$t_+^2 \, (\beta_\pi^2 - 1)$$
$$+ \, t_-^2 \, (\beta_\pi^2 - 1)$$
$$+ \, 2 t_+ t_- \, (\beta_\pi^2 + 1)$$

$$\beta_\pi^2 - 1 = -1/\gamma_\pi^2$$

$$\gamma_\pi^2(\beta_\pi^2 + 1) = \gamma_\pi^2\left(2 - \frac{1}{\gamma_\pi^2}\right) = 2\gamma_\pi^2 - 1$$

$$\Rightarrow d = c\sqrt{-(t_+^2 + t_-^2) + 2 t_+ t_- (2\gamma_\pi^2 - 1)}$$

S8

2004

1 we have $\underline{E}_{\parallel}' = E_{\parallel}$

$$\underline{E}_{\perp}' = \gamma (E_{\perp} + \underline{v} \wedge \underline{B})$$

$$B_{\parallel}' = B_{\parallel}$$

$$\underline{B}_{\perp}' = \gamma (\underline{B}_{\perp} - \underline{v} \wedge \underline{E}/c^2)$$

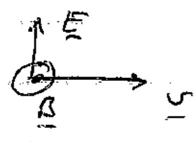
if $D > 0$ can make $B = 0$:

choose $v \perp B$ then $B_{\parallel}' = B_{\parallel} = 0$

choose

$$\underline{v} \wedge \underline{E} = c^2 \underline{B}$$

e.g. $v = c^2 B / E$



we have $E^2/c^2 - B^2 > 0$

$$\Rightarrow \quad E^2 > c^2 B^2$$

$$\Rightarrow \quad E > cB$$

$\Rightarrow \quad v < c$, $\Rightarrow$ possible.

If $D < 0$ then make $E = 0$:

choose $\underline{v} \wedge \underline{B} = - \underline{E}$

e.g. $v = E/B$

$< c$