Lagrangian methods for fields

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This is not a thorough introduction to Lagrangian methods, it is simply some notes on a few issues.

1 Derivation of Euler-Lagrange for a single field

Suppose we have a Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}\left(\phi,\ \boldsymbol{\nabla}\phi,\ \partial_t\phi\right) \tag{1}$$

This expression asserts that $\mathcal{L}$ is an explicit function of a field ϕ, its spatial gradient $\boldsymbol{\nabla}\phi$ and its time derivative $\partial\phi/\partial t$. The notation implies that this $\mathcal{L}$ does not depend on time or spatial coordinates except indirectly through the way that the field $\phi(t, x, y, z)$ may depend on those quantities.

It is important to be clear that when we write $\boldsymbol{\nabla}\phi$ and $\partial_t\phi$ in (1) we mean the following:

$$\boldsymbol{\nabla}\phi = \left.\boldsymbol{\nabla}\phi\right|_t, \qquad \partial_t\phi = \left(\frac{\partial\phi}{\partial t}\right)_{x,y,z} \tag{2}$$

That is, in the first expression the gradient is understood to be taken at fixed t, and in the second expression the time derivative is at constant x, y, z. It is important to be clear about this because later on we may encounter things like a time derivative at constant ϕ or $\boldsymbol{\nabla}\phi$, which is not what we have here.

The Lagrangian of the system is the volume integral of the Lagrangian density:

$$L = \int \mathcal{L}\, \mathrm{d}^3\mathbf{r} \tag{3}$$

(where $\mathrm{d}^3\mathbf{r}$ is a shorthand for the volume element $\mathrm{d}x\mathrm{d}y\mathrm{d}z$). The action is given by the time-integral of the Lagrangian along some given path (more on this path in a moment):

$$S = \int_{t_A}^{t_B} L\, \mathrm{d}t. \tag{4}$$

The path along which the time integral is performed is stipulated by giving the value of $\phi(t, x, y, z)$ at each time t, for all of space. This is a strange sort of path: to imagine it, you have to imagine a plot of ϕ verses time at each x, y, z and it is a path on that graph, but there is a continuous infinity of such graphs: one at each x, y, z. Note, we do not need to stipulate either $\boldsymbol{\nabla}\phi$ or $\partial_t\phi$ in addition, because $\boldsymbol{\nabla}\phi$ is already given by specifying ϕ at all locations (at any given time), and $\partial_t\phi$ is also stipulated by specifying ϕ at all times. A good way to take note of these facts is to specify the path as follows. First we suppose that one particular path is set by some function $\phi(t, x, y, z)$, and then we note that other paths can be specified as

$$\phi + \delta\phi \tag{5}$$

where $\delta\phi$ is some function of (t, x, y, z). In the Euler-Lagrange method we will be interested in cases where $\delta\phi$ is small.

Now let's examine the Lagrangian density when the field is given by (5). We have

$$\mathcal{L}\left(\phi + \delta\phi,\ \boldsymbol{\nabla}(\phi + \delta\phi),\ \partial_t(\phi + \delta\phi)\right) = \mathcal{L}\left(\phi + \delta\phi,\ \boldsymbol{\nabla}\phi + \delta(\boldsymbol{\nabla}\phi),\ \partial_t\phi + \delta(\partial_t\phi)\right) \tag{6}$$

where we introduced the definitions

$$\delta(\boldsymbol{\nabla}\phi) \equiv \boldsymbol{\nabla}(\delta\phi), \qquad \delta(\partial_t\phi) \equiv \partial_t(\delta\phi). \tag{7}$$

Often you see the notation $\partial_i\phi$ instead of $\boldsymbol{\nabla}\phi$. When we have lots of δ and ∂ symbols around, the notation can become confusing. You might find it helpful to introduce the symbols

$$u \equiv \frac{\partial\phi}{\partial t}, \qquad \mathbf{v} \equiv \boldsymbol{\nabla}\phi \tag{8}$$

and then the right hand side of (6) can be written

$$\mathcal{L}(\phi + \delta\phi,\ \mathbf{v} + \delta\mathbf{v},\ u + \delta u). \tag{9}$$

You can think of u and $\mathbf{v}$ as a set of 'velocities' of the field.

Now we are ready to proceed with the argument concerning least action. The idea is that the path specified by ϕ shall be the one giving a stationary value (often a minimum, but it can be a maximum) of the action S with respect to variations $\delta\phi$. So with this in mind, we assume $\delta\phi$ is small and we expand $\mathcal{L}$ in a Taylor series expansion up to first order:

$$\mathcal{L}(\phi + \delta\phi,\ \mathbf{v} + \delta\mathbf{v},\ u + \delta u) \simeq \mathcal{L}(\phi,\ \mathbf{v},\ u) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\mathbf{v}}\cdot\delta\mathbf{v} + \frac{\partial\mathcal{L}}{\partial u}\delta u \tag{10}$$

where you should understand that

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &\quad \text{means} \quad \left(\frac{\partial\mathcal{L}}{\partial\phi}\right)_{\mathbf{v},u} \\
\frac{\partial\mathcal{L}}{\partial\mathbf{v}} &\quad \text{means} \quad \text{the vector with components } \left(\frac{\partial\mathcal{L}}{\partial v_i}\right)_{\phi,v_{j\neq i},u} \\
\frac{\partial\mathcal{L}}{\partial u} &\quad \text{means} \quad \left(\frac{\partial\mathcal{L}}{\partial u}\right)_{\phi,\mathbf{v}}
\end{aligned}$$

Ok, now we can write that the change in the action brought about by the field variation $\delta\phi(t, x, y, z)$ is

$$\begin{aligned} \delta S &= \int_{t_A}^{t_B} \mathrm{d}t \int \mathrm{d}^3\mathbf{r}\, \left[\mathcal{L}(\phi + \delta\phi,\ \mathbf{v} + \delta\mathbf{v},\ u + \delta u) - \mathcal{L}(\phi\ \mathbf{v},\ u)\right] & (11) \\ &\simeq \int_{t_A}^{t_B} \mathrm{d}t \int \mathrm{d}^3\mathbf{r}\, \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \delta\mathbf{v} + \frac{\partial \mathcal{L}}{\partial u}\delta u\right] & (12) \end{aligned}$$

by using (10). You often see this expression written with an equals sign. This is because we shall eventually take the limit where the variation tends to zero, so the 2nd and higher order terms of the Taylor expansion will be negligible. On the right hand side of (12) there is a term involving $\delta\phi$ which we shall leave as it is, and terms involving $\delta\mathbf{v}$ and δu which we can develop by making use of (7):

$$\int \mathrm{d}t \frac{\partial \mathcal{L}}{\partial u}\delta u = \int \mathrm{d}t \frac{\partial \mathcal{L}}{\partial u}\left(\frac{\partial(\delta\phi)}{\partial t}\right)_{x,y,z} \tag{13}$$

Now perform an integration by parts, not forgetting the limits on the integral. We thus obtain

$$\left[\frac{\partial \mathcal{L}}{\partial u}\delta\phi\right]_{t_A}^{t_B} - \int_{t_A}^{t_B} \mathrm{d}t \frac{\partial^2 \mathcal{L}}{\partial t \partial u}\delta\phi. \tag{14}$$

At this point we bring in the fact that the variation $\delta\phi$ is chosen in such a way that it shall be strictly zero at the two limits t_A and t_B, so the first term here is zero and we are left with just the second term (the integral). By similar reasoning one also finds

$$\int_{t_A}^{t_B} \mathrm{d}t \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \delta\mathbf{v} = -\int_{t_A}^{t_B} \mathrm{d}t \left(\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right)\delta\phi. \tag{15}$$

(and see the footnote[1]). Next we substitute the results (14) and (15) into (12), giving

$$\delta S \simeq \int_{t_A}^{t_B} \int \mathrm{d}^3\mathbf{r} \left[\frac{\partial \mathcal{L}}{\partial \phi} - \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \partial_t \frac{\partial \mathcal{L}}{\partial u}\right]\delta\phi \tag{16}$$

where the expression is exact in the limit $\delta\phi \to 0$. Now we can note that the integral has to come out zero no matter what form the variation $\delta\phi$ may take (provided only that it is small), so we can infer that for this to happen the integrand must itself be zero. Hence we obtain

Euler-Lagrange equation for a single field

$$\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}} + \partial_t \frac{\partial \mathcal{L}}{\partial u} = \frac{\partial \mathcal{L}}{\partial \phi} \tag{17}$$

[1] In index notation, with the summation convention, the term in the bracket here would be written

$$\left(\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right) = \partial_i \frac{\partial \mathcal{L}}{\partial v_i}$$

This completes the calculation to obtain the Euler-Lagrange equation. I shall now present the result in two further versions, by modest changes of notation. If we introduce a quantity u_α having four values for the index, such that

$$u_0 \equiv \frac{\partial \phi}{\partial t}, \qquad u_i \equiv \partial_i \phi$$

then the Euler-Lagrange equation can be written

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial u_\mu} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \tag{18}$$

where the notation on the left implies a sum over the four terms $\mu = 0, 1, 2, 3$. Finally, you often see it written the following way:

Euler Lagrange equation (again)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}. \tag{19}$$

Equations (17), (18), (19) are all precisely the same, merely written in terms of different symbols. Speaking for myself, I find (17) and (18) helpful as a way to remind myself precisely what (19) means, with no mistake.

Exercise. What is held constant in each partial derivative in (17)?

Exercise. (harder) In the particle case we have that two Lagrangians differing by a total time derivative describe the same physics. Show that the corresponding result for Lagrangian densities is that one can add a 4-divergence to a Lagrangian density:

$$\mathcal{L} \to \mathcal{L} + \partial_\mu f^\mu \tag{20}$$

without changing the physics (i.e. the path of stationary action is not changed), under the assumption that f vanishes sufficiently quickly at spatial infinity. [Hint: Gauss' theorem in 4 dimensions.]

1.1 Intuition

Let's try to get some physical/mathematical intuition about our result. The Euler-Lagrange equation for a classical dynamical system described by coordinates $\mathbf{q}$ and velocities $\dot{\mathbf{q}}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \tag{21}$$

where the index i picks out one of the coordinates and its associated velocity (the set of q_i and $\dot{q}_i$ make up $\mathbf{q}$ and $\dot{\mathbf{q}}$), and $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the Lagrangian.

This equation has a physical interpretation as (rate of change of)(generalized momentum) = (generalized force)

and the total derivative with respect to time indicates that the derivative is to be taken along the path followed by the system.

The EL equation for a field shares most the features of the one for a particle or set of particles: there is a partial derivative of $\mathcal{L}$ with respect to a 'velocity' (i.e. $\partial_\mu \phi$) and a partial derivative of $\mathcal{L}$ with respect to a 'coordinate' (i.e. the field ϕ). The main difference is that whereas the particle result has a total derivative with respect to time, the field result has a four-divergence (that is, the differential operation ∂_μ acting on each part of $\partial\mathcal{L}/\partial u_\mu$ and a sum is taken). Recall that, in the field case, the 'path' followed by the system is not a single line in coordinate space, but an infinite set of lines, one at each x, y, z. So the 'path' is a 4-dimensional quantity. This may help to remind us that in the field case we need to take a 4-dimensional derivative along the 'path'. The equation can then be read **loosely** as

(4-divergence of)(momentum density) = (force density)

One observes that terms in $\mathcal{L}$ involving ϕ and not its derivatives are to do with potential energy, and terms involving $\partial_t \phi$ but not ϕ itself are to do with kinetic energy.

2 A Lagrangian yielding Schrödinger's equation

Consider now a Lagrangian involving two fields, which we shall call A and B. Each is a scalar field, but we allow that it may be complex-number-valued. Suppose we have a Lagrangian density

$$\mathcal{L} = \mathcal{L}(A,\ \nabla A,\ \partial_t A,\ B,\ \nabla B,\ \partial_t B) \tag{22}$$

This statement is needed in order that it shall be clear in subsequent expressions what is being held constant in any given partial derivative. We consider the case where $\mathcal{L}$ is given by the expression

$$\mathcal{L} = -\frac{\hbar^2}{2m}\nabla A \cdot \nabla B - VAB + i\hbar\left(B\partial_t A - A\partial_t B\right) \tag{23}$$

where $V(t, x, y, z)$ is some function. With two fields there are two Euler-Lagrange equations:

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_A} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}_A} = \frac{\partial \mathcal{L}}{\partial A}, \tag{24}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_B} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}_B} = \frac{\partial \mathcal{L}}{\partial B}. \tag{25}$$

where I introduced $u_A = \partial_t A$, $\mathbf{v}_A = \boldsymbol{\nabla} A$ and similarly for u_B, $\mathbf{v}_B$. Let's first evaluate the first of these. We have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u_A} = \frac{\partial \mathcal{L}}{\partial(\partial_t A)} &= i\hbar B, & (26) \\ \frac{\partial \mathcal{L}}{\partial \mathbf{v}_A} = \frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla} A)} &= \frac{-\hbar^2}{2m}\boldsymbol{\nabla} B, & (27) \\ \frac{\partial \mathcal{L}}{\partial A} &= -VB. & (28) \end{aligned}$$

(The first and last of these are straightforward if you are clear on what is being held constant in each case; the middle one may take a little thought.) Hence the Euler-Lagrange equation reads

$$\frac{\partial}{\partial t}(i\hbar B) + \boldsymbol{\nabla}\cdot\left(-\frac{\hbar^2}{2m}\boldsymbol{\nabla} B\right) = -VB \tag{29}$$

which gives

$$-\frac{\hbar^2}{2m}\nabla^2 B + VB = -i\hbar\frac{\partial B}{\partial t}. \tag{30}$$

Note that this equation only includes one of the fields! This is rather convenient (it won't happen for all Lagrangians). The calculation for field A is now easy since it goes the same apart from the sign in front of i, so we get

$$-\frac{\hbar^2}{2m}\nabla^2 A + VA = i\hbar\frac{\partial A}{\partial t}. \tag{31}$$

This is the Schrödinger equation, and furthermore we can note that whenever A satisfies this equation, then the function $B = A^*$ will satisfy (30). In other words, *after* completing the analysis as far as getting Euler-Lagrange equations, we can think of A and B as a field and its complex conjugate, so in a sense we have just one field not two in the end. This is why you often see (23) written in the form

$$\mathcal{L} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi^* - V\psi\psi^* + i\hbar\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right). \tag{32}$$

This equation should *not* be understood as a complete statement of the Lagrangian function. Rather, it shows the way that function can be expressed *at locations along the solution path* (the one satisfying EL equations), in the case where we have taken $B = A^*$.

One should take a moment to consider what we just did. We took it that the fields A and B were complex-number-valued. They are not operators, so we are not doing quantum field theory. We are doing classical field theory. But one can't help noticing that the equation we arrived at is the one which applies to wavefunctions in quantum mechanics (in its non-relativistic formulation). Are we doing quantum mechanics after all then? The answer is: no, not really. We are just observing some mathematical facts which link a Lagrangian density with an equation of motion for a field, if we assume the field behaves so as to give a stationary value of the action with respect to variation of the field. That has nothing to do with quantum mechanics except that the field theory gave a partial differential equation which we happen to know also appears in quantum mechanics.

3 Energy conservation for a field

In particle mechanics, energy conservation can be related to the Lagrangian via the result

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} \tag{33}$$

where $H = -L + \sum_i p_i \dot{x}_i$ (where $p_i = \partial L / \partial \dot{x}_i$). This gives us the hint that field energy ought to be related to the Lagrangian in a similar way, and its conservation will be related to an absence of time-dependence in the Lagrangian. In the following we will make these remarks more precise.

In the field case, energy conservation takes the form of a continuity equation.

We will show that if a Lagrangian density has no explicit time-dependence, then there is a scalar conserved quantity. By mutual agreement this quantity is called energy. The proof includes not just global energy conservation but also local conservation. Local conservation means that the change in total energy in any region matches the net flow of energy into that region. This means we are looking for two quantities not just one: energy density ρ and energy flux $\mathbf{N}$. Conservation of energy is expressed by

Continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{N} = 0. \tag{34}$$

Here is how it is done for the case of a single field.

We consider a Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}\left(\phi,\ \boldsymbol{\nabla}\phi,\ \partial_t \phi,\ t\right) \tag{35}$$

where the partial derivatives expressed by $\boldsymbol{\nabla}$ and ∂_t are as presented in (2). Note that we allow here that the Lagrangian density might be an explicit function of time. We then have the total derivative

$$\mathrm{d}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\mathrm{d}\phi + \frac{\partial \mathcal{L}}{\partial u}\mathrm{d}u + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \mathrm{d}\mathbf{v} + \left(\frac{\partial L}{\partial t}\right)_{\phi,u,\mathbf{v}} \mathrm{d}t \tag{36}$$

where for convenience we introduced u and $\mathbf{v}$ as defined in (9). We would like to consider the change in $\mathcal{L}$ for a small change in t while holding (x, y, z) constant. By taking the above equation and dividing by $\mathrm{d}t$ at constant (x, y, z) we obtain

$$\left(\frac{\partial \mathcal{L}}{\partial t}\right)_{x,y,z} = \frac{\partial \mathcal{L}}{\partial \phi}\partial_t \phi + \frac{\partial \mathcal{L}}{\partial u}\partial_t u + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot (\partial_t \mathbf{v}) + \left(\frac{\partial L}{\partial t}\right)_{\phi,u,\mathbf{v}} \tag{37}$$

where, throughout, the symbol ∂_t means differentiation at constant (x, y, z). (It is, of course, important to be clear about the distinction between this and constant $\phi, u, \mathbf{v}$.) Next we use $\partial_t \phi = u$ by definition, and also

$$\partial_t \mathbf{v} = \partial_t \boldsymbol{\nabla}\phi = \boldsymbol{\nabla}(\partial_t \phi) = \boldsymbol{\nabla} u, \tag{38}$$

so (37) can be written

$$\partial_t \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} u + \frac{\partial \mathcal{L}}{\partial u} \partial_t u + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot (\boldsymbol{\nabla} u) + \left(\frac{\partial L}{\partial t} \right)_{\phi, u, \mathbf{v}} . \tag{39}$$

Now use the Euler-Lagrange equation to replace the $\partial \mathcal{L}/\partial \phi$ term:[2]

$$\begin{aligned} \partial_t \mathcal{L} &= \left(\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}} + \partial_t \frac{\partial \mathcal{L}}{\partial u} \right) u + \frac{\partial \mathcal{L}}{\partial u} \partial_t u + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot (\boldsymbol{\nabla} u) + \left(\frac{\partial L}{\partial t} \right)_{\phi, u, \mathbf{v}} \\ &= \partial_t \left(\frac{\partial \mathcal{L}}{\partial u} u \right) + \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}} u \right) + \left(\frac{\partial L}{\partial t} \right)_{\phi, u, \mathbf{v}} . \end{aligned} \tag{40}$$

(where we used that the two terms involving $\mathbf{v}$ make the derivative of a product, and similarly for u).

Finally, an insightful way to present this result is:

$$\boxed{ -\left(\frac{\partial L}{\partial t} \right)_{\phi, u, \mathbf{v}} = \partial_t \left(\frac{\partial \mathcal{L}}{\partial u} u - \mathcal{L} \right) + \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}} u \right) . } \tag{41}$$

By comparing this with the continuity equation (34) we find that if we introduce the *definitions*

$$\rho \equiv \frac{\partial \mathcal{L}}{\partial u} u - \mathcal{L}, \tag{42}$$

$$\mathbf{N} \equiv \frac{\partial \mathcal{L}}{\partial \mathbf{v}} u \tag{43}$$

then we have that, in the case where $\mathcal{L}$ has no explicit dependence on time, i.e. when

$$\left(\frac{\partial L}{\partial t} \right)_{\phi, u, \mathbf{v}} = 0 \tag{44}$$

then ρ and $\mathbf{N}$ satisfy the continuity equation (34), and therefore ρ is the density and $\mathbf{N}$ the flux of a scalar conserved quantity. That quantity we choose to call *energy*.

The total energy in the field, at any given time, is

$$H = \int \mathrm{d}^3\mathbf{r}\, \rho = \int \mathrm{d}^3\mathbf{r} \left[\frac{\partial \mathcal{L}}{\partial u} u - \mathcal{L} \right] . \tag{45}$$

This expression has obvious similarities with the relation between hamiltonian and lagrangian in particle mechanics.

The above discussion is sufficient for a consideration of energy conservation when $\mathcal{L}$ has no explicit dependence on t. More generally, one can have a case where $\mathcal{L}$ does have a t-dependence, but such as to have some more general symmetry, for example such that the action does not change under a transformation $t \to t' = t + \tau$ for some function $\tau(t, \mathbf{x})$. To study this case requires a longer argument involving action integrals, in which one allows that τ may depend on time.

[2] In section 1 we derived the EL equation for a Lagrangian with no explicit time-dependence. The reader may wish to revisit the derivation allowing for a time-dependence; one finds that this does not change the result.

4 Momentum conservation

Linear momentum conservation is related to symmetry with respect to displacement in space. In this section we will not explore the most general such symmetry. We will treat the simplest case, which is the one where $\mathcal{L}$ is unaffected by spatial translation. Consider, for example, the case

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_{t,y,z} = 0 \tag{46}$$

The calculation closely matches the one for energy. By starting from (36) we obtain

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_{t,y,z} = \frac{\partial \mathcal{L}}{\partial \phi}\partial_x \phi + \frac{\partial \mathcal{L}}{\partial u}\partial_x u + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot (\partial_x \mathbf{v}) \tag{47}$$

By now using the Euler-Lagrange equation to replace $\partial\mathcal{L}/\partial\phi$ and also noting $\partial_x u = \partial_t \partial_x \phi$ we find

$$\begin{aligned}\left(\frac{\partial \mathcal{L}}{\partial x}\right)_{t,y,z} &= \left(\boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}} + \partial_t \frac{\partial \mathcal{L}}{\partial u}\right)\partial_x \phi + \frac{\partial \mathcal{L}}{\partial u}\partial_t \partial_x \phi + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot (\partial_x \boldsymbol{\nabla}\phi) \\ &= \partial_t \left(\frac{\partial \mathcal{L}}{\partial u}\partial_x \phi\right) + \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\partial_x \phi\right)\end{aligned} \tag{48}$$

Hence when $\partial_x \mathcal{L} = 0$ then there is a conserved quantity whose density is

$$\frac{\partial \mathcal{L}}{\partial u}\partial_x \phi \tag{49}$$

and whose flux is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\partial_x \phi \tag{50}$$

The conserved quantity is called the momentum in the x direction. Similar arguments for $\partial\mathcal{L}/\partial y$ and $\partial\mathcal{L}/\partial z$ give two further components, so the complete momentum density is

$$\frac{\partial \mathcal{L}}{\partial u}\boldsymbol{\nabla}\phi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)}\boldsymbol{\nabla}\phi. \tag{51}$$

The volume integral of this quantity is the complete momentum in the field.

5 Postscript

These notes have merely introduced some of the ideas and methods of field theory. We have mostly restricted attention to a single scalar field. The conservation of energy and momentum is an important concept, even for a single field, but it does not have much physical impact unless we are discussing two or more fields which can interact with one another. In that case one may have a Lagrangian for all the fields together, and then energy and momentum may move between one field and another. For example, classical electrodynamics can be treated using the Lagrangian density

$$\mathcal{L} = -A_\mu j^\mu - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{52}$$

where j^μ is the charge four-current, A^μ the four-potential, $F_{ab} = \partial_a A_b - \partial_b A_a$ the field tensor. It is not obvious, simply from looking at this Lagrangian, whether or not it has a symmetry with respect to spatial or temporal translations. But if one assumes that it does (or if one insists that it must) then one will arrive at equations relating the divergence of a tensor associated with the electromagnetic field to a quantity which can be interpreted as the rate of doing work (per unit volume) on the charges. *This expresses the conservation of energy and momentum in classical electrodynamics.* However a presentation of this derivation is beyond the scope of these notes.