Geometric Brownian motion

Solve the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW \qquad (1)$$

or its integral form

$$S_T = S_0 + \int_0^T S_t \mu\, dt + \int_0^T S_t \sigma\, dW$$

The integral over the stochastic variable W requires a careful definition and is not equivalent to usual calculus. The usual expression for the Taylor expansion of a function of two variables $f(z,t)$ is

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} dz^2 + \ldots dz\,dt + \ldots dt^2$$

If $z = W$ is a random variable such that $dW^2 = dt = dz^2$

(neglect $dx\,dt \sim (dt)^{3/2} \ldots$)

	dt	dW
dt	0	0
dW	0	dt

Hence

$$df = \frac{\partial f}{\partial z} dW + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} dt$$

A trial solution is $f = S(t,W) = S_0 e^{\alpha t + \beta W}$

$$\Rightarrow \quad df = dS = \beta S\, dW + (\alpha + \tfrac{1}{2}\beta^2) S\, dt$$

cf. eqn (1) implies

$$\beta = \sigma, \quad \mu = \alpha + \tfrac{1}{2}\sigma^2$$

$$\alpha = \mu - \sigma^2/2$$

$$\therefore \quad S = S_0\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

Geometric Brownian motion by integration

Solve $df = \mu f dt + \sigma f dt$

$$= \frac{\partial f}{\partial z} dW + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} dt$$

These imply

$$\frac{\partial f}{\partial z}(t,W) = \sigma f(t,W) \quad \text{and}$$

$$\frac{\partial f}{\partial t}(t,W) + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(t,W) = \mu f(t,W)$$

These equations hold for all values of the random variable W and hence must hold for all $z \in \mathbb{R}$.

$$\frac{\partial f}{\partial t}(t,z) = \sigma f(t,z) \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial t}(t,z) + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(t,z) = \mu f(t,z) \quad \text{—— (2)}$$

Let $g(t,z) = \ln(f(t,z))$ then

$$\frac{\partial g}{\partial z} = \frac{\partial}{\partial z}(\ln f) = \frac{1}{f}\frac{\partial f}{\partial z} = \sigma$$

$$\Rightarrow g(t,z) = g(t,0) + \sigma z$$

$$f(t,z) = e^{g(t,0)} e^{\sigma z} = f(t,0) e^{\sigma z}$$

From eqn (2)

$$e^{\sigma z} \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 e^{\sigma z} f(t,0) = \mu f(t,0) e^{\sigma z}$$

$$\Rightarrow \frac{\partial f(t,0)}{\partial t} = \left(\mu - \frac{\sigma^2}{2}\right) f(t,0)$$

$$\frac{\partial g}{\partial t}(t,0) = \mu - \sigma^2/2$$

$$g(t,0) = g(0,0) + \left(\mu - \frac{\sigma^2}{2}\right)t$$

$$f(t,z) = e^{g(t,z)} = e^{g(t,0) + \sigma z} = e^{g(0,0) + \sigma z + (\mu - \frac{\sigma^2}{2})t}$$

$$f(t,z) = f(0,0) e^{\sigma z + (\mu - \frac{\sigma^2}{2})t}$$

$$\boxed{f = f_0 e^{\sigma W + (\mu - \frac{\sigma^2}{2})t}}$$