Pricing of Forward Contracts

These topics are described in Chapters 1-3 of *The Mathematics of Financial Derivatives* (Wilmott, Howison and Dewynne, CUP) and Ch. 5 of *Options, Futures, and other Derivatives* (JC Hull, 8th ed. Pearson).

The time value of Money

Theorem: It is better to be paid 1 million pounds today, than 1 million pounds in 1 years time.

Proof: Given an amount of cash C_0 at time $t = 0$, it can be invested risk-free in a deposit account paying interest at rate r (continuously compounded) so that

$$\frac{dC}{dt} = rC \Rightarrow C(t) = C_0 e^{rt}. \tag{1}$$

This gives the time-varying value of money, which is the background drift against which everything else is compared. (It is a moving 'frame of reference' for prices.).

Pricing a forward contract (using portfolio replication)

We consider the price of forward contract to deliver an asset at time T (in the future) in exchange for a fixed strike price K. The payoff is

$$V_T = S_T - K. \tag{2}$$

where S_T is spot price at time T, which is not known at the start! Therefore the only way to guarantee delivery at known cost is to acquire a holding of the asset now, funded by borrowing cash. The yield on the asset is y, and we immediately reinvest by buying more assets so that the number N of assets grows as

$$\frac{dN}{dt} = yN \Rightarrow N_T = N_0 e^{yT}. \tag{3}$$

Therefore if the number of assets that we need to have at time T is $N_T = 1$ then we need $N_0 = e^{-yT}$ at the start. Thus the cash borrowed initially is $C_0 = N_0 S_0$ where S_0 is the starting price (which is known). Following this through we see that the amount of cash that we have to repay is

$$C_T = C_0 e^{rT} = N_0 S_0 e^{rT} = S_0 e^{(r-y)T}. \tag{4}$$

If the strike price K is equal to C_T then the nett cashflow at time T is zero, i.e.,

A receives an amount K by selling the asset (purchased at $t = 0$) and repays the outstanding bank loan C_T.

B buys the asset for price K at time T as agreed in the contact (no matter what the current price).

Thus there is a hedging strategy that A can adopt (if they want) which eliminates the risk (of losing money from adverse prices changes $S_T > K$), while at the same time giving up the chance of gaining by speculating that $S_T < K$. This strategy is static, i.e., it does require any adjustments to be made during the lifetime of the contract (unlike a hedging strategy for options). The initial value of the contract reflects the present value of the cashflows in the replicating strategy; applying a discount factor (e^{-rT}) to eqn 2 gives the (starting) value of the forward contract:

$$V_0 = e^{-rT}(S_0 e^{(r-y)T} - K) = S_0 e^{-yT} - K e^{-rT}. \tag{5}$$

Put-call parity

Consider a portfolio where we are long one asset S, long one put (of value P), and short one call (of value C). The call and put both have the same exercise price K and the same expiry date (or time) T. The total value of this portfolio at time t is given by

$$\Pi = S_t + P - C. \tag{6}$$

The payoff at expiry is

$$\text{Payoff} = S_T + \max(K - S_T, 0) - \max(S_T - K, 0). \tag{7}$$

If $S_T \leq K$,

$$\text{Payoff} = S_T + (K - S_T) - 0 = K. \tag{8}$$

Or, if $S_T \geq K$,

$$\text{Payoff} = S_T + 0 - (S_T - K) = K. \tag{9}$$

Hence the payoff is always K.

How much should you pay now for a portfolio worth K at time T? Answer: its discounted value $K \exp[-r(T-t)]$. Hence

$$S_t + P - C = K\, e^{-r(T-t)}, \tag{10}$$

or equivalently in notation consistent with the previous section,

$$S_0 + P - C = K\, e^{-rT}, \tag{11}$$

where S_0 is the starting price (i.e. today's price which we know).

If this Put-call Parity did not hold then arbitragers could (and would) make instantaneous riskless profit by buying and selling options and shares, and at the same time borrowing or lending money in the correct proportions.

Further comment on Put-call parity

We can rewrite eqn (11) as

$$C - P = S_0 - K\, e^{-rT}, \tag{12}$$

and on setting the yield to zero ($y = 0$) eqn (5) becomes

$$V_0 = S_0 - Ke^{-rT}. \tag{13}$$

The equivalence of i) a forward contract, and ii) a portfolio where we are long one call and short one put, is clear when one compares the payoff diagrams (for a contract and options that have the same strike price and expiry date.)