Rotating Bose-Einstein condensates in the Lowest Landau Level

JD

September 23, 2008

We consider a 2D Bose gas of N atoms of mass M, confined in an isotropic harmonic potential $V(r) = m\omega^2 r^2/2$, with $r = (x^2 + y^2)^{1/2}$. We use $\hbar$, ω, $\hbar\omega$, $\sqrt{\hbar/(m\omega)}$ as units of angular momentum, frequency, energy and length, respectively. We are interested here in the properties of the system when it is rotating at a frequency Ω close to the trapping frequency ω.

1 Single particle problem

1. Consider the single particle Hamiltonian $\hat{H} = \hat{p}^2/(2M) + \hat{V}(r)$. Show that the energies can be written $E_n = n + 1$ (n non negative integer) and that the dimensions of the associated eigenspace $\mathcal{E}_n$ is $g_n = n + 1$.

2. (lengthy! One can admit the result and continue...)
   Show that one can find a basis of each subspace $\mathcal{E}_n$ that is also an eigenbasis of $\hat{L}_z$, where $\hat{L}_z$ is the z component of the angular momentum operator, with the eigenvalues $L_z = -n, -n+2, \ldots, n-2, n$. Note that $\hat{L}_z = -i\partial_\theta$, where θ is the polar angle in the xy plane.
   Hint 1: consider the (non-normalized) states
   $$\phi_{j,k}(x,y) = e^{r^2/2}\, (\partial_x + i\partial_y)^j\, (\partial_x - i\partial_y)^k \left(e^{-r^2}\right) \tag{1}$$
   where j and k are non-negative integers.
   Hint 2: in case of difficulty, just restrict to the subspaces $\mathcal{E}_0$ and $\mathcal{E}_1$.

3. Show that $(x+iy)^n e^{-r^2/2}$ has the energy $n+1$ and the angular momentum $L_z = n\hbar$.

4. The single particle Hamiltonian in the rotating frame is $\hat{H}' = \hat{H} - \Omega \hat{L}_z$. Give the eigenvalues of $\hat{H}'$ as a function of Ω.

5. What happens for $\Omega = 1$, i.e. a rotation frequency equal to the trapping frequency? Show in particular that the eigenstates form infinitely degenerate manifolds (Landau levels), separated by the energy 2 (i.e. $2\hbar\omega$ in real life).

6. Give the mathematical structure of a state in the Lowest Landau Level.

7. (lengthy! One can admit the result and continue...)
   Show that for a state in the LLL one has the following identity
   $$\langle E_{\rm kin}\rangle = \langle E_{\rm ho}\rangle = \frac{1}{2} + \frac{1}{2}\int \psi^* \left[L_z\psi\right]\, d^2r \tag{2}$$
   where the kinetic and harmonic oscillator energies are:
   $$\langle E_{\rm kin}\rangle = \frac{1}{2}\int |\vec{\nabla}\psi|^2\, d^2r \qquad \langle E_{\rm ho}\rangle = \frac{1}{2}\int r^2\, |\psi|^2\, d^2r\ . \tag{3}$$

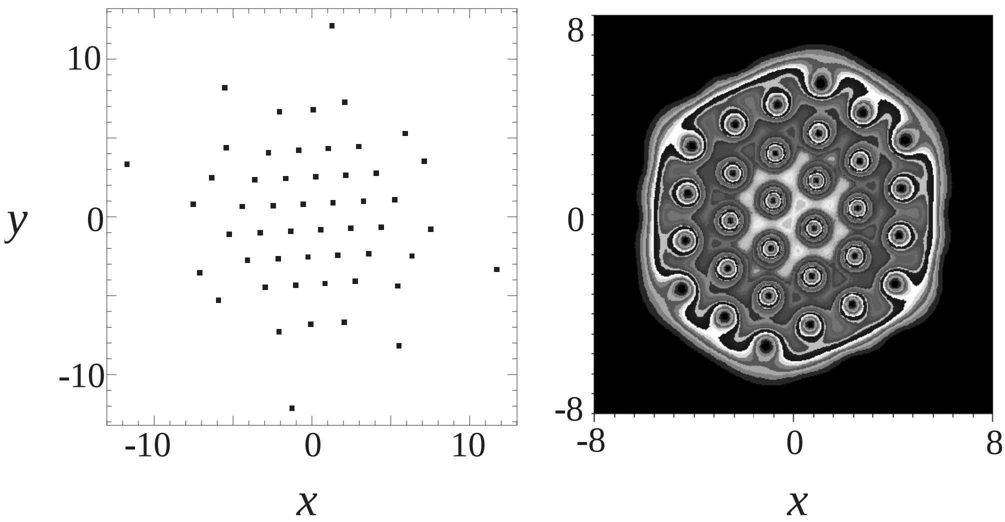

Figure 1: Structure of the ground state of a rotating Bose-Einstein condensate described by a LLL wavefunction ($\Lambda = 3000$). (a) Vortex location; (b) Atomic density profile (with a larger scale). The unit for the positions x and y is $(\hbar/(m\omega))^{1/2}$.

2 A mean-field analysis of the gas

In the following $\tilde{g}$ denotes the dimensionless 2D coupling strength. We choose Ω slightly smaller than ω.

1. Explain the reason for this choice of Ω.
2. Write the energy functional to be minimized for the determination of the ground state.
3. Show that if one restricts to the LLL, the minimization procedure depends only on the parameter
$$\Lambda = \frac{N\tilde{g}}{1-\Omega} \ . \qquad (4)$$
An example of minimization is shown in fig.1 for $\Lambda = 3000$.
4. Once the state ψ minimizing the energy functional is known, how can one find the locations of the vortices?
5. Give an estimate of the radius of the cloud using a coarse graining over the vortex structure.

Reference: A. Aftalion, X. Blanc, and J. Dalibard, Phys. Rev. A **71**, 023611 (2005).