Low dimensional quantum gases, rotation and vortices


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Goal of these lectures
Discuss some aspect of the physics of quantum low dimensional systems

Planar fluids
Quantum wells and MOS structures
High $T_c$ superconductivity

Discuss also the effect of quantum rotation

Neutron stars
Superconductor in a magnetic field

VORTICES

Outline of the lectures

1. Static 2D gases
   Normal to superfluid transition
   Berezinski-Kosterlitz-Thouless mechanism: breaking of vortex pairs

2. Rotating 2D gases
   Vortices are the “simplest way” to set a quantum system in rotation.
   Physics in the Lowest Landau Level and Connection with Quantum Hall phenomenon

Chapter 1: Static 2D gases

§1. The ideal Bose gas in 2D
The ideal uniform 2D Bose gas

Square box \( L \times L \)

Eigenstates: plane waves of momentum: \( \vec{p} = \hbar \vec{k} \)

\[
E_k = \frac{\hbar^2 k^2}{2m} \quad \vec{k} = \frac{2\pi}{L} (n_x, n_y)
\]

Is there a BEC at finite temperature? No!

\[
N = \sum_k n(E_k) \quad n(E) = \frac{1}{\exp [(E - \mu)/k_B T] - 1}
\]

\[
N \approx \frac{L^2}{4\pi^2} \int n(E_k) \, d^2k = -\frac{L^2}{\lambda^2} \ln \left(1 - e^{\mu/(k_B T)}\right)
\]

In 2D, BEC occurs when \( n \lambda^2 \) is large enough.

The ideal uniform 2D Bose gas (2)

The thermodynamic limit

Thermodynamic limit: \( \frac{N}{L^2} = \text{Cst.} \quad N, L \to \infty \)

In 2D: for any phase space density \( n \lambda^2 \), the occupation of the various states in non-singular, and no BEC is expected.

\[
n \lambda^2 = -\ln \left(1 - e^{\mu/(k_B T)}\right)
\]

\( \mu \) varies from \(-\infty\) to 0

In 3D: when the phase space density reaches \( n \lambda^3 = 2.612 \), BEC occurs

The ideal Bose in a harmonic potential

In 3D, BEC occurs when \( N = 1.2 \left(\frac{k_B T}{\hbar\omega}\right)^3 \)

In 2D, BEC occurs when \( N = 1.6 \left(\frac{k_B T}{\hbar\omega}\right)^2 \)

\[1.6 \approx \frac{\pi^2}{6}\]

Bagnato – Kleppner (1991)

Does harmonic trapping make 2D and 3D equivalent?

The density in the trap (semi-classical approach)

\[
\rho(\vec{r}, \vec{p}) = \frac{1}{h^d} \frac{1}{\exp \left(\frac{(\vec{p}_m^2 + V(r) - \mu)/k_B T}{\hbar\omega} - 1\right)}
\]

\[
n(\vec{r}) = \int \rho(\vec{r}, \vec{p}) \, d^dp
\]

\[n(\vec{r}) \lambda^3 = g_{3/2} \left(e^{(\mu - V(r))/k_B T}\right) \quad n(\vec{r}) \lambda^2 = -\ln \left(1 - e^{(\mu - V(r))/k_B T}\right)\]

At BEC:

\[
n(0) \lambda^3 = 2.612 \quad n(0) \lambda^2 = \infty
\]

Note: \( \int n(\vec{r}) \, d^2r = \frac{\pi^2}{6} \left(\frac{k_B T}{\hbar\omega}\right)^2 \)
§2. Interacting particles in (quasi) 2D

The 3D Gross-Pitaevski equation

\[
-\frac{\hbar^2}{2m} \Delta \psi + V_{\text{trap}}(\vec{r}) \psi(\vec{r}) + g^{(3D)} |\psi(\vec{r})|^2 \psi(\vec{r}) = i\hbar \frac{\partial \psi}{\partial t}
\]

\[g^{(3D)} = \text{energy} \times \text{volume} = \frac{4\pi \hbar^2}{m} a \quad a: \text{scattering length}\]

Two-body problem:

\[\phi(\vec{r}) \sim e^{i \vec{k} \cdot \vec{r}} + f(k) e^{ikr} \]

\[f(k): \text{scattering amplitude} \quad \lim_{k \to 0} f(k) = -a\]

The 2D version of the GP equation

A simple-minded (but efficient) approach

\[
-\frac{\hbar^2}{2m} \Delta \Phi + V_{\text{trap}}(\vec{r}) \Phi(\vec{r}) + g^{(2D)} |\Phi(\vec{r})|^2 \Phi(\vec{r}) = i\hbar \frac{\partial \Phi}{\partial t}
\]

\[g^{(2D)} = \text{energy} \times \text{area} = \frac{\hbar^2}{m} \tilde{g}\]

where \( \tilde{g} \) is dimension-less

How a 3D pseudopotential is transformed in 2D

3D interaction energy:

\[E_{\text{int}} = \frac{N(N-1)}{2} g^{(3D)} \int |\psi(\vec{r})|^4 \, d^3r \quad g^{(3D)} = \frac{4\pi \hbar^2 a}{m}\]

Trial wave function for the 2D problem:

\[\psi(x, y, z) = \Phi(x, y) e^{-z^2/2\ell_z^2} \quad \Phi(x, y) = \frac{e^{-\tilde{g} x^2/\ell_z^2}}{(\pi \ell_z^2)^{1/4}}\]

\[E_{\text{int}} = \frac{N(N-1)}{2} g^{(2D)} \int |\Phi(\vec{r})|^4 \, d^2r \quad \text{with} \quad g^{(2D)} = \frac{\hbar^2}{m} \sqrt{8\pi} \frac{a}{\ell_z}\]

\[\tilde{g} = \sqrt{8\pi} \frac{a}{\ell_z}\]

\{ \begin{aligned}
\text{Liquid helium film}: & \quad \tilde{g} \sim 1 \\
\text{ENS experiment (Rb)}: & \quad \tilde{g} \sim 0.1 \\
\text{NIST experiment (Na)}: & \quad \tilde{g} \sim 0.01
\end{aligned} \]

For a more rigorous approach, see Petrov-Holzmann-Shlyapnikov
§3. Is there true long range order in 2D?

\[ g_1(|\vec{r} - \vec{r}'|) = \langle \psi^*(\vec{r}) \psi(\vec{r}') \rangle; \]

Does this function tends to a non-zero value when \( \vec{r} - \vec{r}' \) ?

Is there true long range order in 2D?

At zero temperature? YES!
At non-zero temperature? NO!

Schick 1971

Small parameter characterizing the interactions and giving the “quantum depletion”

\[ \frac{1}{\ln(1/\sqrt{n}a^2)} \]

This parameter plays the role of \( \sqrt{n}a^3 \) in 3D

Phase fluctuations in a 2D uniform quasi-BEC

Assume that the system is well described by an order parameter

\[ \psi(\vec{r}) = \sqrt{n(\vec{r})} e^{i\theta(\vec{r})} \]

Assume also that the repulsive interactions are strong enough to “freeze” density fluctuations, so that we deal only with phase fluctuations:

\[ \psi(\vec{r}) \sim \sqrt{n_0} e^{i\theta(\vec{r})} \]

Kinetic energy

\[ E = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 \, d^2r \approx \frac{\hbar^2 n_0}{2m} \int |\nabla \theta|^2 \, d^2r \]

To proceed, we now make a Fourier expansion of the phase

\[ \psi(\vec{r}) = \sum_{k} C_k e^{i\vec{k} \cdot \vec{r}} \]

\[ \theta(\vec{r}) = \sum_{k} \tilde{C}_k e^{i\vec{k} \cdot \vec{r}} \]

Energy:

\[ E = \frac{\hbar^2 n_0}{2m} \int |\nabla \theta|^2 \, d^2r = \frac{\hbar^2 n_0 L^2}{2m} \sum_{k} k^2 |C_k|^2 \]

Thermal equilibrium for each mode:

\[ \frac{\hbar^2 n_0 L^2}{2m} k^2 |C_k|^2 = \frac{1}{2} k_B T \]
The correlation function of the phase

We introduce the imaginary part \( C''_{k} \) of the Fourier coefficient \( C_{k} \)

\[
\langle (\theta(\vec{r}/2) - \theta(-\vec{r}/2))^2 \rangle = 8 \sum_{\vec{k}} \langle |C''_{\vec{k}}|^2 \rangle \sin^2(\vec{k} \cdot \vec{r}/2)
\]

Using the thermal average for \( C_{\vec{k}} \) and going to an integral, we get

\[
\langle (\theta(\vec{r}/2) - \theta(-\vec{r}/2))^2 \rangle = \frac{2}{\pi n_0 \lambda^2} \int \frac{\sin^2(\vec{k} \cdot \vec{r}/2)}{k^2} d^2k \approx \frac{\pi n_0 \lambda^2}{\frac{1}{k^2} d^2k} = \frac{2}{n_0 \lambda^2} \int dk \frac{1}{k}
\]

Upper bound of the integral: \( 1/\xi \) or \( 1/\lambda \)
Lower bound of the integral: \( 1/r \)

The result of the calculation:

For a degenerate 3D gas: \( n_0 \lambda^3 > 1 \)
and \( n_0 \xi^3 \sim \frac{1}{(n_0 a^3)^{1/2}} \) so that in the dilute regime \( n_0 \xi^3 \gg 1 \)

Therefore \( \langle (\theta(\vec{r}/2) - \theta(-\vec{r}/2))^2 \rangle \ll 1 \) in 3D for any \( r \)

Summary for the correlation function of the phase

Assuming that the phase varies "slowly" enough for the Fourier expansion to be valid

\[
\langle (\theta(\vec{r}) - \theta(0))^2 \rangle \simeq \frac{2}{n_0 \lambda^2} \ln(r/\xi) \quad r \gg \xi
\]


- No long range order at finite temperature
- \( \vec{u}_0 \)
- \( \vec{u}_n \) in 2D: \( \langle (\vec{u}_n - \vec{u}_0)^2 \rangle \propto \log(n) \)
- Whereas this tends to a finite value in 3D
- The kind of order a physical system can possess is profoundly affected by its dimensionality.

Calculation of the \( g_1 \) function

We still neglect density fluctuations

\[
g_1(r) = \langle \psi^*(\vec{r}) \psi(0) \rangle = n_0 \langle e^{i(\theta(\vec{r}) - \theta(0))} \rangle
\]

Gaussian fluctuations for the phase: \( \langle e^{iG} \rangle = e^{-\langle G^2 \rangle/2} \)

Using \( \langle (\theta(\vec{r}) - \theta(0))^2 \rangle \simeq \frac{2}{n_0 \lambda^2} \ln(r/\xi) \)
we obtain finally:

\[
g_1(r) = n_0 \left( \frac{\xi}{r} \right)^{1/(n_0 \lambda^2)}
\]

Valid at low temperature, where the Fourier expansion of the phase is valid.
Indication for a phase transition in a 2D Bose fluid

\[ g_1(|r - r'|) = \langle \psi^*(r) \psi(r') \rangle \]

A phase transition in a 2D quantum fluid

Summary for this first part on the static 2D Bose gas

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<th>uniform</th>
<th>Harmonic trap</th>
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<td>No BEC</td>
<td>BEC ( N &gt; N_c )</td>
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<td>Interacting</td>
<td>No BEC (algebraic decay of ( g_1 )) but a superfluid transition</td>
<td>?</td>
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