# Cold atoms in optical lattices from a CMP perspective: exercises

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1. Consider a single particle moving in one dimension, in a strong optical lattice potential plus harmonic trap. (This is the case we considered in Thursday's lecture; its Hamiltonian is $H = -t\sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right) + \frac{1}{2}\kappa a^2 \sum_j j^2 c_j^\dagger c_j$.) For large energies ($E \gg t$), the kinetic term may be neglected.

   (a) Show that, in this limit, the density of states has the form

   $$\nu(E) \sim \frac{1}{\sqrt{E}} \qquad\qquad E \gg t.$$

   (b) What is the analogous result in $d$ dimensions, where $d > 1$?

2. (a) Prove the following recursion relation for the density of states of a separable[1] system:

   $$\nu_d(E) = \int_{-\infty}^{\infty} \nu_{d-1}(E-\chi)\nu_1(\chi)d\chi,$$

   where $\nu_d(E)$ is the density of states of the $d$-dimensional system.

   (b) Hence, given the form of $\nu_1(E)$ shown in Fig. 1, sketch the densities of states of the $d = 2$ and $d = 3$ systems. Carefully identify the energies at which any significant features occur, and check that your results agree with the limiting $E \gg t$ forms identified above.

3. Consider a tight-binding particle moving in one dimension under the influence of the trapping potential

   $$V(x) = C|x|.$$

   (a) Obtain the semiclassical orbits, $p(x)$, for this problem.

   (b) Hence construct the WKB quantisation condition, and differentiate it with respect to energy to obtain the density of states, $\nu(E)$. Sketch your result.

   (c) What is the behaviour of $\nu(E)$ for $E \gg t$? Comment on your answer.

[1] A system is separable if its Hamiltonian can be divided into a sum of parts, each of which refers to only one spatial dimension. The $d$-dimensional harmonic oscillator is separable, as is the $d$-dimensional tight-binding model on a hypercubic lattice.

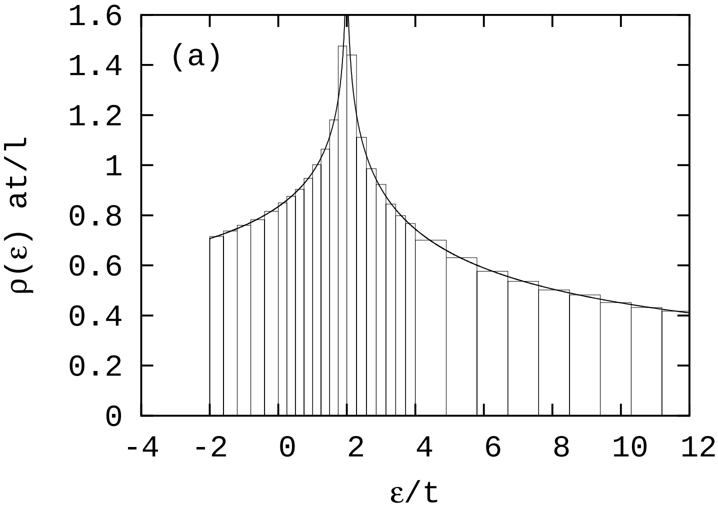


Figure 1: The density of states for a single tight-binding particle moving in a harmonic trapping potential. Solid line: analytical result. Boxes: coarse-grained density of states from numerical solution.

4. Consider a system that undergoes a metal-insulator transition as a parameter, $x$, is tuned through zero. The system is metallic for $x < 0$, and insulating for $x > 0$, with a gap $\Delta(x) = A\sqrt{x}$, where $A$ is a constant. The density of states, $\nu(E)$, is approximated as

$$\nu(E) = \begin{cases} \nu_0 & E > \Delta, \\ 0 & E < \Delta, \end{cases}$$

where $\nu_0$ is a constant.

   (a) Assuming that the insulator breaks down (i.e. reverts to metallic behaviour) for $k_B T > \Delta$, draw the $x - T$ phase diagram of this system.

   (b) A crude estimate of the entropy at a temperature $T$ is $S = k_B \ln \Omega$, where $\Omega$ is the number of excitations with energy $E \leqslant k_B T$. Using this approximation, obtain an expression for the lines of constant entropy $T_S(x)$, and sketch them on your phase diagram. Comment on their behaviour in or near the insulating phase.