The AC-Stark Shift

Consider a many-level atom in its ground state $|g\rangle$ interacting with a classical laser field $\vec{E}(t) = \vec{\varepsilon}E_0(t)\exp(i\omega t) +$ $c.c$, which has frequency ω, polarisation vector $\vec{\varepsilon}$ and electric field amplitude $E_0(t)$ at the position of the atom $x = 0$, which we assume to be fixed, or at least very slowly varying on the timescales under consideration. Assume that $E_0(t)$ is slowly varying on the timescale given by ω.

The Hamiltonian for this system can be expressed as $\hat{H} = \hat{H}_0 + \hat{H}_I$. Here $\hat{H}_0$ is the Hamiltonian for the atom, including all fine structure terms etc., and we can write the eigenstates of $\hat{H}_0$

$$\hat{H}_0 |k\rangle = \hbar\omega_k |k\rangle .$$

The interaction between the field and the atom is given by

$$\hat{H}_I = -\hat{\mu}.\vec{E}(t)$$

where we can express the operator $\hat{\mu} = -e\vec{d}$ in terms of the dipole matrix elements

$$\vec{\mu} = \sum_{n,k} |n\rangle\langle k| \langle n|\vec{\mu}|k\rangle = \sum_{n,k} |n\rangle\langle k| \vec{\mu}_{nk}.$$

($\vec{\mu}$ is sometimes also written as $\vec{d}$).

Our goal is to calculate the energy shift of the ground state due to the applied electromagnetic field (the so-called AC Stark shift). If we expand the state in terms of the unperturbed eigenstates, $|\psi(t)\rangle = \sum_n a_n(t)\exp(-i\omega_n t)|n\rangle$, then we are interested in the coefficient of he ground state, $a_g(t)$. In particular, if the coupling is weak so that $|a_g(t)| \approx 1$ for all times t, we would like to write $a_g = \exp(i\phi(t))$, and extract the time dependence of the phase $\phi(t)$, averaged over a single period $2\pi/\omega$. The steps required are:

1. Explain why the first-order correction to the energy shift is equal to zero.

2. Use time-dependent perturbation theory to write the expansion for the coefficient of $|g\rangle$, $a_g(t)$ to second order in H_I. Assume at time $t = 0$, $|a_g(t = 0)| = 1$, and that all other states are unoccupied.

3. Using the fact that $E_0(t)$ is slowly varying with respect to ω, perform the innermost time integral.

4. Perform the second time integral so as to compute the average $\langle\dot{\phi}\rangle$ over one period $t = 2\pi/\omega$. Explain why certain terms do not contribute to the final expression, and show that the AC Stark shift is given by:

$$\Delta E = -\hbar\langle\dot{\phi}\rangle = -\frac{1}{2\hbar}|E_0(t)|^2 \sum_n \left(\frac{1}{\omega_n - \omega_g + \omega} + \frac{1}{\omega_n - \omega_g - \omega}\right)|\vec{\mu}_{ng}\cdot\vec{\varepsilon}|^2$$

5. How does this expression simplify when the frequency ω is tuned very close to one transition, e.g., $\omega_m - \omega_g$? Under what conditions can we make a rotating wave approximation?