Low dimensional quantum gases, rotation and vortices

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Review article:
I. Bloch, J. Dalibard, W. Zwerger,
Many-Body Physics with Ultracold Gases
Rev. Mod. Phys. **80**, 885 (2008)

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Goal of these lectures

Discuss some aspect of the physics of quantum low dimensional systems

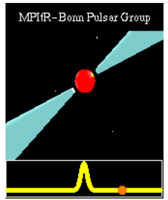
Planar fluids

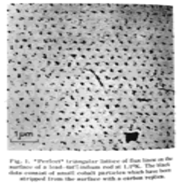
Quantum wells
and MOS structures

High T_c
superconductivity

Discuss also the effect of quantum rotation

Neutron stars





Superconductor
in a magnetic field

VORTICES

Outline of the lectures

1. Static 2D gases

   Normal to superfluid transition

   Berezinski-Kosterlitz-Thouless mechanism:
   breaking of vortex pairs

2. Rotating 2D gases

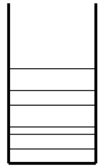
   Vortices are the “simplest way” to set a quantum system in rotation.

   Physics in the Lowest Landau Level and
   Connection with Quantum Hall phenomenon

Chapter 1 : Static 2D gases

§1. The ideal Bose gas in 2D

The ideal uniform 2D Bose gas



Square box $L \times L$

Eigenstates: plane waves of momentum: $\vec{p} = \hbar \vec{k}$

$$E_k = \frac{\hbar^2 k^2}{2m} \qquad \vec{k} = \frac{2\pi}{L}(n_x, n_y)$$

Is there a BEC at finite temperature? No!

$$N = \sum_{\vec{k}} n(E_k) \qquad n(E) = \frac{1}{\exp[(E-\mu)/k_B T] - 1}$$

$$N \simeq \frac{L^2}{4\pi^2} \int n(E_k)\, d^2k \;=\; -\frac{L^2}{\lambda^2} \ln\left(1 - e^{\mu/(k_B T)}\right)$$

The ideal uniform 2D Bose gas (2)

The thermodynamic limit

Thermodynamic limit: $n = \frac{N}{L^2} = \text{Cst.} \qquad N, L \to \infty$

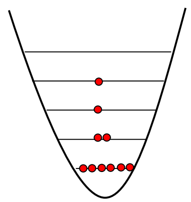
In 2D: for any phase space density $n\lambda^2$, the occupation of the various states in non singular, and no BEC is expected.

$$\boxed{n\lambda^2 = -\ln\left(1 - e^{\mu / k_B T}\right)}$$

μ varies from $-\infty$ to 0

In 3D: when the phase space density reaches $n\lambda^3 = 2.612$, BEC occurs

The ideal Bose in a harmonic potential



In 3D, BEC occurs when $N = 1.2 \left(\frac{k_B T}{\hbar\omega}\right)^3$

In 2D, BEC occurs when $\boxed{N = 1.6 \left(\frac{k_B T}{\hbar\omega}\right)^2} \qquad 1.6 \simeq \frac{\pi^2}{6}$

Bagnato – Kleppner (1991)

Does harmonic trapping make 2D and 3D equivalent?

The density in the trap (semi-classical approach)

$$\rho(\vec{r}, \vec{p}) = \frac{1}{h^d} \frac{1}{\exp\left(\left(\frac{p^2}{2m} + V(r) - \mu\right)/kT\right) - 1}$$

$$n(\vec{r}) = \int \rho(\vec{r}, \vec{p})\, d^d p$$

3D:

$$n(\vec{r})\, \lambda^3 = g_{3/2}\left(e^{(\mu - V(r))/kT}\right)$$

At BEC:

$$n(0)\, \lambda^3 = 2.612$$

2D:

$$n(\vec{r})\, \lambda^2 = -\ln\left(1 - e^{(\mu - V(r))/kT}\right)$$

At BEC:

$$n(0)\, \lambda^2 = \infty$$

Note: $\int n(\vec{r})\, d^2r = \frac{\pi^2}{6} \left(\frac{kT}{\hbar\omega}\right)^2$

§2. Interacting particles in (quasi) 2D

The 3D Gross-Pitaevski equation

$$-\frac{\hbar^2}{2m}\Delta\psi + V_{\text{trap}}(\vec{r})\psi(\vec{r}) + g^{(3D)}|\psi(\vec{r})|^2\psi(\vec{r}) = i\hbar\frac{\partial\psi}{\partial t}$$

$$g^{(3D)} = \text{energy} \times \text{volume} = \frac{4\pi\hbar^2}{m}\,a \qquad a : \text{scattering length}$$

Two-body problem: $\phi(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}} + f(k)\,\frac{e^{ikr}}{r}$

$f(k)$: scattering amplitude $\qquad \lim_{k\to 0} f(k) = -a$

The 2D version of the GP equation

A simple-minded (but efficient) approach

$$-\frac{\hbar^2}{2m}\Delta\Phi + V_{\text{trap}}(\vec{r})\Phi(\vec{r}) + g^{(2D)}|\Phi(\vec{r})|^2\Phi(\vec{r}) = i\hbar\frac{\partial\Phi}{\partial t}$$

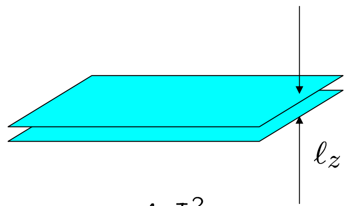
$$g^{(3D)} = \text{energy} \times \text{volume} = \frac{4\pi\hbar^2}{m}\,a$$

⬇

$$g^{(2D)} = \text{energy} \times \text{area} = \frac{\hbar^2}{m}\,\tilde{g}$$

where $\tilde{g}$ is dimension-less

How a 3D pseudopotential is transformed in 2D



3D interaction energy:

$$E_{\text{int}} = \frac{N(N-1)}{2}\,g^{(3D)}\int|\psi(\vec{r})|^4\,d^3r \qquad g^{(3D)} = \frac{4\pi\hbar^2 a}{m}$$

Trial wave function for the 2D problem: $\psi(x,y,z) = \Phi(x,y)\,\frac{e^{-z^2/2\ell_z^2}}{(\pi\ell_z^2)^{1/4}}$

$$E_{\text{int}} = \frac{N(N-1)}{2}\,g^{(2D)}\int|\Phi(\vec{r})|^4\,d^2r \qquad \text{with } g^{(2D)} = \frac{\hbar^2}{m}\sqrt{8\pi}\frac{a}{\ell_z}$$

$\tilde{g} = \sqrt{8\pi}\frac{a}{\ell_z}$

- Liquid helium film: $\tilde{g} \sim 1$
- ENS experiment (Rb): $\tilde{g} \sim 0.1$
- NIST experiment (Na): $\tilde{g} \sim 0.01$

For a more rigorous approach, see Petrov-Holzmann-Shlyapnikov

§3. Is there true long range order in 2D?

$$g_1(|\vec{r}-\vec{r}'|) = \langle \psi^*(\vec{r})\psi(\vec{r}')\rangle$$

Does this function tends to a non-zero value when $|\vec{r}-\vec{r}'|$?

Is there true long range order in 2D?

At zero temperature ? YES ! Schick 1971

Small parameter characterizing the interactions and giving the "quantum depletion" $\dfrac{1}{\ln(1/na^2)}$

This parameter plays the role of $\sqrt{na^3}$ in 3D

At non-zero temperature ? NO !

At low temperature, algebraic decay of $g_1(|\vec{r}-\vec{r}'|) = \langle \psi^*(\vec{r})\psi(\vec{r}')\rangle$

Phase fluctuations in a 2D uniform quasi-BEC

Assume that the system is well described by an order parameter

$$\psi(\vec{r}) = \sqrt{n(\vec{r})}\; e^{i\theta(\vec{r})}$$

Assume also that the repulsive interactions are strong enough to "freeze" density fluctuations, so that we deal only with phase fluctuations:

$$\psi(\vec{r}) \simeq \sqrt{n_0}\; e^{i\theta(\vec{r})}$$

Kinetic energy

$$E = \frac{\hbar^2}{2m}\int |\vec{\nabla}\psi|^2\, d^2r \simeq \frac{\hbar^2 n_0}{2m}\int |\vec{\nabla}\theta|^2\, d^2r$$

To proceed, we now make a Fourier expansion of the phase

Fourier expansion of the phase

Assume that the phase of the wave function varies "slowly" enough:

$$\psi(\vec{r}) \simeq \sqrt{n_0}\; e^{i\theta(\vec{r})} \quad \text{with} \quad \begin{cases} \theta(\vec{r}) = \sum_{\vec{k}} C_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} \\ \vec{k} = \dfrac{2\pi}{L}(n_x, n_y) \end{cases}$$

$$\theta(\vec{r}) \text{ real} \Rightarrow C_{\vec{k}} = C^*_{-\vec{k}}$$

Energy: $$E = \frac{\hbar^2 n_0}{2m}\int |\vec{\nabla}\theta|^2\, d^2r = \frac{\hbar^2 n_0}{2m} L^2 \sum_{\vec{k}} k^2\, |C_{\vec{k}}|^2$$

Thermal equilibrium for each mode: $$\frac{\hbar^2 n_0}{2m} L^2\, k^2\, |C_{\vec{k}}|^2 = \frac{1}{2} k_B T$$

The correlation function of the phase

We introduce the imaginary part $C''_{\vec{k}}$ of the Fourier coefficient $C_{\vec{k}}$

$$\langle(\theta(\vec{r}/2)-\theta(-\vec{r}/2))^2\rangle = 8\sum_{\vec{k}}\langle|C''_{\vec{k}}|^2\rangle\ \sin^2(\vec{k}\cdot\vec{r}/2)$$

Using the thermal average for $C_{\vec{k}}$ and going to an integral, we get

$$\begin{aligned}\langle(\theta(\vec{r}/2)-\theta(-\vec{r}/2))^2\rangle &= \frac{2}{\pi n_0\lambda^2}\iint\frac{\sin^2(\vec{k}\cdot\vec{r}/2)}{k^2}\,d^2k \\ &\simeq \frac{1}{\pi n_0\lambda^2}\iint\frac{1}{k^2}\,d^2k \\ &= \frac{2}{n_0\lambda^2}\int\frac{dk}{k}\end{aligned}$$

Upper bound of the integral: $1/\xi$ or $1/\lambda$

Lower bound of the integral: $1/r$

The correlation function of the phase (2)

Result of the calculation:

$$\langle(\theta(\vec{r}/2)-\theta(-\vec{r}/2))^2\rangle \simeq \frac{2}{n_0\lambda^2}\int\frac{dk}{k} = \frac{2}{n_0\lambda^2}\ln(r/\xi)$$

whereas one gets in 3D

$$\langle(\theta(\vec{r}/2)-\theta(-\vec{r}/2))^2\rangle \simeq \frac{4}{n_0\lambda^2}\int dk = \frac{1}{n_0\lambda^2\xi}$$

For a degenerate 3D gas: $n_0\lambda^3 > 1$

and $n_0\xi^3 \sim \dfrac{1}{(n_0a^3)^{1/2}}$ so that in the dilute regime $n_0\xi^3 \gg 1$

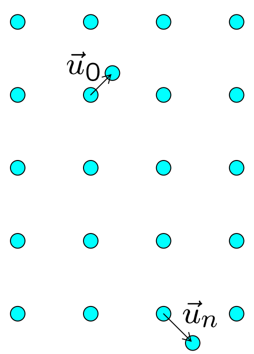
Therefore $\langle(\theta(\vec{r}/2)-\theta(-\vec{r}/2))^2\rangle \ll 1$ in 3D for any r

Summary for the correlation function of the phase

Assuming that the phase varies “slowly” enough for the Fourier expansion to be valid

$$\langle(\theta(\vec{r})-\theta(0))^2\rangle \simeq \frac{2}{n_0\lambda^2}\ln(r/\xi) \qquad r \gg \xi$$

Characteristic of 2D [Peierls 1935, Mermin – Wagner - Hohenberg 1966]



No long range order at finite temperature

in 2D: $\langle(\vec{u}_n-\vec{u}_0)^2\rangle \propto \log(n)$

whereas this tends to a finite value in 3D

The kind of order a physical system can possess is profoundly affected by its dimensionality.

Calculation of the g_1 function

We still neglect density fluctuations

$$g_1(r) = \langle\psi^*(\vec{r})\ \psi(0)\rangle = n_0\langle e^{i(\theta(\vec{r})-\theta(0))}\rangle$$

Gaussian fluctuations for the phase: $\langle e^{iG}\rangle = e^{-\langle G^2\rangle/2}$

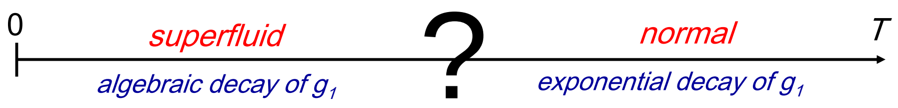
Using $\langle(\theta(\vec{r})-\theta(0))^2\rangle \simeq \dfrac{2}{n_0\lambda^2}\ln(r/\xi)$

we obtain finally:

$$\boxed{g_1(r) = n_0\left(\frac{\xi}{r}\right)^{1/(n_0\lambda^2)}}$$

Valid at low temperature, where the Fourier expansion of the phase is valid

Indication for a phase transition in a 2D Bose fluid



$$g_1(|\vec{r}-\vec{r}\,'|) = \langle \psi^*(\vec{r})\psi(\vec{r}\,')\rangle$$

A phase transition in a 2D quantum fluid

VOLUME 40, NUMBER 26 PHYSICAL REVIEW LETTERS 26 JUNE 1978

Study of the Superfluid Transition in Two-Dimensional ^{4}He Films

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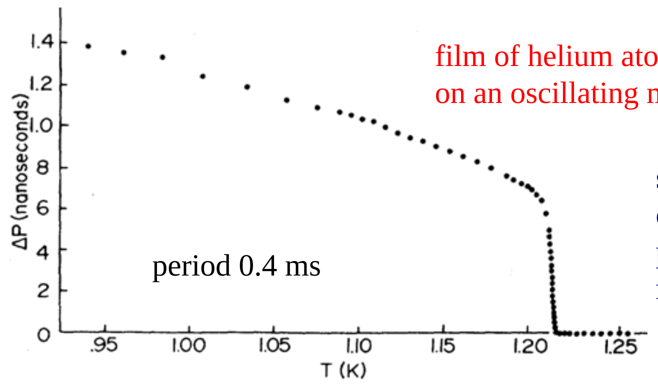
(Received 20 April 1978)



film of helium atoms adsorbed on an oscillating mylar substrate

sudden reduction of the oscillation period when T is reduced!

Summary for this first part on the static 2D Bose gas

	uniform	Harmonic trap
Ideal	No BEC	BEC $N > N_c$
Interacting	No BEC (algebraic decay of g_1) but a superfluid transition Origin?	?