6 Non-zero risk in the real world

HANDOUT 6 OF 7

6.1 The other side of derivatives

In Chapter 2 we looked at how an investor might use derivatives to manage risk. In order for the Black-Scholes pricing theory to work, we needed to make several major assumptions about how financial markets behave. Here we re-visit the whole question of risk and derivatives for real-world markets, without automatically making these assumptions. Consequently the formalism in this Chapter is more complicated than Chapter 2: we therefore present it in a pedagogical manner while emphasizing the practical steps that one needs to take to implement it. The formalism is built upon the landmark work of Bouchaud and Sornette[1]. It is possible to take things even further: one can generalize this approach to address the crucial issue of managing portfolios in the presence of non-zero transaction costs -- but this is beyond the scope of the present course.

We start by re-examining the whole topic of derivative pricing and risk. Consider the following example scenario:

> *An investor predicts that the price of asset A will increase significantly over the next three months. He therefore buys a large amount of the asset, hoping to sell it back at a profit after that period. The investor can insure his position by also buying an equal quantity of put-options dated three months in the future, with a strike price equal to today's asset value. Even if the investor is wrong and asset A falls in value, he will be able to 'unwind' his unfavourable position by selling the assets to the option writer at the same value as he bought them. Hence the investor will only suffer a loss equal to the original put-option value, which is essentially his insurance premium on the investment.*

Hence the investor holding the portfolio of assets A and the put-options, can be sure that his portfolio will not lose more than the original option value. However, the position for the *writer* of those options is reversed: the maximum possible achievable profit is the original option value while a large loss could be faced. This can be seen from the payoff function for the put-option[2]:

$$\text{payoff} = V_T\left[x_T, X\right] = \max\left[X - x_T, 0\right] \qquad (6.1)$$

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[1] J.P. Bouchaud and D. Sornette, Journal de Physique I, **4**, 863 (1994). See also [BP].

[2] Recall from Chapter 2 that we denote the value of an option at time t since the contract was written as V_t. Hence, V_0 is the option premium and V_T the option payout. Also as before, X is the option 'exercise' or 'strike' price and T is the maturity.

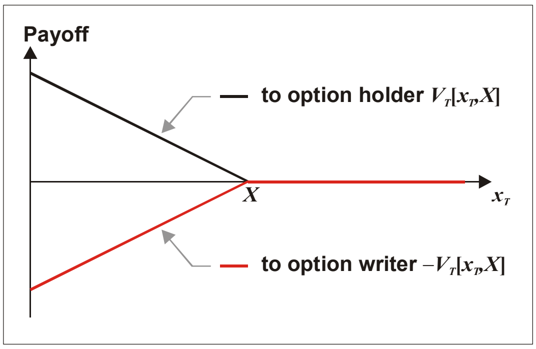

Figure 6-1: Put-option payoff at expiry for the option contract holder and writer.

In the event that the price of the underlying asset at expiry is less than the strike price ($x_T < X$), Figure 6-1 shows that the writer of the option could be faced with a large payout to the holder. Let's denote $p[x_t \mid x_0]$ as the conditional probability distribution function such that $p[x_t \mid x_0]dx_t$ is the probability that the underlying asset price[3] at time t is in the range $x_t \to x_t + dx_t$, given that the asset price at the time of writing the contract was x_0. We can then calculate the distribution of the option writer's profit or loss on the contract at expiry, i.e. his 'variation of wealth' ΔW_T, as follows. We know that the option writer gets to keep the initial option value (the insurance premium) whatever the asset price does. Additionally he has to pay out $V_T[x_T, X]$ if $x_T \le X$. Hence, using Equation (6.1):

$$\Delta W_T = \begin{cases} V_0[x_0, X, T] & x_T \ge X \quad ; \\ V_0[x_0, X, T] - (X - x_T) & x_T \le X \quad . \end{cases} \tag{6.2}$$

The PDF for variations of the option writer's wealth is given by:

$$\begin{aligned} p\big[\Delta W_T > V_0[x_0, X, T]\big] &= 0 \\ p\big[\Delta W_T = V_0[x_0, X, T]\big] &= \int_X^{\infty} p[x_T \mid x_0]\, dx_T \\ p\big[\Delta W_T = y < V_0[x_0, X, T]\big] d\Delta W_T &= p\big[x_T = (y - V_0[x_0, X, T] + X) \mid x_0\big] dx_T \end{aligned} \tag{6.3}$$

A histogram showing an example of $p[\Delta W_T]$ is shown in Figure 6-2:

[3] To simplify the appearance of formulae in this Chapter, we will use x_t instead of $x[t]$ to denote the asset price at time t. Hence $x_t \equiv x[t]$.

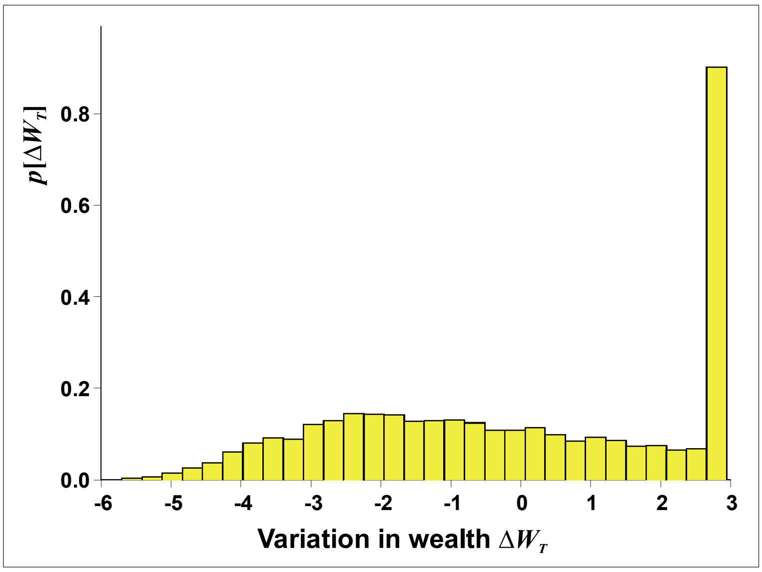

Figure 6-2: Histogram representing the probability distribution function (PDF) of the variation of wealth $p\left[\Delta W_T\right]$ resulting from a Monte-Carlo simulation of writing 5000 put options with parameters $x_0 = 8,\ X = 10,\ T = 100$ with an interest-rate $r = 0$. The PDF of the underlying asset's price movement[4] $p\left[R_{t,t-1}\right]$ was taken to be lognormal with volatility $\sigma = 5\%$.

6.2 Hedging to reduce risk

As demonstrated above, the position of an option writer is one in which large potential losses may arise. This is why option writers hedge their position by strategically buying a certain quantity $\phi\left[x_t, t\right]$ of the underlying asset. Take the example scenario in which the investor buys put-options:

> *A bank has sold to an investor a large quantity n of put-options on asset A with a strike price equal to the initial asset value $X = x_0 = \$10$. The options were sold at $V_0 = \$1$ each. The asset price falls to $x_{t_1} = \$9$ at time t_1, and the bank starts to worry it will have to payout on the option. It thus short-sells $\phi = n$ of asset A as a hedge. By the expiry time of the option, asset A has fallen to a value of $x_T = \$5$ and the payout to the investor is thus $V_T = X - x_T = \$5$. The option writer has made a loss on each of the options of the value minus the payout $V_0 - V_T = -\$4$, but has also made a profit on each of the hedging assets of $-\left(x_T - x_{t_1}\right) = \$4$. Overall the option writer has neither made a profit nor a loss: his variation in wealth $\Delta W_T = 0$. If the bank had not hedged, its loss would have been $-\$4$ per*

This demonstrates how hedging can reduce the option writer's potential loss, essentially his risk. In this case the bank made only one re-hedging at time t_1, deciding that they would prepare themselves

[4] The PDF $p\left[R_{t,t-1}\right]$ is the probability density function of returns, where the return is defined in Equation (1.3).

for what they considered to be a certain payout to the investor at the time of expiry. However, in general a bank would never be so sure about the outcome of an asset movement, hence it would be more reasonable to accrue the hedging position in smaller chunks, selling and buying the underlying asset when it became more or less likely that the payout would have to be made to the investor.

6.3 Zero risk?

By making assumptions about how an underlying financial asset will move, the Black-Scholes analysis of Chapter 2 shows how *in theory* it is possible to *never* lose any capital through writing an option, i.e. the variation of the option writer's wealth always remains zero: $\Delta W_T = 0$ and hence 'zero-risk'. With the confidence of this outcome behind them, banks are able to justify their exposure to huge derivatives portfolios – and the more derivatives contracts, the more commissions. So what then are the magic ingredients of this theory that guarantee zero risk? Let's recall the main assumptions:

1. Continuous time: continuous trading
2. Efficient markets: no arbitrage
3. Underlying assets follow a random walk

These assumptions are questionable for the reasons discussed earlier in this book. However the one which stands out most in the context of hedging is the first -- the assumption of continuous time and hence continuous trading -- since it implies the use of a strategy for *continuous* re-hedging. This does not simply mean re-hedging every time the asset price moves: it actually means re-hedging every time *time itself moves*, which is impossible. In addition, the presence of transaction costs gives rise to a financial barrier to high-frequency trading: the greater the number of re-hedgings, the greater the cost to the bank. Presumably there will be a trade off that banks have to make: the more they re-hedge, the closer they will approach the zero-risk limit -- however the cost of their transactions will imply more expensive options which in turn implies fewer customers. The third assumption -- that of a lognormal[5] random walk of the asset price -- is closely related to the assumption of continuous time employed in Section 2.4.3. Suppose that in an infinitesimal time dt, the asset return dx/x has a probability density function $\mathrm{p}\left[dx/x\right]$ that is highly non-Gaussian. Then in any arbitrarily small but finite time interval Δt, there will have been an infinite number of trials of $\mathrm{p}\left[dx/x\right]$. The Central Limit Theorem discussed in

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[5] Equation (2.36) which was used to derive the Black-Scholes equation, is a log-normal random walk as opposed to Equation (2.34) which is a random walk. The only difference lies in the variable performing the random walk. In the first case, the variable is dx/x which is the price return (Equation (1.3) in the continuous-time limit). In the second case, the variable is just dx. The distinction is unimportant. Typically $dx \ll x$ and hence the price-return dx/x behaves like the price-change dx.

Section 2.2.3.4 implies that for virtually every choice of $\mathrm{p}\left[dx/x\right]$, the resulting distribution $\mathrm{p}\left[\Delta x/x\right]$ should be Gaussian. However, as discussed in Chapter 2 and demonstrated in Chapter 3, the distribution of real asset price returns can be non-Gaussian up to very large time intervals Δt. Moreover asset returns can show a non-negligible degree of higher-order temporal correlations. The failure of the random walk assumption must therefore also impact on this result of zero-risk derivative portfolios.

6.4 Pricing and hedging with real-world asset movements

The fundamental question is: *If we cannot achieve zero-risk, then how much risk do we actually have - and how can it be minimized*? With news concerning large financial losses and inadequate risk-control arriving increasingly often in the media, and the requirements of the international Basel II agreement needing to be implemented, these are questions which banks are increasingly keen to answer. However, it is not obvious *how* to answer such questions. Perhaps adding on corrections to Black-Scholes? Unfortunately it is not easy to do perturbation theory around zero – and zero is the magical value of risk underpinning Black-Scholes. What is clear is that one must avoid making the same implicit initial assumptions as Black-Scholes, by instead going ‘back to basics’. In the following sections, we will present such a back-to-basics approach which focuses on minimizing the option writer’s risk.

6.4.1 Variation of wealth

We start with the option writer’s variation of wealth at the time of the contract expiry:

$$\Delta W_T = value - payout + hedging\ profit \tag{6.4}$$

The $value$ term comes from the cost of the option contract (the premium) which is paid by the holder to the writer at the time of writing, $t = 0$. The option cost is kept by the option writer irrespective of any subsequent underlying asset movement and is thus not a function of $x_{t\neq 0}$. The option cost V_0 is banked at the risk-free interest rate r until time of expiry T, giving:

$$value = V_0\left[x_0, X, T\right]\left(1+r\right)^T \tag{6.5}$$

The $payout$ term comes from the payout which the option writer must give to the option holder at the point of expiry of the option $t = T$, and is thus the final value of the option contract. This $payout$ must be paid to the option holder irrespective of any preceeding underlying asset movement and is thus not a function of $x_{t\neq T}$.

$$payout = V_T = V_T\left[x_T, X\right] \tag{6.6}$$

The *hedging profit* term comes from the profit or loss realised on the ϕ_t underlying assets which are held by the option writer at time t for the purposes of hedging. If the writer has not managed to obtain any information about where the asset price will be in the future based on past prices, then the quantity of assets he chooses to hold for hedging will only be a function of time and the current asset price, i.e. $\phi_t = \phi_t\left[x_t\right]$. Without loss of generality, we formulate the problem in *discrete time* by writing $t = i\tau$ with $i = 1,2,3\ldots$ etc.: later we will comment on the limiting case of $\tau \to 0$ corresponding to the Black-Scholes assumption of continuous-time. Between two consecutive times $t-\tau$ and t, the option writer holds $\phi_{t-\tau}$ hedging assets: the profit on these assets is therefore $\phi_{t-\tau}\left(x_t - x_{t-\tau}\right)$. However, the capital $\phi_{t-\tau}x_{t-\tau}$ held in the underlying asset could have been gaining interest at rate r. The interest lost in this period is thus $\phi_{t-\tau}x_{t-\tau}\left(1+r\right)^{\tau} - \phi_{t-\tau}x_{t-\tau}$. Gathering together these contributions, we have the hedging profit from $t-\tau$ to t equal to:

$$\phi_{t-\tau}\left(\left(x_t - x_{t-\tau}\right) - x_{t-\tau}\left(\left(1+r\right)^{\tau} - 1\right)\right) = \phi_{t-\tau}\left(x_t - \left(1+r\right)^{\tau} x_{t-\tau}\right)$$

At each timestep t, the option writer banks this hedging profit at the risk-free rate until expiry, giving a net variation in wealth due to hedging as:

$$\textit{hedging profit} = \sum_{i=1}^{T/\tau} \phi_{(i-1)\tau}\left(x_{i\tau} - \left(1+r\right)^{\tau} x_{(i-1)\tau}\right)\left(1+r\right)^{T-i\tau} \qquad (6.7)$$

Combining Equations (6.4), (6.5), (6.6), and (6.7) gives us an equation for the variation in the option writer's wealth at expiry:

$$\Delta W_T = V_0\left(1+r\right)^T - V_T + \sum_{i=1}^{T/\tau} \phi_{(i-1)\tau}\left(x_{i\tau} - \left(1+r\right)^{\tau} x_{(i-1)\tau}\right)\left(1+r\right)^{T-i\tau} \qquad (6.8)$$

So far we have made *no assumptions* about the actual movement of the underlying asset: x_t could represent any arbitrary process. At the end of this Chapter we will include an additional contribution from the cost of transacting the underlying at each timestep $t = i\tau$. This contribution aside, Equation (6.8) is general.

If we had made the Black-Scholes assumption of continuous time, which implies taking the limit $\tau \to 0$, then the summation would turn into an integral and Equation (6.8) would become:

$$\Delta W_T = V_0 e^{rT} - V_T + \int_0^T \phi_t\left(\frac{dx_t}{dt} - r x_t\right) e^{r(T-t)} dt \qquad (6.9)$$

However, we do not wish to make this assumption: the formalism does not require it and, amongst other things, we wish to investigate the effects of *discrete hedging* on the risk of writing an option. Discrete hedging simply refers to the process of changing the number of assets held to hedge the option, at discrete time intervals. Note that as written, Equation (6.8) treats the intervals between successive re-hedges as being of equal length τ, but this need not mean the hedge must change at each time $t = i\tau$ because we can easily have $\phi_{i\tau} = \phi_{(i-1)\tau}$. Hence the regularity of the discreteness in time does not limit the applicability of the formalism.

We can now use Equation (6.8) to examine the risk of option writing under different schemes of hedging, different underlying asset movements and different option types (e.g. different payout functions). As an illustration, let us examine the distribution of variation in wealth for different values of the trading time τ. As for Figure 6-2, we simulate repeatedly the process of writing and (this time) hedging an option on an asset that moves with a random walk. We keep constant the option parameters such as the initial asset value x_0, the strike price X, the expiry time T and the volatility of the underlying asset's movement σ. Instead it is the 'realization' of the underlying asset's price $\{x_t\}\big|_{t=0\to T}$ which changes, each time we simulate the option writing and hedging process. 'Realization' refers to the specific evolution in time, which in this case is random (lognormal[5]). Figure 6-3 shows an example of five different lognormal asset price realizations:

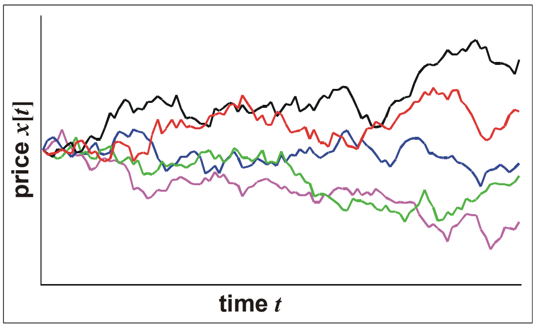

Figure 6-3: Five different 'realizations' $\{x_t\}\big|_{t=0\to T}$ of a random (lognormal) underlying asset price movement.

For each re-hedging time $t = i\tau$ during the asset price realization x_t, the hedge $\phi_{i\tau}\left[x_{i\tau}\right]$ is calculated using the Black-Scholes delta-hedging recipe (recall Section 2.4.3). At the end of the realization, i.e. at $t = T$, we use Equation (6.8) to calculate the overall variation of the option writer's wealth ΔW_T. This process is repeated, and a histogram constructed of all the ΔW_T values. Figure 6-4 below shows an example of such a 'Monte-Carlo' simulation. Each of the histograms was constructed for 5000 realizations of the underlying asset price movement. The four different histograms correspond to four different values for the trading time τ:

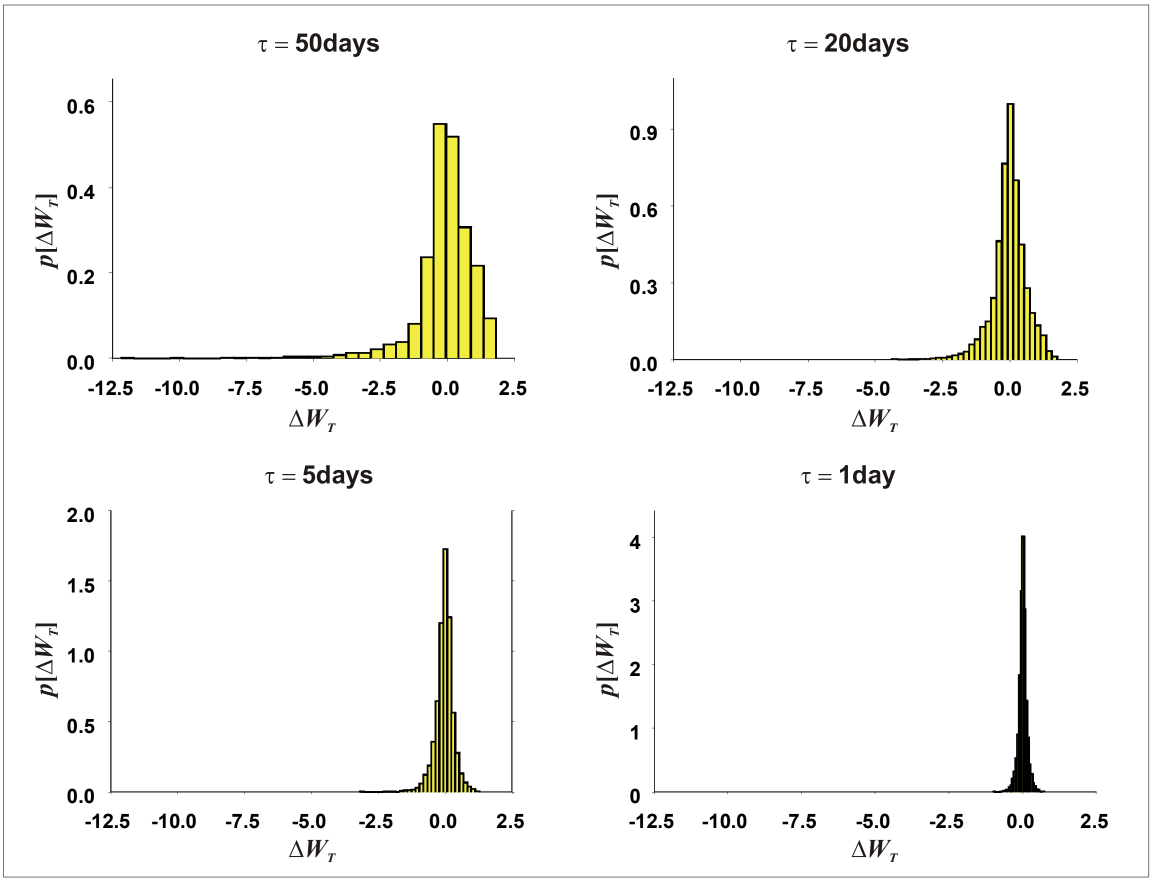

Figure 6-4: Histograms showing the PDF $p\left[\Delta W_T\right]$ due to writing an option with different frequencies of re-hedging. These results were produced using a Monte-Carlo simulation of 5000 underlying asset price random walks. The option contract considered was a European put with $x_0 = 8,\ X = 10,\ T = 100\,\text{days},\ r = 0$. The option was priced and hedged in accordance with the Black-Scholes theory (Chapter 2). The PDF of the underlying's movement $p\left[R_{t,t-1}\right]$ was taken to be lognormal with $\sigma = 5\%$.

Figure 6-4 shows that as the trading time decreases (i.e. the frequency of re-hedging increases) the spread in the variation of wealth decreases. This means less risk for the option writer. Let us now examine this dependence of the risk on trading time more closely by constructing a graph of the standard deviation of the distribution for the variation in wealth $\sigma\left[\Delta W_T\right]$ as a function of the trading time τ :

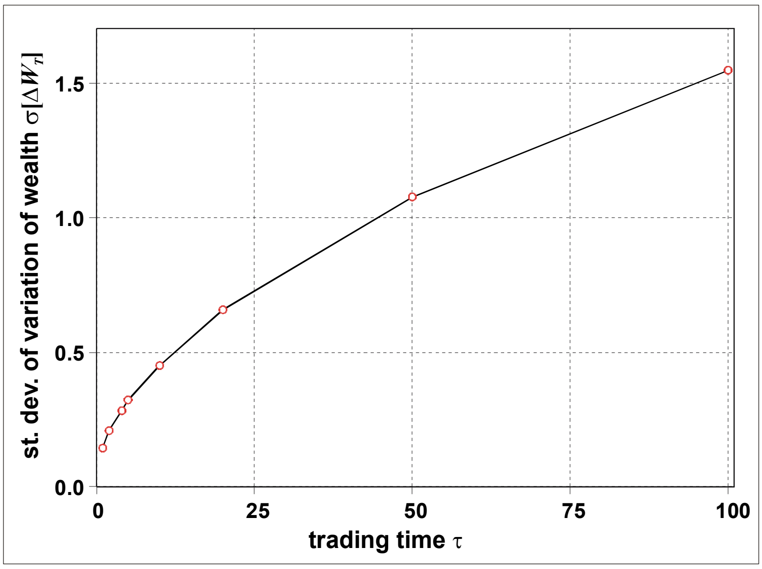

Figure 6-5: The standard deviation of the variation of the option writer's wealth $\sigma\left[\Delta W_T\right]$ as a function of the trading time τ. The option contract considered was a European put with $x_0 = 8,\ X = 10,\ T = 100$ days and $r = 0$. The PDF of the underlying's movement $p\left[R_{t,t-1}\right]$ was taken to be lognormal with $\sigma = 5\%$.

Figure 6-5 shows very clearly that as we decrease the trading time (i.e. increase our frequency of re-hedging) the spread in the distribution of ΔW_T drops markedly. In fact, the dependence of the standard deviation $\sigma\left[\Delta W_T\right]$ on trading time τ essentially follows a square-root dependence:

$$\sigma\left[\Delta W_T\right] \propto \sqrt{\tau} \tag{6.10}$$

Equation (6.10) carries the implication that as the trading time reduces to zero (i.e. $\tau \to 0$) the spread in the distribution of wealth variation $\sigma\left[\Delta W_T\right]$ also reduces to zero. This essentially recovers the Black-Scholes result, where continuous re-hedging using the delta-hedging strategy removes all of the stochastic variation from the option writer's portfolio, yielding zero risk. This is expected: our Monte-Carlo simulation was consistent with the Black-Scholes theory in that we modelled the underlying asset's price movement as a random walk, with $p\left[R_{t,t-1}\right]$ being lognormal.

We know that the random walk model for the underlying asset price movement, is not in general a good one (recall Chapter 3). What would happen if we made another choice for $p\left[R_{t,t-1}\right]$? Will the Black-Scholes recipe still work its magic of zero-risk with continuous re-hedging? To answer this, we can repeat the Monte-Carlo simulation exactly as before, but this time using a slightly more realistic model for the underlying asset price movement. As a demonstration, we will use a process known as the Hull-White model for the underlying asset price movement. This belongs to a class of models having *stochastic volatility*[6]. Stochastic volatility refers to the fact that in the random evolution of the asset price,

$$\frac{dx}{x} = \mu dt + \sigma dX_1 , \qquad (6.11)$$

the volatility σ *also* undergoes random evolution. In the Hull-White model, the volatility performs a mean-reverting random walk given by:

$$d\left(\sigma^2\right) = a\left(b - \sigma^2\right)dt + c\sigma^2 dX_2 \qquad (6.12)$$

where a, b, c are constants and dX_1, dX_2 are observations of uncorrelated Gaussian variables with zero mean and variance proportional to dt. Our main concern is not the price process itself, but rather to examine the risk of option writing when $p\left[R_{t,t-1}\right]$ incorporates some flavour of the 'stylized facts' observed in empirical price movements (recall Chapter 3). In particular with suitable choices of a, b, c, the probability density function $p\left[R_{t,t-1}\right]$ has a higher kurtosis (i.e. more peaked, with fatter tails) than the lognormal, as shown below:

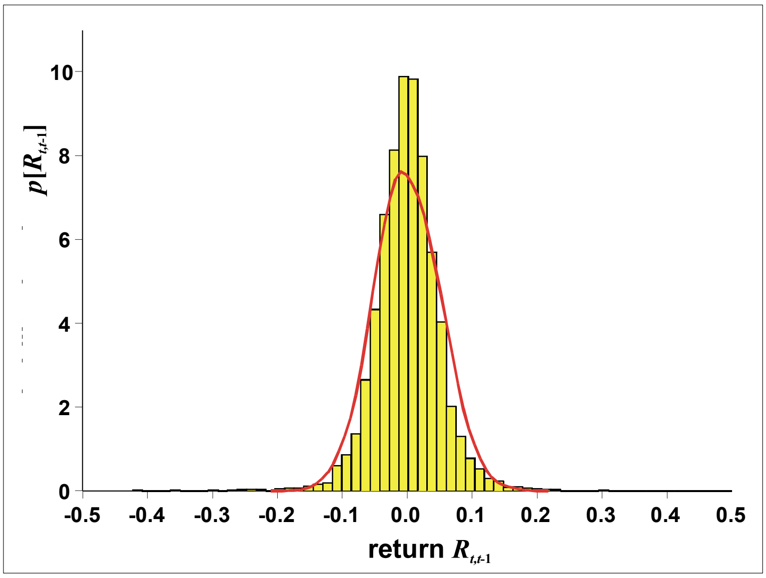

Figure 6-6: PDF of returns $p\left[R_{t,t-1}\right]$ for a Hull-White stochastic volatility model and a lognormal model. The yellow histogram represents the Hull-White model with $a = 0.05,\ b = 0.05^2,\ c = 0.25$. The solid red line represents the lognormal model with $\sigma = 0.05$.

[6] For a description of stochastic differential equations like Equation (6.11) and (6.12) see Section 2.2.5 and [WDH]. Although stochastic volatility models represent some improvement over a straightforward random walk, they still cannot capture all the higher-order temporal correlations and scaling properties found in Chapter 3 for real market data.

Figure 6-7 compares the resulting dependence of $\sigma\left[\Delta W_T\right]$ on the trading time for the Hull-White model, and the lognormal price process from Figure 6-5:

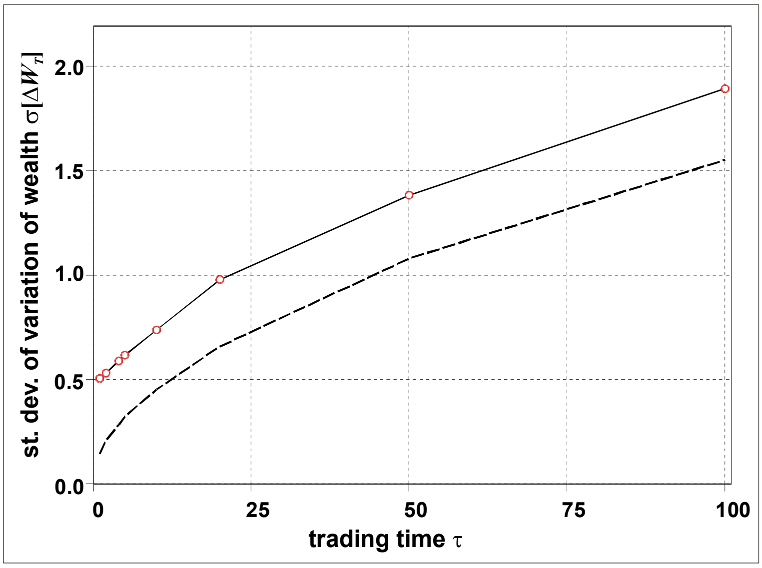

Figure 6-7: The standard deviation of the variation of the option writer's wealth $\sigma\left[\Delta W_T\right]$ as a function of the trading time τ for Hull-White (solid line) and lognormal (dashed line) asset models. The option contract considered was a European put with $x_0 = 8,\ X = 10,\ T = 100\,\text{days},\ r = 0,\ \sigma = 5\%$.

Figure 6-7 shows a marked *increase* of risk for *all* trading times when using the more realistic Hull-White stochastic volatility model for the underlying. Most importantly, when we extrapolate $\sigma\left[\Delta W_T\right]$ back to $\tau = 0$ *we no longer get the zero-risk result of the Black-Scholes continuous delta-hedging recipe*. Followers of the Black-Scholes philosophy might claim that the delta-hedge clearly needs to be modified in light of the new stochastic differential equation governing the asset price movements. In fact, the Black-Scholes formula *can* be re-formulated for stochastic volatility models like the Hull-White model: tricks such as hedging the option not only with a quantity of underlying assets but also a quantity of other options, can be used to *theoretically* reduce the risk for continuous hedging to zero once more. However, here's the big problem for such Black-Scholes followers: in general there is *no* such differential equation model, like the coupled equations (6.11) and (6.12), which can adequately reproduce all the important features of the financial market price-series of interest. So how can one even start to generalize Black-Scholes under these conditions? As suggested by the discussions in Chapters 2 and 3, the 'no model' situation will be the rule rather than the exception. Hence we need to move to the next stage of Bouchaud and Sornette's formalism. We already have a general analytical formula for calculating the variation of the option writer's wealth: now we want to see what happens when we average over all possible *real-world* realizations. This will give us an expression for the *real-*

world option price, i.e. an option price formula in the absence of a local-time differential model for the asset price evolution.

6.4.2 Price for a real-world option

Instead of using a particular model for the underlying, we just consider the financial data itself in order to obtain the option price. We first simplify Equation (6.8) for the variation of the option writer's wealth, by assuming that the risk-free interest rate $r = 0$. This approximation is simply a shift of 'reference frame': we are moving everything to a world where the current value of cash is equal to the future value[7]. We do not *need* to make this approximation here, but the resulting mathematics is made far clearer with it. Hence Equation (6.8) becomes

$$\Delta W_T = V_0 - V_T + \sum_{i=0}^{T/\tau-1} \phi_{i\tau}\left(x_{(i+1)\tau} - x_{i\tau}\right) \tag{6.13}$$

where for convenience we have re-defined $i \to i+1$, which changes nothing. Now let's find the mean at $t = 0$ of this variation in wealth over all possible realizations of the underlying asset's price movement $\{x_t\}\big|_{t=0\to T}$ during the lifetime of the option, i.e. $\langle \Delta W_T \rangle_{x_0 \dots x_T}$. We average over Equation (6.13), making the substitution $t = i\tau$ to make the result easier to read:

$$\langle \Delta W_T \rangle_{x_0 \dots x_T} = V_0\left[x_0, X, T\right] - \left\langle V_T\left[x_T, X\right]\right\rangle_{x_T} + \sum_{t/\tau=0}^{T/\tau-1} \left\langle \phi_t\left[x_t\right]\left(x_{t+\tau} - x_t\right)\right\rangle_{x_{t+\tau}|x_t, x_t} \tag{6.14}$$

We have explicitly written out the functional dependencies of V_0, V_T and ϕ_t, to illustrate that the different terms depend on the underlying asset's price at different times: for example the payoff function $V_T\left[x_T, X\right]$ only depends on the asset price at expiry. Equation (6.14) expresses the averaging over realizations $\langle\cdots\rangle_{x_0 \dots x_T}$ as an averaging over just the explicit dependence of the term considered (e.g. for the payoff function we have $\langle\cdots\rangle_{x_0 \dots x_T} \to \langle\cdots\rangle_{x_T}$). This notation is demonstrated in Equation (6.15) below:

$$\begin{aligned}
\left\langle f\left[x_t\right]\right\rangle_{x_0 \dots x_T} &= \int f\left[x_t\right] p\left[x_T, x_{T-\tau}, \dots, x_\tau \mid x_0\right] dx_T dx_{T-\tau} \dots dx_\tau \\
&= \int f\left[x_t\right] p\left[x_t \mid x_0\right] dx_t \\
&\equiv \left\langle f\left[x_t\right]\right\rangle_{x_t}
\end{aligned} \tag{6.15}$$

[7] In fact at the time of writing, interest rates are low across the industrialized world, hence making this a very good approximation.

given that the asset price starts off from x_0 at time $t=0$ with probability unity. For a term that depends on the underlying asset's price at more than one time, for example $f\left[x_{t_1}, x_{t_2}\right]$ with $t_2 > t_1$, the averaging process of Equation (6.15) is expressed as:

$$\begin{aligned}\left\langle f\left[x_{t_1}, x_{t_2}\right]\right\rangle_{x_0 \ldots x_T} &= \int f\left[x_{t_1}, x_{t_2}\right] p\left[x_T, x_{T-\tau}, \ldots, x_\tau \mid x_0\right] dx_T dx_{T-\tau} \ldots dx_\tau \\ &= \int f\left[x_{t_1}, x_{t_2}\right] p\left[x_{t_2} \mid x_{t_1}, x_0\right] p\left[x_{t_1} \mid x_0\right] dx_{t_2} dx_{t_1} \\ &\equiv \left\langle f\left[x_{t_1}, x_{t_2}\right]\right\rangle_{x_{t_2} \mid x_{t_1}, x_{t_1}}\end{aligned} \tag{6.16}$$

We are able to express the averaging as in Equations (6.15) and (6.16), since the value of the function $f\left[x_t\right]$ cannot be conditional on the value of the underlying asset at any *future* time $t' > t$. Note that we have *not* yet made any assumptions regarding statistical independence. Unless otherwise stated, we will drop the explicit dependence of $p\left[x_{t+\tau} \mid x_t, x_0\right]$ on x_0, since x_0 is fixed throughout. The option cost V_0 is not a function of the underlying asset price at any point except $t=0$. Since we assume that all asset price realizations $\left\{x_t\right\}\big|_{t=0 \to T}$ start at the same fixed price x_0, a function of x_0 alone is constant under the averaging process. Let us now expand the summand of Equation (6.14):

$$\begin{aligned}\left\langle \phi_t\left[x_t\right]\left(x_{t+\tau} - x_t\right)\right\rangle_{x_{t+\tau} \mid x_t, x_t} &= \left\langle \phi_t\left[x_t\right]\left\langle x_{t+\tau}\right\rangle_{x_{t+\tau} \mid x_t}\right\rangle_{x_t} - \left\langle \phi_t\left[x_t\right] x_t\right\rangle_{x_t} \\ &= \left\langle \phi_t\left[x_t\right]\left(x_t + \mu_t\right)\right\rangle_{x_t} - \left\langle \phi_t\left[x_t\right] x_t\right\rangle_{x_t} \\ &= \left\langle \phi_t\left[x_t\right] \mu_t\right\rangle_{x_t}\end{aligned} \tag{6.17}$$

In the second line of Equation (6.17) we equate the average value of the underlying price at time $t+\tau$ (i.e. $\left\langle x_{t+\tau}\right\rangle_{x_{t+\tau} \mid x_t}$) to the asset price at time t (i.e. x_t) plus a conditional drift term μ_t. Below is a schematic diagram showing the position of this average value:

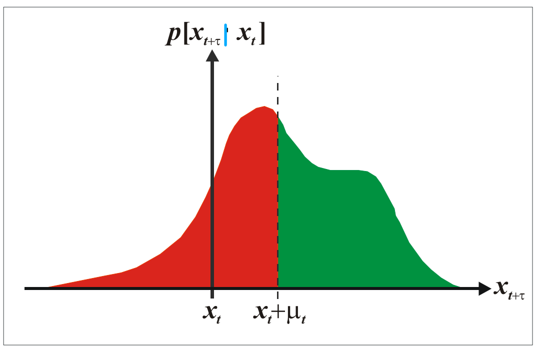

Figure 6-8: Schematic diagram showing the position of the mean of the PDF $p\left[x_{t+\tau} \mid x_t\right]$. The two shaded portions have equal area.

More formally: $$\left\langle x_{t+\tau} \right\rangle_{x_{t+\tau}|x_t} = \int_0^\infty x_{t+\tau}\, p\left[x_{t+\tau} \mid x_t\right] dx_{t+\tau} = x_t + \mu_t \qquad (6.18)$$

However, under un-biased movement of the underlying asset price, the conditional drift term μ_t will be equal to zero. We are now going to make our *first major assumption*, that $\mu_t = 0$. This gives us: $\left\langle x_{t+\tau} \right\rangle_{x_{t+\tau}|x_t} = x_t$, and consequently, the summation term of Equation (6.14) is equal to zero. Later we will discuss relaxing this assumption to account for *biased* asset price movements. We can now go ahead and assert the principle of no arbitrage, which basically states that the option should be written at a 'fair' price - hence neither the writer nor the holder will on average make any money from the contract:

$$\left\langle \Delta W_T \right\rangle_{x_0 \ldots x_T} = 0 \qquad (6.19)$$

Combining equations (6.14) and (6.19), we obtain the price of the contract V_0:

$$\begin{aligned} V_0\left[x_0, X, T\right] &= \left\langle V_T\left[x_T, X\right] \right\rangle_{x_T} \\ &= \int_0^\infty V_T\left[x_T, X\right] p\left[x_T \mid x_0\right] dx_T \end{aligned} \qquad (6.20)$$

If, for example, we set the payoff function to be that of a European vanilla call, i.e. $V_T\left[x_T, X\right] = \max\left[x_T - X, 0\right]$ then the option price from Equation (6.20) becomes:

$$V_0 = \int_X^\infty \left(x_T - X\right) p\left[x_T \mid x_0\right] dx_T \qquad (6.21)$$

6.4.3 Implementing the real-world pricing formula

We have constructed a pricing formula for options *without* the need for an underlying asset price model. The only assumption has been that the increments of the underlying asset's price have zero mean. However unlike the Black-Scholes option price formula for a given payoff function, the pricing formula of Equation (6.20) is not in a 'closed-form': we cannot simply drop the formula into a spreadsheet and expect it to spit out a number. This has been a common criticism of the Bouchaud-Sornette approach: practitioners managing portfolios of thousands of contracts need a very fast pricing system. However, it is easy to be scared-off unnecessarily by the integral sign of Equation (6.20). It really can be a very quick process to numerically obtain the probability density function $p\left[x_T \mid x_0\right]$ and integrate over it. We will demonstrate this with an example using real data. The dataset we will use here is the same as that analyzed in Chapter 3: the daily closing values for the NYSE composite index which, at the time of writing, is freely available from http://www.unifr.ch/econophysics. Our analysis of this data will parallel some of the discussion of data-analysis in Chapter 3. However in contrast to Chapter 3 where we used the entire data-set to characterize the statistical properties, we here want to mimic the typical scenario faced in practice whereby the available dataset is not particularly large. Furthermore we want to provide a step-by-step cookbook of how to implement the statistical analysis of this data in preparation for its use in the formalism, by contrast to the discussion in Chapter 3 which just focused on the end results of this statistical analysis. We will therefore be using a relatively small subset of data. Specifically we will use daily data for the period 1990-1998, instead of the entire record from 1966 onwards.

The first step is to use the series of prices to generate a series of returns over a time increment $\Delta t = T$, the expiry time of the option. This is achieved using the definition of returns (Equation (1.3)):

$$R_t \equiv R_{t,t-T} = \frac{x_t - x_{t-T}}{x_{t-T}} \qquad (6.22)$$

Let's consider a one-month (i.e. $T = 21$ trading days) European call-option. Recalling our assumption of an underlying movement with no drift, we need to de-trend these returns by subtracting the mean, i.e. $R'_t = R_t - \langle R_t \rangle_t$. Next we need to build a histogram of the de-trended return probability, in order to simulate the probability density function $p[x_T \mid x_0]$. Most spreadsheets and analysis packages come with tools to do this automatically, however the process is simple:

a. Identify the minimum $R_{\min}$ and maximum $R_{\max}$ de-trended returns in the series

b. Define a bin-size $\Delta R = (R_{\max} - R_{\min})/\sqrt{n}$ as a guide, where n is the length of the series

c. Define a function $n[R']$, which is the number of detrended returns in the series $\{R'\}$ which have a value in the range $R' \to R' + \Delta R$

d. Calculate $n[R']$ at the discrete values $R' = R_{\min} + j\Delta R$ for each integer j in the range $j \le 0 \le \sqrt{n}$ and assume $n[R']$ is constant within the range $R' \to R' + \Delta R$.

e. Calculate the frequency $n[R']/n$ for each bin

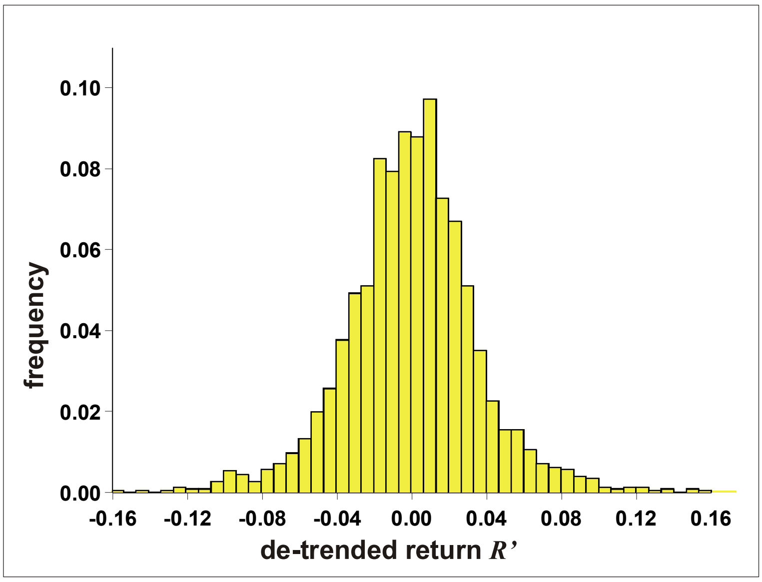

Figure 6-9: Histogram of one-month de-trended returns (i.e. $n[R']/n$) on the NYSE composite index for 1990-8.

The histogram of returns looks non-Gaussian, as also found in Chapter 3 for the larger dataset. In fact the kurtosis $\kappa > 5$ is well in excess of that for a Gaussian. This histogram of returns can be mapped to the PDF we require in the following way. We use the fact that $x_T = x_0 + x_0 R'$ to give

$$p[x_T \mid x_0]dx_T = p\left[R' = \tfrac{x_T - x_0}{x_0}\right]dx_T$$

The probability $p\left[R' = \tfrac{x_T - x_0}{x_0}\right]$ can be obtained from our histogram (see Figure 6-9). As $n[R']/n$ gives the frequency of occurrence of returns within a return-interval ΔR, then the density of this occurrence is given by dividing by the interval length. In price space (x_T) this interval length is $x_0\,\Delta R$ thus:

$$p[x_T \mid x_0]dx_T = p\left[R' = \tfrac{x_T - x_0}{x_0}\right]dx_T = \frac{n\left[\tfrac{x_T - x_0}{x_0}\right]}{n\,x_0\,\Delta R}dx_T \qquad (6.23)$$

This gives us a PDF of the form needed to calculate the option price: so let's now use it in Equation (6.21). First let $x[j] = x_0(1 + R_{\min} + j\,\Delta R)$. Then, using Equation (6.21) and (6.23):

$$\begin{aligned} V_0[x_0, X, T] &= \int_X^\infty (x_T - X)\,p[x_T \mid x_0]dx_T \\ &= \sum_{j=0}^{\sqrt{n}-1}\left(\mathrm{H}[x[j]-X]\frac{n[R_{\min} + j\,\Delta R]}{n\,x_0\,\Delta R}\int_{x[j]}^{x[j+1]}(x_T - X)dx_T\right) \qquad (6.24) \\ &= \sum_{j=0}^{\sqrt{n}-1}\left(\mathrm{H}[x[j]-X]\frac{n[R_{\min} + j\,\Delta R]}{n}\left((x[j]-X) + \frac{x_0\,\Delta R}{2}\right)\right) \end{aligned}$$

where $\mathrm{H}[x]$ is the Heaviside function. We have used the fact that, due to our method of forming the histogram, the probability is constant in any bin. Equation (6.24) may look complicated, but it is really quite simple and fast to evaluate numerically since the number of terms in the sum (i.e. $\sqrt{n}$) is typically small. The results from pricing an option with initial asset value $x_0 = 500$, and our one month maturity $T = 21$, are shown below in Figure 6-10 for a range of values of the strike price X :

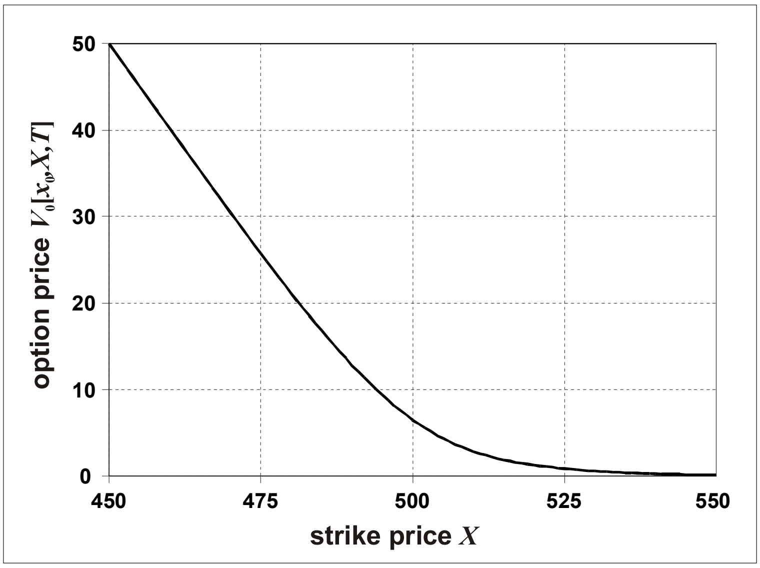

Figure 6-10: The calculated real-world option price V_0 as a function of the strike price X for a European call option of initial spot value $x_0 = 500$ and expiry $T = 21$ days on the NYSE composite index.

This general variation of call option value with strike price, is as expected. What we really want to do at this point is to compare this generated 'real-world option price' with the Black-Scholes result from Chapter 2. Instead of just putting the two prices side-by side, let's use the common trick in finance of running the Black-Scholes formula *backwards* from our supposed 'real-world option price'. We can then solve for the Black-Scholes volatility σ that *would* have resulted in this same price. This quantity is known as the 'implied-volatility':

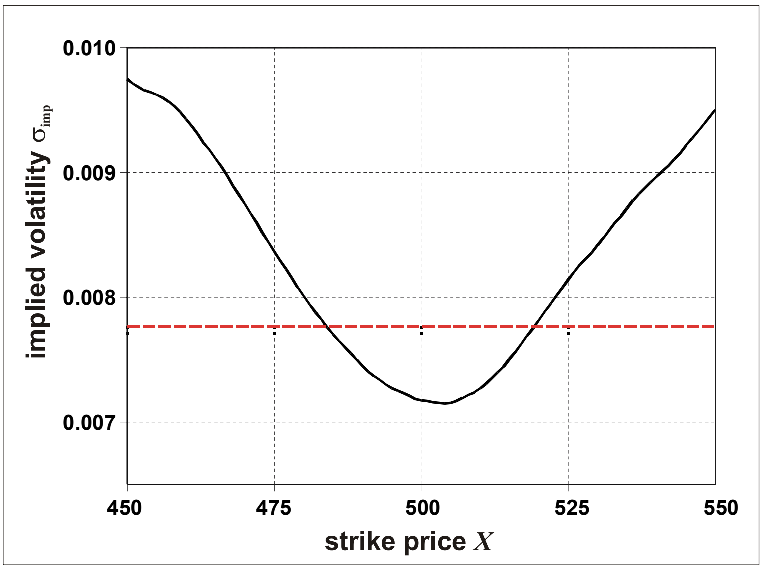

Figure 6-11: Implied volatility as a function of option strike price for a European call option of initial spot value $x_0 = 500$ and expiry $T = 21$ days on the NYSE composite index. The prices used to generate the implied volatility curve were calculated using Equation (6.24). The horizontal dashed line shows the value of the historical volatility.

Figure 6-11 provides a wonderful example of the 'implied-volatility smile' (though its shape is often more of a 'smirk' or even a 'frown'.). This is the same type of result that one obtains from calculating the implied volatility from actual traded option prices, thereby giving us confidence in the new formalism[8].

6.4.4 Quantifying the risk analytically

We now turn our attention to minimizing the *spread* of the wealth distribution, and in turn the option writer's 'risk'. We use the term 'risk' rather casually here to imply the uncertainty in the option writer's profit-and-loss situation. We want to keep our approach reasonably general, hence we will consider an adequate measure of uncertainty in an outcome to be an increasing function of the outcome's variance. With this in mind, the minimization of the outcome's risk becomes a minimization of the variance of that outcome. Since we are considering our outcome to be the profit or loss the option writer experiences, i.e. the 'variation of wealth' ΔW_T, we therefore need to minimize the variance $\left\langle (\Delta W_T)^2 \right\rangle_{x_0 \ldots x_T} - \left(\langle \Delta W_T \rangle_{x_0 \ldots x_T} \right)^2$. We have already asserted however that $\langle \Delta W_T \rangle_{x_0 \ldots x_T} = 0$

[8] As with all numerical implementations, the evaluation of Equation (6.20) described in this section has some intrinsic numerical error. This error arises from the discrete binning of the data when forming the PDF. Equation (6.24) approximates the lower limit of the integral in Equation (6.20) to the nearest bin. For simplicity, we suggested a number of bins of order $\sqrt{n}$. In general the greater the number of bins, the more sensitive the calculation is to the finite nature of the dataset.

by the principal of no arbitrage (or equivalently setting a 'fair' option price), hence the variance of the option writer's variation in wealth at expiry is given by:

$$\begin{aligned}\operatorname{var}\left[\Delta W_T\right] &= \left\langle \left(\Delta W_T\right)^2 \right\rangle_{x_0 \dots x_T} \\ &= \left\langle \left(V_0\left[x_0, X, T\right] - V_T\left[x_T, X\right] + \sum_{t/\tau=0}^{T/\tau-1} \phi_t\left[x_t\right]\left(x_{t+\tau} - x_t\right) \right)^2 \right\rangle_{x_0 \dots x_T}\end{aligned} \tag{6.25}$$

Expanding Equation (6.25) gives six distinct terms. Remembering that the form of Equation (6.4) is

$$\Delta W_T = value - payout + hedging\ profit$$

leads us to treat these six terms one by one:

value $\times$ *value*

$$\left\langle V_0\left[x_0, X, T\right]^2 \right\rangle_{x_0 \dots x_T} = V_0\left[x_0, X, T\right]^2 = \left(\left\langle V_T\left[x_T, X\right] \right\rangle_{x_T} \right)^2 \tag{6.26}$$

The option price is given by Equation (6.20) and is constant over all realizations of the underlying price process. This is because x_0, which is constant for all realizations, is the only asset price on which it depends.

payout $\times$ *payout*

$$\left\langle \left(V_T\left[x_T, X\right] \right)^2 \right\rangle_{x_0 \dots x_T} = \left\langle \left(V_T\left[x_T, X\right] \right)^2 \right\rangle_{x_T} \tag{6.27}$$

The payoff function is only a function of the underlying asset's price at expiry x_T, hence the averaging is over this price alone.

hedging profit $\times$ *hedging profit*

$$\begin{aligned}\left\langle \left(\sum_{t/\tau=0}^{T/\tau-1} \phi_t\left[x_t\right]\left(x_{t+\tau} - x_t\right) \right)^2 \right\rangle_{x_0 \dots x_T} &= \sum_{t/\tau, t'/\tau=0}^{T/\tau-1} \left\langle \phi_t\left[x_t\right]\left(x_{t+\tau} - x_t\right) \phi_{t'}\left[x_{t'}\right]\left(x_{t'+\tau} - x_{t'}\right) \right\rangle_{x_0, \dots, x_T} \\ &= \sum_{t/\tau=0}^{T/\tau-1} \left\langle \left(\phi_t\left[x_t\right]\right)^2 \left(x_{t+\tau} - x_t\right)^2 \right\rangle_{x_{t+\tau}|x_t, x_t} + \sum_{t/\tau \neq t'/\tau}^{T/\tau-1} \left\langle \phi_t\left[x_t\right]\left(x_{t+\tau} - x_t\right) \phi_{t'}\left[x_{t'}\right]\left(x_{t'+\tau} - x_{t'}\right) \right\rangle_{x_0, \dots, x_T}\end{aligned} \tag{6.28}$$

We first represent the squared sum as a double sum over the time labels t and t', and then split this into two separate parts: one where $t = t'$ and one where $t \neq t'$. We could then use Equation (6.16) on the second of these sums, but this gets messy for the general scenario. Instead we choose to make an assumption about the underlying asset's movement. This will be our *second major assumption*, that the price increments $\Delta x_{t,t-\tau} = x_t - x_{t-\tau}$ and $\Delta x_{t' \neq t, t'-\tau} = x_{t'} - x_{t'-\tau}$ are uncorrelated. With this assumption, the second sum in Equation (6.28) vanishes. We again use our first assumption, i.e. $\mu_t = 0$, and perform the averaging over $x_{t+\tau}$ in the first sum:

$$\left\langle \left(\sum_{t/\tau=0}^{T/\tau-1} \phi_t[x_t](x_{t+\tau} - x_t) \right)^2 \right\rangle_{x_0 \ldots x_T} \begin{aligned} &= \sum_{t/\tau=0}^{T/\tau-1} \left\langle (\phi_t[x_t])^2 (x_{t+\tau} - x_t)^2 \right\rangle_{x_{t+\tau}|x_t, x_t} \\ &= \sum_{t/\tau=0}^{T/\tau-1} \left\langle (\phi_t[x_t])^2 \int_0^\infty \left(x_{t+\tau} - \langle x_{t+\tau} \rangle_{x_{t+\tau}|x_t} \right)^2 p[x_{t+\tau} \mid x_t] dx_{t+\tau} \right\rangle_{x_t} \\ &= \sum_{t/\tau=0}^{T/\tau-1} \left\langle (\phi_t[x_t])^2 \sigma_{t+\tau,t}{}^2 \right\rangle_{x_t} \end{aligned} \quad (6.29)$$

In the second line of Equation (6.29) where we explicitly carry out the averaging over $x_{t+\tau} \mid x_t$, we use $x_t = \langle x_{t+\tau} \rangle_{x_{t+\tau}|x_t}$. This makes it easy to identify the integral as the variance of the distribution of the underlying asset price between times t and $t+\tau$, which we will call $\sigma_{t+\tau,t}{}^2$.

$-2\times$ *value* $\times$ *payout*

$$\begin{aligned} \left\langle -2V_0[x_0, X, T] V_T[x_T, X] \right\rangle_{x_0 \ldots x_T} &= -2 \left\langle \langle V_T[x_T, X] \rangle_{x_T} V_T[x_T, X] \right\rangle_{x_0 \ldots x_T} \\ &= -2 \left(\langle V_T[x_T, X] \rangle_{x_T} \right)^2 \end{aligned} \quad (6.30)$$

The option price is constant, as given by Equation (6.20). The payoff is just a function of x_T.

$2\times$ *value* $\times$ *hedging profit*

$$\begin{aligned} \left\langle 2V_0[x_0, X, T] \sum_{t/\tau=0}^{T/\tau-1} \phi_t[x_t](x_{t+\tau} - x_t) \right\rangle_{x_0 \ldots x_T} &= 2V_0[x_0, X, T] \sum_{t/\tau=0}^{T/\tau-1} \left\langle \phi_t[x_t](x_{t+\tau} - x_t) \right\rangle_{x_{t+\tau}|x_t, x_t} \\ &= 0 \end{aligned} \quad (6.31)$$

The option price is constant, hence is unaffected by averaging over realizations. We then used Equation (6.17) and our assumption of zero conditional drift ($\mu_t = 0$) to reduce the summand and hence entire term to zero.

$-2\times$ *payout* $\times$ *hedging profit*

$$\left\langle -2V_T[x_T, X] \sum_{t/\tau=0}^{T/\tau-1} \phi_t[x_t](x_{t+\tau} - x_t) \right\rangle_{x_0 \ldots x_T} = -2 \sum_{t/\tau=0}^{T/\tau-1} \left\langle V_T[x_T, X] \phi_t[x_t](x_{t+\tau} - x_t) \right\rangle_{x_0, \ldots, x_T} \quad (6.32)$$

The summand on the right hand side of Equation (6.32) contains the asset price at times t, $t+\tau$ and T. This means that we need to consider realizations that start at x_t, pass through $x_{t+\tau}$ and end at x_T. We evaluate this complicated conditional average in the following way:

$$\left\langle -2V_T\left[x_T,X\right]\sum_{t/\tau=0}^{T/\tau-1}\phi_t\left[x_t\right]\left(x_{t+\tau}-x_t\right)\right\rangle_{x_0\ldots x_T} = -2\sum_{t/\tau=0}^{T/\tau-1}\left\langle \begin{array}{l} V_T\left[x_T,X\right]\phi_t\left[x_t\right]\times \\ \int_0^\infty \left(x_{t+\tau}-x_t\right)p\left[x_{t+\tau}\mid x_t,x_T\right]dx_{t+\tau}\end{array}\right\rangle_{x_T\mid x_t,x_t} \qquad (6.33)$$

$$= -2\sum_{t/\tau=0}^{T/\tau-1}\left\langle V_T\left[x_T,X\right]\phi_t\left[x_t\right]\left\langle \Delta x_{t+\tau,t}\right\rangle_{x_t\to x_T}\right\rangle_{x_T\mid x_t,x_t}$$

where $\left\langle \Delta x_{t+\tau,t}\right\rangle_{x_t\to x_T}$ represents an average increment in a realization of the underlying asset's price evolution which starts at price x_t and ends at price x_T. The price increment $\Delta x_{t+\tau,t}=x_{t+\tau}-x_t$.

We now have all the terms in the equation for the variance of the variation in the option writer's wealth. We can finally put all these contributing terms (Equations (6.26), (6.27), (6.29), (6.30), (6.31) and (6.33)) together to give:

$$\mathrm{var}\left[\Delta W_T\right]=\left\langle \left(V_T\left[x_T,X\right]\right)^2\right\rangle_{x_T}-\left(\left\langle V_T\left[x_T,X\right]\right\rangle_{x_T}\right)^2+\sum_{t/\tau=0}^{T/\tau-1}\left(\begin{array}{l}\left\langle \left(\phi_t\left[x_t\right]\right)^2\sigma_{t+\tau,t}{}^2\right\rangle_{x_t} - \\ 2\left\langle V_T\left[x_T,X\right]\phi_t\left[x_t\right]\left\langle \Delta x_{t+\tau,t}\right\rangle_{x_t\to x_T}\right\rangle_{x_T\mid x_t,x_t}\end{array}\right) \qquad (6.34)$$

Writing the averages in Equation (6.34) out explicitly gives us:

$$\begin{aligned}\mathrm{var}\left[\Delta W_T\right]&=\mathrm{R}\\ &=\mathrm{R}_c+\sum_{t/\tau=0}^{T/\tau-1}\int_0^\infty p\left[x_t\mid x_0\right]\left(\begin{array}{l}\sigma_{t+\tau,t}{}^2\left(\phi_t\left[x_t\right]\right)^2- \\ 2\phi_t\left[x_t\right]\int_0^\infty V_T\left[x_T,X\right]\left\langle \Delta x_{t+\tau,t}\right\rangle_{x_t\to x_T}\, p\left[x_T\mid x_t\right]dx_T\end{array}\right)dx_t\end{aligned} \qquad (6.35)$$

where R_c is given by:

$$\mathrm{R}_c=\int_0^\infty \left(V_T\left[x_T,X\right]\right)^2 p\left[x_T\mid x_0\right]dx_T-\left(\int_0^\infty V_T\left[x_T,X\right]p\left[x_T\mid x_0\right]dx_T\right)^2$$

Equation (6.35) gives us an analytical expression for calculating the variance of the variation of wealth distribution. This variance measure can then be used in our chosen model for calculating risk.

6.4.5 Risk-minimizing hedging strategy

Our measure of the variance in the variation of the option writer's wealth, depends on the hedging strategy $\phi_t\left[x_t\right]$ which is adopted. It is now our objective to find a form for the hedging strategy which minimizes this variance, and hence our chosen risk measure. This is accomplished by means of a functional minimization of Equation (6.35). In practical terms, this corresponds to a simple differentiation with respect to the function ϕ:

$$\frac{\partial \mathrm{R}}{\partial \phi_t[x_t]} = 0 \tag{6.36}$$

Combining Equations (6.35) and (6.36) gives us:

$$\sum_{t/\tau=0}^{T/\tau-1} \int_0^\infty p[x_t \mid x_0] \left(2\sigma_{t+\tau,t}{}^2 \phi_t[x_t] - 2\int_0^\infty V_T[x_T, X] \langle \Delta x_{t+\tau,t} \rangle_{x_t \to x_T} p[x_T \mid x_t] dx_T \right) dx_t = 0 \tag{6.37}$$

The simplest way to satisfy Equation (6.37) is to set the integrand equal to zero. This ensures that for any general choice of price process PDF $p[x_t \mid x_0]$, the equation will be satisfied and a minimum risk measure then assured. Note here that Equation (6.35) is an upward-curving parabola in $\phi_t[x_t]$ and hence a minimum (rather than a maximum) in the risk is assured by Equation (6.36). Hence:

$$\phi_t^*[x_t] = \frac{1}{\sigma_{t+\tau,t}{}^2} \int_0^\infty V_T[x_T, X] \langle \Delta x_{t+\tau,t} \rangle_{x_t \to x_T} p[x_T \mid x_t] dx_T \tag{6.38}$$

Equation (6.38) gives us the 'optimal' hedging strategy $\phi_t^*[x_t]$. The strategy is optimal in the sense that it is the single strategy which minimizes the variance of the option writer's wealth and hence our chosen measure of risk. We can see how much risk remains by using this form of the optimal strategy in Equation (6.35) for the variance. This gives us a 'residual risk' R^* given by:

$$\mathrm{R}^* = \mathrm{R}_c - \sum_{t/\tau=0}^{T/\tau-1} \int_0^\infty \sigma_{t+\tau,t}{}^2 \left(\phi_t^*[x_t] \right)^2 p[x_t \mid x_0]\, dx_t \tag{6.39}$$

The optimal strategy of Equation (6.38) can be simplified further if we make the *additional assumption* that the increments $\Delta x_{t,t-\tau} = x_t - x_{t-\tau}$ are independent and identically distributed[9]. This means that the evolution of the underlying asset does not change behaviour during the life of the option. Under this assumption we have:

$$\sigma_{t+\tau,t}{}^2 = \sigma^2 \tau, \quad \langle \Delta x_{t+\tau,t} \rangle_{x_t \to x_T} = \frac{x_T - x_t}{T - t} \tau \tag{6.40}$$

where σ is the stationary standard deviation of increments $\Delta x_{t,t-1} = x_t - x_{t-1}$. In Equation (6.38) this gives:

$$\phi_t^*[x_t] = \frac{1}{\sigma^2 (T-t)} \int_0^\infty (x_T - x_t) V_T[x_T, X]\, p[x_T \mid x_t] dx_T \tag{6.41}$$

For the case of a European call option, Equation (6.41) gives the risk-minimizing optimal hedging strategy to be:

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[9] It is possible that a weaker condition could suffice. Depending on the PDFs of price-increments, it may be enough that the increments are uncorrelated and have identical means and variances. For simplicity, we will impose the more general assumption of i.i.d. increments.

$$\phi_t^*[x_t] = \frac{1}{\sigma^2(T-t)} \int_X^{\infty} (x_T - x_t)(x_T - X)\, p[x_T \mid x_t]\, dx_T \qquad (6.42)$$

6.4.6 Implementing the optimal strategy

The process of taking a set of real financial data and using it to generate a risk-minimizing optimal hedging strategy according to Equation (6.41), seems at first to be similar to the process of implementing the option pricing equation. We have to construct the PDF $p[x_T \mid x_t]$ and calculate the variance of price increments σ^2. Although this seems straightforward following Section 6.4.3, there are some pitfalls when dealing with real financial data. These pitfalls arise mainly due to our assumptions about the data[10]: first that it has zero-mean and is uncorrelated, and then that it is i.i.d.

6.4.6.1 Using real data

Let's return to look at the data itself: the NYSE composite index daily values 1990-8 which we used in Section 6.4.3 to price an option. We will take our assumptions one by one, and test their validity for this dataset. Our first assumption was that μ_t, the conditional mean of the increments $\Delta x_{t,t-\tau} = x_t - x_{t-\tau}$, was equal to zero (recall Equation (6.18)). Let's examine this by first forming the series of returns $R_t \equiv R_{t,t-1} = \left((x_t - x_{t-1})/x_{t-1}\right)$. The value of μ_t can then be calculated and compared to σ:

$$\mu_t \approx x_0 \tau \langle R_t \rangle_t = 5.1\times10^{-4} x_0 \tau \qquad \sigma \approx x_0 \sqrt{\tau \left\langle \left(R - \langle R_t \rangle_t\right)^2 \right\rangle_t} = 7.8\times10^{-3} x_0 \sqrt{\tau} \qquad (6.43)$$

Hence for relatively small values of the interval τ, the mean increment in the underlying's price is indeed much smaller than its fluctuations. Next, let's look at our assumption of uncorrelated increments in price: we used this assumption in quantifying the risk analytically in Section 6.4.4. We will calculate the linear correlation coefficient $\rho[x, y]$ defined for timeseries $\{x_t\}$ and $\{y_t\}$ in a similar way to Chapter 3:

$$\rho[\{x_t\},\{y_t\}] = \frac{\left\langle \left(x_t - \langle x_t \rangle_t\right)\left(y_t - \langle y_t \rangle_t\right) \right\rangle_t}{\sqrt{\left\langle \left(x_t - \langle x_t \rangle_t\right)^2 \right\rangle_t \left\langle \left(y_t - \langle y_t \rangle_t\right)^2 \right\rangle_t}} \qquad (6.44)$$

We wish to investigate whether the increment in price at time t, i.e. $\Delta x_{t,t-\tau}$, is correlated with the increment in price at an earlier time $t' < t$. We therefore examine the autocorrelation, i.e.

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[10] The attraction of the more general formalism developed in this Chapter, is that these assumptions do not *need* to be made. The formalism becomes more cumbersome if they are not made, but the approach remains valid.

$\rho\left[\{R_t\},\{R_{t'}\}\right]$ where $\{R_{t'}\}$ represents the series of returns $\{R_t\}$ shifted in time such that $t' = t - \Delta t$.

Figure 6-12 below shows the autocorrelation of the series of returns $\{R_t\}$ and absolute returns $\{|R_t|\}$:

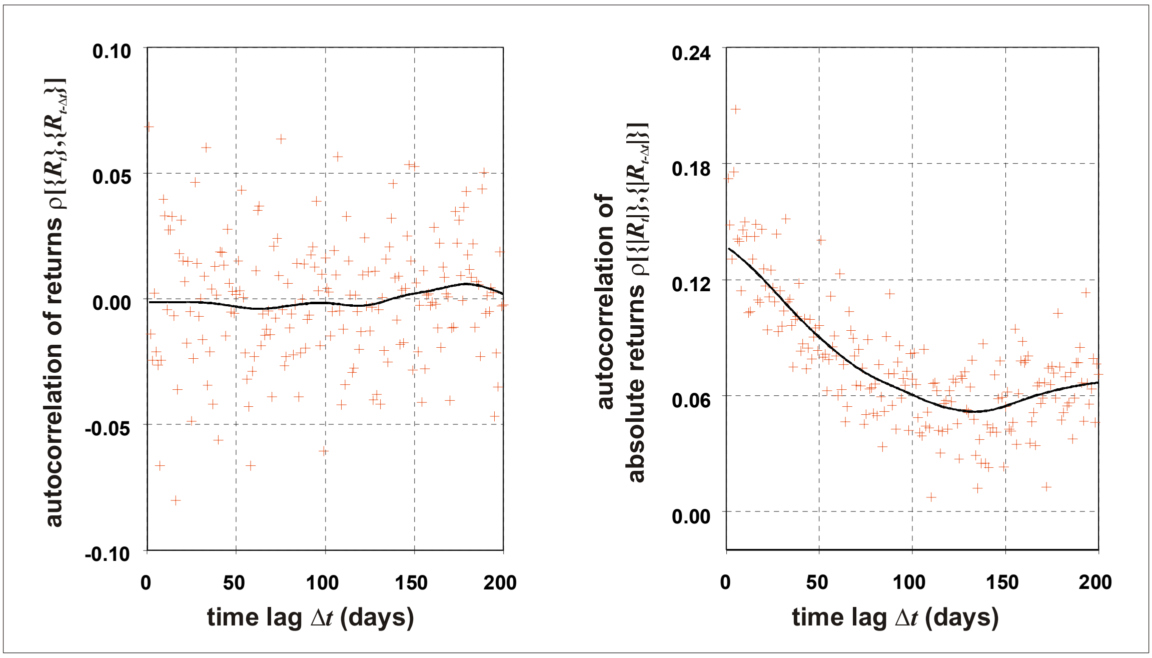

Figure 6-12: Autocorrelation of returns (left panel) and absolute returns (right panel) against lag time. Red crosses represent empirical results, black line gives a moving-average trendline.

Figure 6-12 shows that there is essentially no evidence of linear correlation in the returns series in our dataset. However there is a significant degree of correlation in the absolute returns, as reported earlier in Chapter 3 (see Fig. 3.4). This correlation in the absolute returns only decays very slowly and is still non-negligible even after a whole year of trading. Similar results are found for longer increments $\tau > 1$. This then brings us to our assumption that the increments in the underlying's price $\Delta x_{t,t-\tau}$, are independent and identically distributed (i.i.d.). We used this assumption at the end of Section 6.4.5 to simplify our expressions for the risk-minimizing optimal strategy. The presence of correlations in the absolute returns in Figure 6-12, implies that the price increments are not i.i.d. In Chapter 2, we showed that for uncorrelated increments with identical variances (and hence for i.i.d. increments as well) we expect the variance to scale as $\sigma_{\tau,0}{}^2 = \sigma^2\tau$. The numerical results for the NYSE dataset yield the following graph:

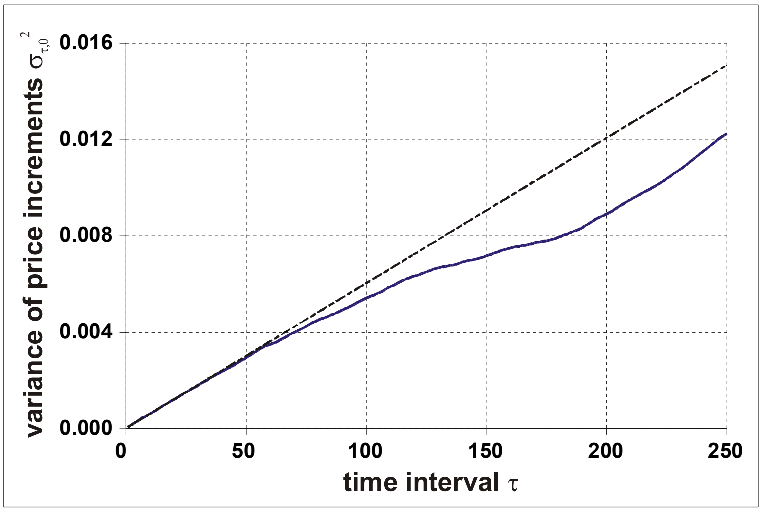

Figure 6-13: The non-linear growth of the variance of returns $\sigma_{\tau,0}^{\;2}$ as a function of time interval τ for the NYSE data (solid line). The dashed line represents growth for a random walk model, i.e. $\sigma_{\tau,0}^{\;2} = \sigma^2\tau$.

As can be seen in Figure 6-13, there is a marked departure in the real data from the i.i.d. prediction of linear growth of variance. However, this departure seems relatively small for increments $\tau \le 50$ trading days. Let us also examine the stationarity of the PDF, since this is also part of the assumption of an i.i.d. price process. To do this we split our dataset into three roughly equal parts, each spanning three years. We then construct the PDF of returns $p\left[R_{t,t-1}\right]$ for each period by constructing a histogram in the same way as detailed in Section 6.4.3:

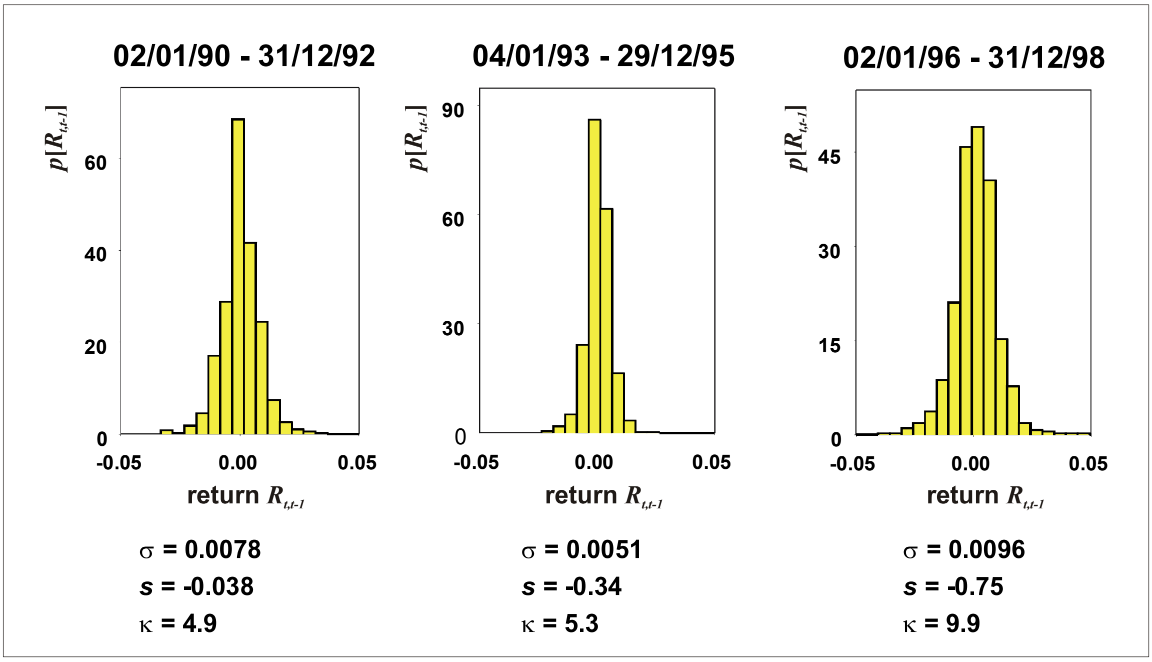

Figure 6-14: Histograms showing probability distribution function (PDF) of daily returns from three periods in the NYSE dataset. Under each plot are the standard deviation σ , skewness (asymmetry) s and the kurtosis (peakedness) κ .

Although the three distributions in Figure 6-14 have a similarly peaked, non-Gaussian shape, the shape parameters σ, s, κ demonstrate that the distributions are actually quite dissimilar.

Having discussed the assumptions and their possible limitations, our job is now to develop insight into how deviations from these assumptions will affect our implementation. Recalling Equation (6.42) for the risk-minimizing hedging strategy for a European call option, we see that there are two empirical forms that we need to obtain from our dataset: σ which is the standard deviation of increments $\Delta x_{t,t-1} = x_t - x_{t-1}$, and $p\left[x_T \mid x_t\right]$ which is the probability density function. Let's start with σ. The standard deviation of price increments can be approximated as $\sigma = x_t \sigma\left[R_{t,t-1}\right]$, where $\sigma\left[R_{t,t-1}\right]$ is the standard deviation of the one-timestep returns. However, we need to be careful here when dealing with real financial data. Although we made the assumption of i.i.d. increments in order to analytically construct the optimal hedging strategy, we have shown above that the real data exhibit some departure from this assumption. Let's explain why this might concern us here: consider a heavily 'in-the-money' option (i.e. $x_t \gg X$). We would expect that the writer's hedge would be very close to $\phi_t = 1$ since it is almost certain that the asset would be due to be delivered to the option holder at expiry. Let's use the substitution $x_T' = x_T - X$ in Equation (6.42):

$$\phi_t^*\left[x_t\right] = \frac{1}{\sigma^2\left(T-t\right)} \int_0^\infty \left(x_T' + X - x_t\right) x_T' \, p\left[x_T' + X \mid x_t\right] dx_T' \tag{6.45}$$

The limit $x_t \gg X$ gives:

$$\begin{aligned} \phi_t^*\left[x_t\right] &\to \frac{1}{\sigma^2\left(T-t\right)} \int_0^\infty \left(x_T' - x_t\right) x_T' \, p\left[x_T' \mid x_t\right] dx_T' \\ &\to \frac{1}{\sigma^2\left(T-t\right)} \left(\left\langle x_T^2 \right\rangle_{x_T|x_t} - \left(\left\langle x_T \right\rangle_{x_T|x_t} \right)^2 \right) \\ &\to \frac{\sigma_{T,t}^{\;2}}{\sigma^2\left(T-t\right)} \end{aligned} \tag{6.46}$$

where $\sigma_{T,t}^{\;2}$ is the variance of price increments $\Delta x_{T,t} = x_T - x_t$. Equation (6.46) would of course give $\phi_t\left[x_t\right] \to 1$ in this limit under the assumption of i.i.d. increments, because we then have $\sigma_{T,t}^{\;2} = \sigma^2\left(T-t\right)$. However, if the real data deviates from the i.i.d. assumption[9] as demonstrated above, then Equation (6.46) will not give the desired limit of $\phi_t\left[x_t\right] \to 1$ for $x_t \gg X$. This incorrect evaluation of the limit of the hedging strategy can seriously affect further numerical calculations. One could correct the limit by choosing a value for the one-timestep volatility σ given by:

$$\sigma = \frac{\sigma_{T,t}}{\sqrt{T-t}} \tag{6.47}$$

where $\sigma_{T,t}$ is the volatility of price increments $\Delta x_{T,t} = x_T - x_t$. However, this misses the point that we used our assumption of independent increments to arrive at $\phi_t^*[x_t]$. If our data deviates from our assumptions, we can no longer assume that the hedging strategy of Equation (6.42) is at all 'optimal'.

Let us now turn our attention to the construction of the probability density function $p[x_T \mid x_t]$. This is a distribution of price changes over an interval of $T-t$ timesteps (in our case days). Recall our assumption from Section 6.4.5 that the distribution of price increments is identical for all t; if this were true, we would then only have to worry about the time interval of our returns and not the absolute times. We could therefore first generate a series of returns

$$R_{t',t'-(T-t)} \equiv R_{t'} = \frac{x_{t'} - x_{t'-(T-t)}}{x_{t'-(T-t)}} \tag{6.48}$$

and then, following the same procedure as Section 6.4.3, de-trend the returns ($R'_{t'} = R_{t'} - \langle R_{t'} \rangle_{t'}$) and bin the data to generate a histogram giving $n[R']$. We then could construct our PDF by analogy with Equation (6.23) as:

$$p[x_T \mid x_t]\, dx_T = \frac{n[R']}{n\, x_t\, \Delta R}\, dx_T \tag{6.49}$$

However, we saw earlier in this section that the distribution of returns was not in fact identical for all times within our dataset. We therefore ought to use a sufficiently small window of past times during which we believe the distribution is indeed stationary. However this has the associated problem that without a large amount of data, the error in constructing the PDF is large. It seems therefore that we either should generalize our assumptions to cater for the nature of the dataset we're handling, or use a 'surrogate' dataset containing some of the features we observed in the real data, but lacking the features which compromised our assumptions. We will follow the second of these approaches below in order to illustrate the method, and return at the end of this Chapter to consider the effect of generalizing the underlying assumptions.

6.4.6.2 Using surrogate data

The basic aim is to construct a new dataset having the same PDF of one-timestep returns $p[R]$ as the original dataset, but with i.i.d. increments. We can then use this dataset in place of the original in an implementation of Equation (6.42) since it won't break any of the assumptions we have made. We generate the surrogate dataset as follows. First we need to construct the PDF of one-timestep returns from the original data. We follow the procedure of Section 6.4.3, by first forming the return time-series $\{R_t\}$ such that:

$$R_t \equiv R_{t,t-1} = \frac{x_t - x_{t-1}}{x_{t-1}} \tag{6.50}$$

Then we de-trend the data, $R_t' = R_t - \langle R_t \rangle_t$ and bin it between $R_{\min}$ and $R_{\max}$ such that:

$$p[R']dR' = \frac{n[R']}{n\,\Delta R}dR' \tag{6.51}$$

Having constructed $p[R']$, we need to sample it randomly in order to generate a timeseries of independent, identically distributed returns $\{R_t^{\text{surrogate}}\}$. We do this in the following way:

a. Define $p_{\max} = \max\left[p[R']\right]$, the maximum likelihood value of the de-trended returns PDF

b. Choose a random number r uniformly distributed between $R_{\min}$ and $R_{\max}$.

c. Choose a random number p uniformly distributed between 0 and $p_{\max}$

d. If $p[R' = r] \geq p$ then append r to the series $\{R_t^{\text{surrogate}}\}$

e. Loop back to step b. until $\{R_t^{\text{surrogate}}\}$ reaches required length

Once we have our surrogate i.i.d. timeseries $\{R_t^{\text{surrogate}}\}$ we can compare it to the original de-trended returns timeseries generated from the financial dataset. As required, we find that $p\left[R^{\text{surrogate}}\right] = p[R']$ and that the variance of the surrogate timeseries $\sigma^{\text{surrogate}}$ scales as $\sigma^{\text{surrogate}}_{\Delta t,0} = \sigma\sqrt{\Delta t}$. Also since we have generated each $R_t^{\text{surrogate}}$ independently, there will be no autocorrelation between any function of the increments. However, since we have eliminated the subtle correlations in the financial data, we will lose the unique scaling behaviour shown in the original dataset. For example, if we look at how the kurtosis (peakedness) of the returns over Δt timesteps scales with Δt, we find that the original financial data does *not* manage to decay to the Gaussian value of $\kappa = 3$ as suggested by the Central Limit Theorem, in contrast to the i.i.d. surrogate data:

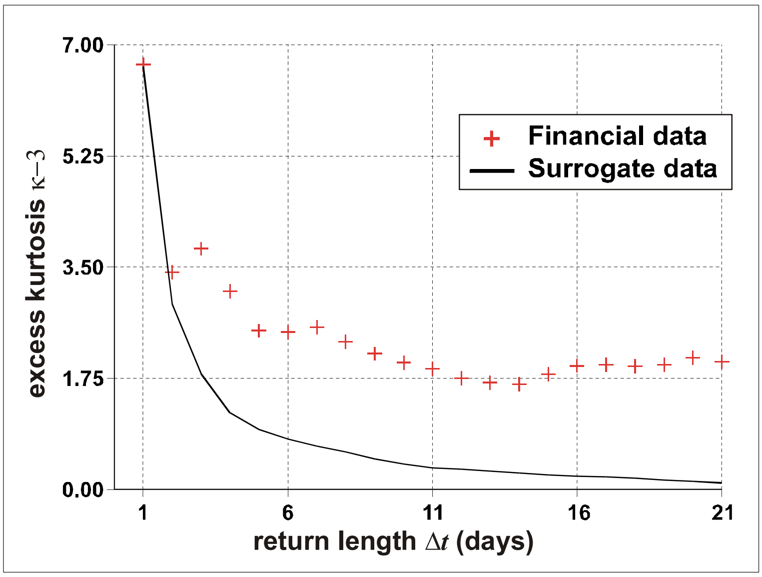

Figure 6-15: Excess kurtosis (i.e. $\kappa - 3$) of returns over Δt days. Crosses show anomalous decay for NYSE data. Solid line shows (power law) decay for surrogate timeseries.

6.4.6.3 Implementation

We will now implement the optimal strategy. We will do this using the surrogate data since we have manufactured it to obey the assumptions we made earlier in the analytical formalism. We will discuss the optimization process for the original, non-i.i.d. data later on. The process of implementing the hedging strategy is similar to that of implementing the fair option price (Section 6.4.3). We begin by taking the surrogate timeseries $\left\{R_t^{\text{surrogate}}\right\}$ and binning it to generate a histogram $n\left[R^{\text{surrogate}}\right]$. We then construct our PDF as:

$$p\left[x_T \mid x_t\right]dx_T = \frac{n\left[R^{\text{surrogate}} = \frac{x_T - x_t}{x_t}\right]}{n\,x_t\,\Delta R}dx_T \qquad (6.52)$$

Defining $x\left[j\right] = x_t\left(1 + R_{\min} + j\Delta R\right)$ and using Equation (6.52) in Equation (6.42), we arrive[11] at an expression for the optimal hedging strategy:

[11] We again consider a European call option as an example.

$$\begin{aligned}\phi_t[x_t] &= \frac{1}{\sigma^2(T-t)}\int_X^{\infty}(x_T - X)(x_T - x_t)\,p[x_T \mid x_t]\,dx_T \\ &= \frac{1}{\sigma^2(T-t)}\sum_{j=0}^{\sqrt{n}-1}\left(\mathrm{H}\big[x[j]-X\big]\frac{n[R_{\min}+j\Delta R]}{n\,x_t\,\Delta R}\int_{x[j]}^{x[j+1]}(x_T - X)(x_T - x_t)\,dx_T\right) \qquad (6.53) \\ &= \frac{1}{\sigma^2(T-t)}\sum_{j=0}^{\sqrt{n}-1}\left(\begin{array}{l}\mathrm{H}\big[x[j]-X\big]\dfrac{n[R_{\min}+j\Delta R]}{n}\times \\ \left(\dfrac{x_t^{\,2}\Delta R^2}{3}-\dfrac{x_t\Delta R}{2}\big(x_t+X-2x[j]\big)+\big(x[j]-X\big)\big(x[j]-x_t\big)\right)\end{array}\right)\end{aligned}$$

Figure 6-16 compares the optimal hedging strategy - implemented using Equation (6.53) together with the surrogate data - to the Black-Scholes delta, at two different times during the option's lifetime. The forms of the two functions are similar as expected, but the risk-minimizing optimal strategy shows a markedly *lower sensitivity* to the underlying asset movement near the expiry time of the contract. At the start of the contract, the Black-Scholes delta-hedging strategy and the risk-minimizing optimal strategy are very similar. This is due to the fact that, with the i.i.d. surrogate data, the distribution $p[x_T \mid x_0]$ has become essentially Gaussian due to convergence under the Central Limit Theorem.

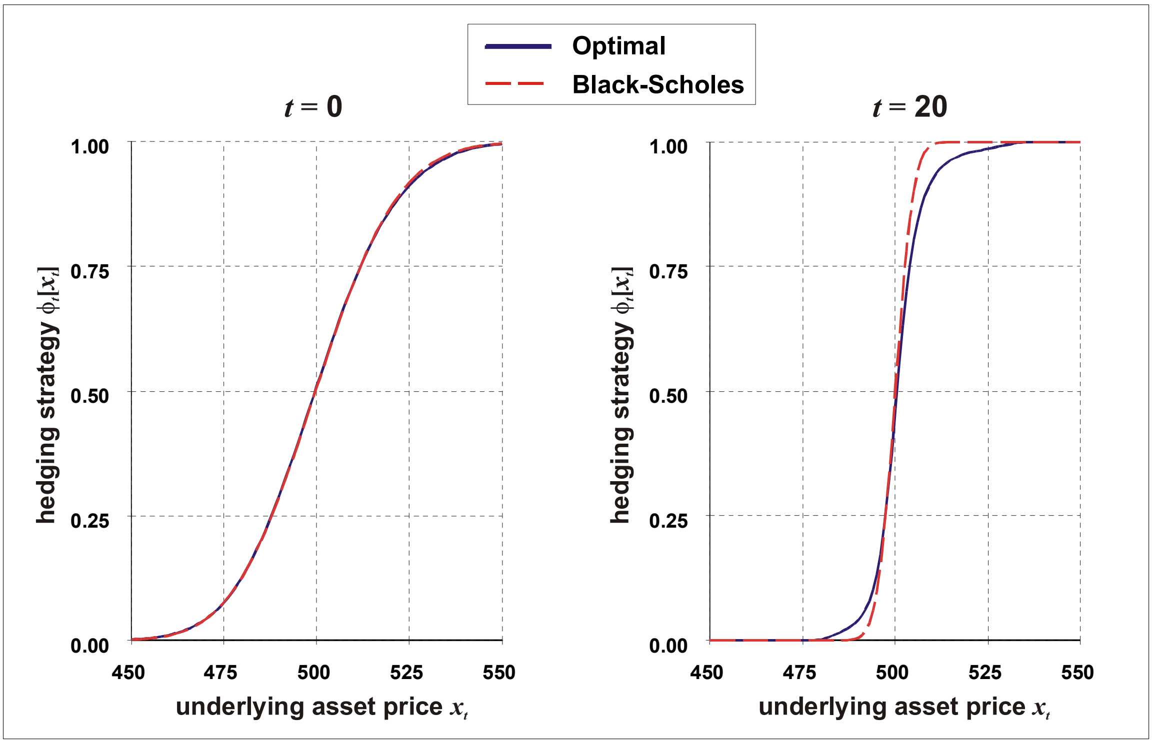

Figure 6-16: Comparison of the risk-minimizing optimal strategy (solid curve) and the Black-Scholes delta-hedge (dashed curve). The option is a European call option with strike $X = 500$ and expiry $T = 21$ days, on the NYSE composite index.

6.4.7 The residual risk

We have shown how to derive and implement a method for pricing and hedging options, based on just the historical underlying asset price data. The formalism we used[1] had the aim of minimizing the spread of the option writer's variation in wealth. We demonstrated earlier in Section 6.4.1 that in the general case, even if the option writer were able to re-hedge continuously, his/her spread in variation of wealth would be non-zero. Essentially the option writer's portfolio has a non-zero risk. In Section 6.4.5 we showed that this risk could be minimized with a suitable choice of hedging strategy leaving a minimum, or 'residual' risk. We now examine the behaviour of this residual risk as a function of different option parameters. This calculation of the residual risk for a real financial dataset will involve a numerical implementation of Equation (6.39). Following the same method as Section 6.4.6 for the implementation of the optimal strategy, we arrive at[12]:

$$\begin{aligned} \mathrm{R}^* &= \mathrm{R}_c - \sum_{\frac{t}{\tau}=0}^{\frac{T}{\tau}-1} \int_0^{\infty} \sigma_{t+\tau,t}^2 \left(\phi_t^*\left[x_t\right]\right)^2 p\left[x_t \mid x_0\right] dx_t \\ &= \mathrm{R}_c - \sigma^2 \tau \sum_{\frac{t}{\tau}=0}^{\frac{T}{\tau}-1} \sum_{j=0}^{\sqrt{n}-1} \left(\frac{n\left[R_{\min} + j\Delta R\right]}{n\, x_0 \Delta R} \int_{x[j]}^{x[j+1]} \left(\phi_t^*\left[x_t\right]\right)^2 dx_t \right) \end{aligned} \qquad (6.54)$$

with $x\left[j\right] = x_0\left(1 + R_{\min} + j\Delta R\right)$, and with the returns to be binned given by $R_t^{\text{surrogate}}$. Unlike our earlier numerical implementations, the integral in Equation (6.54) must be evaluated numerically, since the form of the optimal strategy $\phi_t^*\left[x_t\right]$ for a given real financial dataset is not a simple analytic function (see Equation (6.53)). This makes the numerical calculation of the residual risk computationally intensive and subject to numerical error. **Figure 6-17** shows the dependence of the residual risk on the two parameters of the option, the maturity T and the strike price X :

[12] We use the surrogate dataset, since the expression for risk was generated with the same assumptions regarding the behaviour of the underlying asset.

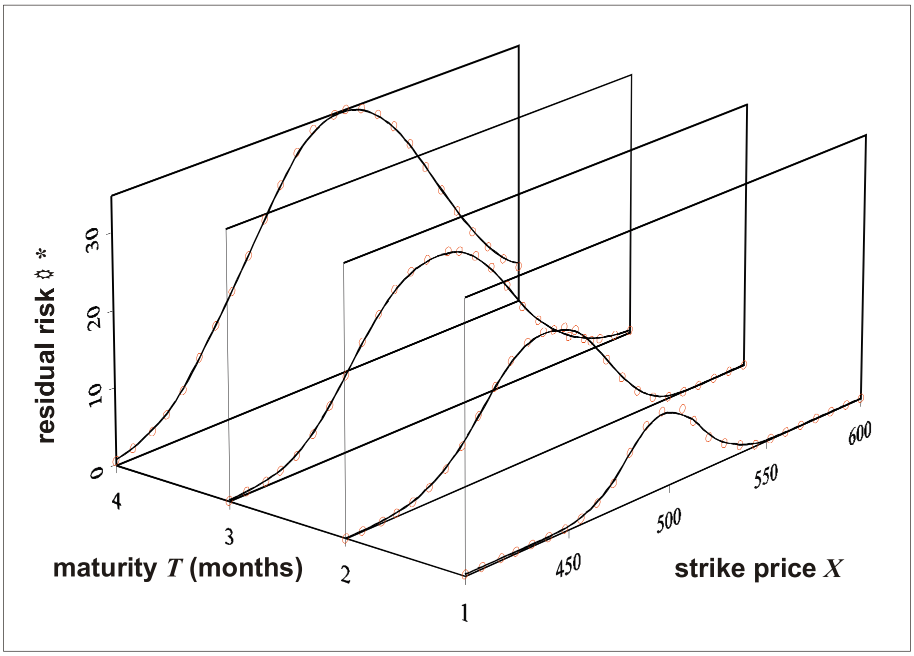

Figure 6-17: Variation of the residual risk R^* vs. the two option parameters, maturity T and strike price X. The option was re-hedged every day ($\tau = 1$). The initial spot value was $x_0 = 500$, and the data used to produce the risk-minimizing hedge was the surrogate timeseries $\left\{R_t^{\text{surrogate}}\right\}$ generated from the NYSE data.

6.4.8 Risk premium

In an efficient market, the prices of the same contract offered by many different suppliers should be the same. However, in practice this is not always the case. If the contract is risky for the supplier (i.e. the spread of his/her probable returns is non-zero), the supplier will tend to add a so-called 'risk-premium' to the contract price. This seems reasonable, since it is generally accepted that people are 'risk-averse': they view uncertainty as a bad thing, and thus require monetary compensation for accepting more uncertainty. However the extent and manner in which different suppliers of a contract will judge this risk, can be very different: after all, there are many ways of assessing risk and hence calculating an adequate compensatory 'risk premium'. A popular technique involves using the variance of the portfolio in order to calculate a risk premium. For example, under certain assumptions the risk-compensation described by Equation (6.55) below can be arrived at either from a utility maximization argument, or from a Value-at-Risk (VaR) approach:

$$\left\langle \Delta W_T \right\rangle = \lambda \sqrt{\mathrm{var}\left[\Delta W_T\right]} \qquad (6.55)$$

Here λ represents the degree of risk-aversion that the option writer desires. Equation (6.19) in Section 6.4.2, gave the 'fair' option price by setting $\left\langle \Delta W_T \right\rangle_{x_0 \ldots x_T} = 0$. We now ask what would change if instead of simply using the no-arbitrage 'fair' condition for pricing, we used a risk-averse pricing scheme such as Equation (6.55). Recall the equation for the variation of the option writer's wealth ΔW_T, in compact form:

$$\Delta W_T = V_0 - V_T + H \tag{6.56}$$

where H is the term corresponding to the gain or loss from hedging assets. We can express the variance of Equation (6.55) above as:

$$\left\langle \Delta W_T{}^2 \right\rangle - \left\langle \Delta W_T \right\rangle^2 = \begin{array}{l} \left(V_0{}^2 + \left\langle V_T{}^2 \right\rangle + \left\langle H^2 \right\rangle - 2V_0 \left\langle V_T \right\rangle - 2\left\langle V_T H \right\rangle + 2V_0 \left\langle H \right\rangle \right) - \\ \left(V_0{}^2 + \left\langle V_T \right\rangle^2 + \left\langle H \right\rangle^2 - 2V_0 \left\langle V_T \right\rangle - 2\left\langle V_T \right\rangle \left\langle H \right\rangle + 2V_0 \left\langle H \right\rangle \right) \end{array} \tag{6.57}$$

where $\langle \ldots \rangle$ is our shorthand for averaging over all underlying asset price realizations $\langle \ldots \rangle_{x_0, \ldots x_T}$.

Cancelling, and using the fact that for unbiased increments of the underlying asset we have $\langle H \rangle = 0$ (recall Equation (6.17)), we get:

$$\begin{aligned} \left\langle \Delta W_T{}^2 \right\rangle - \left\langle \Delta W_T \right\rangle^2 &= \left(\left\langle V_T{}^2 \right\rangle - \left\langle V_T \right\rangle^2 \right) + \left(\left\langle H^2 \right\rangle - 2\left\langle V_T H \right\rangle \right) \\ &= \mathrm{R}_c + \left(\left\langle H^2 \right\rangle - 2\left\langle V_T H \right\rangle \right) = \mathrm{R} \end{aligned} \tag{6.58}$$

which is exactly the same result as Equation (6.35). Hence

$$\left\langle \Delta W_T \right\rangle = \lambda \sqrt{\mathrm{var}\left[\Delta W_T \right]} \;\Rightarrow\; V_0 - \left\langle V_T \right\rangle = \lambda \sqrt{\mathrm{R}} \;\Rightarrow\; V_0 = \left\langle V_T \right\rangle + \lambda \sqrt{\mathrm{R}} \tag{6.59}$$

Our risk-averse pricing scheme given by Equation (6.55) has simply resulted in an *additive term* to the earlier option price (Equation (6.20)). This additive term is proportional to the standard deviation in the option writer's variation of wealth. Interestingly, one could use Equation (6.59) to assess an option writer's degree of risk-aversion based on traded market option prices V_0. This gives an idea of how 'expensive' the option is: the higher the risk-aversion λ, the more the option will cost in excess of the 'fair' price $\langle V_T \rangle$.

6.4.9 Black-Scholes as a special case

The numerical results in Section 6.4.1 suggested that if the underlying asset's price movement *was* i.i.d. lognormal, *and* we hedged continuously with the Black-Scholes delta recipe, then the risk of the contract would vanish completely. Here we show that the formalism of Bouchaud and Sornette also predicts this miraculous result, but as a special case: in particular, for a special choice of underlying asset price PDF $p\left[x_{t_2} \mid x_{t_1} \right]$. This special form of PDF can be shown to be lognormal, normal or quasi-normally distributed. To reproduce the Black-Scholes formula we will here assume a lognormal form for the underlying's distribution of returns such that:

$$p\left[x_{t_2} \mid x_{t_1} \right] = \frac{1}{x_{t_2} \, \sigma^{\mathrm{BS}} \sqrt{2\pi \left(t_2 - t_1 \right)}} e^{-\frac{\left(\ln\left[x_{t_2} / x_{t_1} \right] + \left(\sigma^{\mathrm{BS}} \right)^2 \left(t_2 - t_1 \right) / 2 \right)^2}{2\left(\sigma^{\mathrm{BS}} \right)^2 \left(t_2 - t_1 \right)}} \tag{6.60}$$

where σ^{BS} is the Black-Scholes volatility.

6.4.9.1 The option price

We start by looking at the Black-Scholes option price for a European call option. Equation (6.21) gives this price as:

$$V_0 = \int_X^{\infty} \left(x_T - X\right) p\left[x_T \mid x_0\right] dx_T \tag{6.61}$$

Now let's make the substitution $x_t = x_0 e^{y_t}$ in Equation (6.61):

$$V_0 = x_0 \int_{\ln[X/x_0]}^{\infty} \left(e^{y_T} - e^{\ln[X/x_0]}\right) p\left[y_T \mid 0_0\right] dy_T \tag{6.62}$$

Now we use the return $R_t \equiv R_{t,t-1} = \left(x_t - x_{t-1}\right)/x_{t-1}$ to give:

$$x_t - x_{t-1} = R_t x_{t-1} \;\Rightarrow\; e^{y_t} - e^{y_{t-1}} = R_t e^{y_{t-1}} \;\Rightarrow\; y_t - y_{t-1} = \ln[1 + R_t] \approx R_t - \frac{R_t^{\,2}}{2} \tag{6.63}$$

Since $R_t \ll 1$, we have neglected terms greater than the second power of the return in the Taylor expansion of the logarithm in Equation (6.63). For Black-Scholes, the elementary probability distribution of returns $p\left[R_t\right]$ is Gaussian: hence from Equation (6.63) the probability density function of y_t, $p\left[y_t \mid y_{t-1}\right]$, can be calculated using:

$$p\left[y_t = z\right] = \frac{\partial}{\partial z} p\left[R_t - \frac{R_t^{\,2}}{2} \le z - y_{t-1}\right] \tag{6.64}$$

The mean of this PDF is $y_{t-1} - \frac{1}{2}\left(\sigma^{\mathrm{BS}}\right)^2$ and the variance is $\left(\sigma^{\mathrm{BS}}\right)^2$. We can now exploit the Black-Scholes assumption of continuous time to say that there have been an infinitely large number of increments y_t in the time interval $0 \le t \le T$. The Central Limit Theorem then implies that the distribution $p\left[y_T \mid 0_0\right]$ has converged to a Gaussian with mean $-\frac{1}{2}T\left(\sigma^{\mathrm{BS}}\right)^2$ and variance $T\left(\sigma^{\mathrm{BS}}\right)^2$. Therefore, we can now calculate Equation (6.62) to be:

$$\begin{aligned} V_0 &= x_0 \int_{\ln[X/x_0]}^{\infty} \frac{\left(e^{y_T} - e^{\ln[X/x_0]}\right)}{\sqrt{2\pi T}\sigma^{\mathrm{BS}}} e^{-\left(y_T + T\left(\sigma^{\mathrm{BS}}\right)^2/2\right)^2 \Big/ 2T\left(\sigma^{\mathrm{BS}}\right)^2} dy_T \\ &= x_0\, \Phi\left[\frac{\ln\left[x_0/X\right] + T\left(\sigma^{\mathrm{BS}}\right)^2/2}{\sqrt{T}\sigma^{\mathrm{BS}}}\right] - X\, \Phi\left[\frac{\ln\left[x_0/X\right] - T\left(\sigma^{\mathrm{BS}}\right)^2/2}{\sqrt{T}\sigma^{\mathrm{BS}}}\right] \end{aligned} \tag{6.65}$$

where $\Phi\left[x\right]$ is the cumulative normal distribution function. Equation (6.65) is identical to the Black-Scholes formula for the price of a European call option (see Equation (2.62) in Chapter 2 with $r = 0$).

Hence by inserting the assumptions about the underlying asset and continuous-time into the present formalism, the Black-Scholes result appears as a special case.

6.4.9.2 The hedging strategy

We saw in Section 6.4.6.2 that when our data had almost converged to a Gaussian distribution in the limit of large time-increments, the risk-minimizing optimal strategy of the present formalism almost coincided with the Black-Scholes delta-hedging strategy. We now show that, given the Black-Scholes assumptions, the risk-minimizing optimal strategy analytically reproduces the delta-hedging strategy exactly. We start by recalling that the lognormal distribution of the underlying asset's price movement, which is assumed in Black-Scholes, is very well approximated by a standard Gaussian distribution if the underlying asset price is sufficiently large: in particular $x_0 \gg \sigma\sqrt{T}$. We will use this as a simplifying assumption in what follows. We first then take the Gaussian form of the PDF $p\left[x_{t_2} \mid x_{t_1}\right]$

$$p\left[x_{t_2} \mid x_{t_1}\right] = \frac{1}{\sqrt{2\pi\left(t_2 - t_1\right)}\,\sigma} e^{-\left(\left(x_{t_2} - x_{t_1}\right)^2\right) / 2(t_2 - t_1)\sigma^2}$$

Then we differentiate to get:

$$\frac{\partial p\left[x_{t_2} \mid x_{t_1}\right]}{\partial x_{t_1}} = \frac{x_{t_2} - x_{t_1}}{\sigma^2\left(t_2 - t_1\right)} p\left[x_{t_2} \mid x_{t_1}\right] \tag{6.66}$$

The risk-minimizing optimal strategy for a European call option, Equation (6.42), is given by:

$$\phi_t^*\left[x_t\right] = \frac{1}{\sigma^2\left(T - t\right)} \int_X^\infty \left(x_T - x_t\right)\left(x_T - X\right) p\left[x_T \mid x_t\right] dx_T \tag{6.67}$$

Comparing Equations (6.66) and (6.67) we get:

$$\phi_t^*\left[x_t\right] = \frac{\partial}{\partial x_t} \int_X^\infty \left(x_T - X\right) p\left[x_T \mid x_t\right] dx_T = \frac{\partial V_t}{\partial x_t} \tag{6.68}$$

where we have identified

$$V_t = \int_X^\infty \left(x_T - X\right) p\left[x_T \mid x_t\right] dx_T \tag{6.69}$$

as the option price at time t, by comparison with Equation (6.21). Hence Equation (6.68) gives us exactly the Black-Scholes result from Equation (2.55), that the optimal hedging strategy should be $\phi_t^*\left[x_t\right] = \dfrac{\partial V_t}{\partial x_t}$.

6.4.9.3 The residual risk

All that remains to be done now is to show that by using the Black-Scholes pricing and hedging formulae, the risk of option writing disappears altogether. If we differentiate two Gaussian underlying price PDFs, using Equation (6.66) of the previous section, we get:

$$\frac{\partial p\left[x_{t_2} \mid x_t\right]}{\partial x_t}\frac{\partial p\left[x_{t_1} \mid x_t\right]}{\partial x_t} = \frac{x_{t_2}-x_t}{\sigma^2\left(t_2-t\right)}\frac{x_{t_1}-x_t}{\sigma^2\left(t_1-t\right)}p\left[x_{t_2} \mid x_t\right]p\left[x_{t_1} \mid x_t\right] \qquad (6.70)$$

Multiplying by $\sigma^2 p\left[x_t \mid x_0\right]$, and integrating over the intermediate asset value x_t and time, gives:

$$\begin{aligned}&\sigma^2\int_0^T\int_0^\infty \frac{\partial p\left[x_{t_2} \mid x_t\right]}{\partial x_t}\frac{\partial p\left[x_{t_1} \mid x_t\right]}{\partial x_t}p\left[x_t \mid x_0\right]dx_t dt \\ &= p\left[x_{t_1} \mid x_0\right]\delta\left[x_{t_2}, x_{t_1}\right] - p\left[x_{t_2} \mid x_0\right]p\left[x_{t_1} \mid x_0\right]\end{aligned} \qquad (6.71)$$

Recall the form of the residual risk, Equation (6.39). The continuous hedging scenario, where the discrete sum turns into an integral as the step size $\tau \to 0$, gives:

$$\mathrm{R}^* = \mathrm{R}_c - \sigma^2\int_0^T\int_0^\infty \left(\phi_t^*\left[x_t\right]\right)^2 p\left[x_t \mid x_0\right]dx_t dt \qquad (6.72)$$

If we multiply the identity of Equation (6.71) by the payoff function, and use the fact that $\phi_t^*\left[x_t\right] = \partial V_t / \partial x_t$, then the right-hand side becomes R_c while the left-hand side yields $\sigma^2\int_0^T\int_0^\infty \left(\phi_t^*\left[x_t\right]\right)^2 p\left[x_t \mid x_0\right]dx_t dt$. This implies that the residual risk becomes *zero*, in accordance with the miraculous Black-Scholes result. However this result of zero-risk is *not* general: if we had not assumed an underlying process that was a member of the Gaussian family of processes, *or* we had not assumed that we could hedge continuously, then we would *not* have found this special-case result. Indeed one can show using the Euler-McLaurin formula for the difference between an integral and a discrete sum, that for small re-hedging times τ the residual risk is given by:

$$\mathrm{R}^* = \frac{\sigma^2\tau}{2}P_{>X}\left(1-P_{>X}\right) \qquad (6.73)$$

where the cumulative probability distribution $P_{>X} = \int_X^\infty p\left[x_T \mid x_0\right]dx_T$ gives the probability that the option is exercised at time of expiry. This *remaining risk is not in general small*: for example for an at-the money contract ($X = x_0$) we have (using the substitution of $y_T = x_T - X$ in Equation (6.21))

$$V_0 = \int_X^\infty \left(x_T - X\right)p\left[x_T \mid X\right]dx_T = \int_0^\infty y_T p\left[y_T \mid 0_0\right]dy_T = \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \qquad (6.74)$$

whereas the residual standard deviation in wealth variation can be obtained from Equation (6.73) by using $P_{>X} = \frac{1}{2}$ for an at-the-money option:

$$\sigma\left[\Delta W_T\right]=\sqrt{\mathsf{R}^*}=\sigma\sqrt{\frac{\tau}{2}P_{>X}\left(1-P_{>X}\right)}=\sigma\sqrt{\frac{\tau}{8}}=\sqrt{\frac{\pi}{4T/\tau}}V_0 \tag{6.75}$$

Hence if we have a one-month contract that we hedge every day (i.e. $T/\tau \simeq 21$) then the spread in the option writer's variation of wealth is approximately 20% of the option value. Of course with non-Gaussian underlying asset price processes, the residual risk is much higher and can *never* be hedged away: the differential form of the Black-Scholes formulation cannot take into account the large jumps a real underlying asset may perform, hence the option portfolio cannot be replicated perfectly.

6.4.10 Expanding around the Black-Scholes result

We now turn to look at the effect of *non*-Gaussian underlying asset price distributions, by expanding the PDF about the Gaussian. In this way, we will develop systematic corrections to the Black-Scholes results.

6.4.10.1 Expansion of the option price

We again consider a European call option. The 'fair' option price is given by Equation (6.21):

$$V_0=\int_X^\infty\left(x_T-X\right)p\left[x_T\,|\,x_0\right]dx_T \tag{6.76}$$

Rewriting Equation (6.76) as

$$V_0=\lim_{Y\to\infty}\int_X^Y\left(x_T-X\right)p\left[x_T\,|\,x_0\right]dx_T \tag{6.77}$$

and integrating by parts, we have

$$\begin{aligned}V_0&=\lim_{Y\to\infty}\left(\left[\left(x_T-X\right)\int_{-\infty}^{x_T}p\left[x'_T\,|\,x_0\right]dx'_T\right]_X^Y-\int_X^Y\int_{-\infty}^{x_T}p\left[x'_T\,|\,x_0\right]dx'_T dx_T\right)\\&=\lim_{Y\to\infty}\left(\left(Y-X\right)\left(1-P_>\left[Y\right]\right)-\int_X^Y\left(1-P_>\left[x_T\right]\right)dx_T\right)\\&=\lim_{Y\to\infty}\left(\left(X-Y\right)P_>\left[Y\right]\right)+\int_X^\infty P_>\left[x_T\right]dx_T\\&=\int_X^\infty P_>\left[x_T\right]dx_T\end{aligned} \tag{6.78}$$

where $P_>\left[z\right]=\int_z^\infty p\left[x'_T\,|\,x_0\right]dx'_T$. In the last line of Equation (6.78) we used the fact that $P_>\left[Y\right]\to 0$ faster than $\left(Y-X\right)\to\infty$, which must hold to guarantee the correct normalization of the probability distribution. Hence we have made the transformation of the 'fair' option price in Equation (6.78) such that we can now expand $P_>\left[x_T\right]$ around the Gaussian. If we make the substitution $y=\left(x_T-X\right)/\sigma\sqrt{T}$, we can expand around the cumulative Gaussian $P_{G>}\left[y\right]$ which has mean $z=\left(x_0-X\right)/\sigma\sqrt{T}$ and unit variance. We expand in (rooted) powers of the number of timesteps between writing and expiry $\left(T/\tau\right)^{k/2}$ as:

$$P_>\left[y\right]=P_{G>}\left[y\right]+\frac{e^{-(y-z)^2/2}}{\sqrt{2\pi}}\left(\frac{Q_1\left[y\right]}{\left(T/\tau\right)^{1/2}}+\frac{Q_2\left[y\right]}{\left(T/\tau\right)^{1}}+\ldots\right) \tag{6.79}$$

where the functions $Q_k\left[y\right]$ are polynomials of the normalized cumulants[13] λ_k. The first two of these polynomials can be written as:

$$Q_1\left[y\right]e^{-(y-z)^2/2}=\frac{\lambda_3}{6}\frac{d^2}{dy^2}e^{-(y-z)^2/2}\ ,\quad Q_2\left[y\right]e^{-(y-z)^2/2}=-\frac{\lambda_4}{24}\frac{d^3}{dy^3}e^{-(y-z)^2/2}-\frac{\lambda_3{}^2}{72}\frac{d^5}{dy^5}e^{-(y-z)^2/2} \tag{6.80}$$

Now we can combine Equations (6.80), (6.79), and (6.78) to give:

$$\begin{aligned}V_0=V_G&+\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\sqrt{\frac{\tau}{T}}\frac{\lambda_3}{6}\int_0^\infty\frac{d^2}{dy^2}e^{-(y-z)^2/2}dy-\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left(\frac{\tau}{T}\right)\frac{\lambda_4}{24}\int_0^\infty\frac{d^3}{dy^3}e^{-(y-z)^2/2}dy\\&-\frac{\sigma\sqrt{T}}{\sqrt{2\pi}}\left(\frac{\tau}{T}\right)\frac{\lambda_3{}^2}{72}\int_0^\infty\frac{d^5}{dy^5}e^{-(y-z)^2/2}dy+\ldots\end{aligned} \tag{6.81}$$

The integrals in Equation (6.81) are standard[14]. Hence we can easily obtain a cumulant expansion of the 'fair' option price around the Black-Scholes (Gaussian) price V_G, as follows:

$$\begin{aligned}V_0=V_G&+\sigma\sqrt{T}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\\&\times\left(-\frac{\lambda_3}{6\sqrt{T/\tau}}z+\frac{\lambda_4}{24\left(T/\tau\right)}\left(z^2-1\right)+\frac{\lambda_3{}^2}{72\left(T/\tau\right)}\left(z^4-6z^2+3\right)+\ldots\right)\end{aligned} \tag{6.82}$$

We can use Equation (6.82) directly from our knowledge of the moments (and hence normalized cumulants λ_k) of the probability density function of underlying asset price movements, in order to obtain the price correction to the Black-Scholes option price. Alternatively we can extract from Equation (6.82) an 'implied volatility' as we did in Section 6.4.3, as a function of the moments of the underlying asset's distribution. First we expand the option price to first order, using Equation (6.78):

---

[13] Cumulants c_n are standard parameters characterizing the moments of a PDF. The normalised cumulants are given by $\lambda_n=c_n/\sigma^n$. For example, the third and fourth normalised cumulants λ_3 and λ_4 describe the skewness and kurtosis of the distribution: $\lambda_3=\left\langle\left(x-\left\langle x\right\rangle\right)^3\right\rangle/\sigma^3$ and $\lambda_4+3=\left\langle\left(x-\left\langle x\right\rangle\right)^4\right\rangle/\sigma^4\equiv\kappa$ respectively. For more details see [G] and [BP].

[14] 'Standard' in the sense that they can be found in formulae books readily so needn't be explicitly calculated here.

$$
\begin{aligned}
V_0 &= V_G + \frac{\partial V_G}{\partial \sigma}\delta\sigma + \ldots \\
&= V_G + \delta\sigma \frac{\partial}{\partial \sigma}\int_X^\infty P_{G>}\left[x_T\right]dx_T + \ldots = V_G + \delta\sigma \int_X^\infty \frac{x_T - x_0}{\sqrt{2\pi\sigma^2}\sqrt{T}} e^{-\left(x_T - x_0\right)^2/2\sigma^2 T} dx_T + \ldots \quad (6.83) \\
&= V_G + \frac{\sqrt{T}}{\sqrt{2\pi}} e^{-z^2/2}\delta\sigma + \ldots
\end{aligned}
$$

Comparing Equations (6.83) and (6.82), and considering the typical case in which the skewness of the underlying asset price distribution is not the dominant feature as compared to the kurtosis (i.e. $\lambda_3^{\ 2} \ll \lambda_4$), we find the implied volatility $\sigma_{\text{imp}} = \sigma + \delta\sigma$ to be given by:

$$
\sigma_{\text{imp}} = \sigma\left(1 + \frac{\kappa_T - 3}{24}\left(\frac{\left(X - x_0\right)^2}{\sigma^2 T} - 1\right)\right) \quad (6.84)
$$

where κ_T is the kurtosis of the PDF $p\left[x_T \mid x_0\right]$. Equation (6.84) is parabolic in the strike price X. The effect of the skewness term is to skew the parabola to one side or the other. This explains the origin of the implied volatility smile, as seen for example in Figure 6-11. For typical asset price PDFs where the excess kurtosis is positive, Equation (6.84) predicts the 'smile' seen in traded option-price implied volatilities. A large skewness and/or anomalous negative excess kurtosis, can turn this 'smile' into a 'smirk' or 'frown'.

6.4.10.2 Expansion of the optimal hedging strategy

We will again expand the general result about the Black-Scholes Gaussian case, focusing on the risk-minimizing hedging strategy for a European call option given by Equation (6.42):

$$
\phi_t^*\left[x_t\right] = \frac{1}{\sigma^2\left(T - t\right)}\int_X^\infty \left(x_T - x_t\right)\left(x_T - X\right)p\left[x_T \mid x_t\right]dx_T \quad (6.85)
$$

We can transform the probability distribution $p\left[x_T \mid x_t\right]$ into a sum over the distribution's cumulants[13] $c_{n,T-t}$. Firstly we use the definition of the Fourier Transform of the probability distribution $p\left[x_T \mid x_t\right]$, $\widehat{p}_{T-t}\left[z\right]$:

$$
p\left[x_T \mid x_t\right] = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{p}_{T-t}\left[z\right]e^{-iz\left(x_T - x_t\right)}dz
$$

to give

$$
\left(x_T - x_t\right)p\left[x_T \mid x_t\right] = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{p}_{T-t}\left[z\right]\frac{\partial}{\partial\left(-iz\right)}e^{-iz\left(x_T - x_t\right)}dz \quad (6.86)
$$

We can express the Fourier Transform of the PDF as a sum of cumulants:

$$\hat{p}_{T-t}[z] = e^{\sum_{n=2}^{\infty} \frac{c_{n,T-t}(iz)^n}{n!}} \tag{6.87}$$

Inserting Equation (6.87) into Equation (6.86), and integrating by parts, we get:

$$\begin{aligned}
(x_T - x_t)\, p[x_T \mid x_t] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sum_{n=2}^{\infty} \frac{c_{n,T-t}(iz)^n}{n!}} \frac{\partial}{\partial(-iz)} e^{-iz(x_T - x_t)} dz \\
&= \frac{1}{2\pi} \left[ie^{-iz(x_T - x_t) + \sum_{n=2}^{\infty} \frac{c_{n,T-t}(iz)^n}{n!}} \right]_{-\infty}^{\infty} \\
&\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=2}^{\infty} \frac{c_{n,T-t}\, n i^{n+1} z^{n-1}}{n!} e^{\sum_{n=2}^{\infty} \frac{c_{n,T-t}(iz)^n}{n!}} e^{-iz(x_T - x_t)} dz \\
&= \sum_{n=2}^{\infty} \frac{c_{n,T-t}}{(n-1)!} \frac{1}{2\pi} \int_{-\infty}^{\infty} (iz)^{n-1} e^{\sum_{n=2}^{\infty} \frac{c_{n,T-t}(iz)^n}{n!}} e^{-iz(x_T - x_t)} dz
\end{aligned} \tag{6.88}$$

The integral in the last line of Equation (6.88) is the $(n-1)^{\text{th}}$ derivative of the probability distribution $p[x_T \mid x_t]$ with respect to x_t. This is evident if we look again at the Fourier expansion of the PDF:

$$p[x_T \mid x_t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sum_{n=2}^{\infty} \frac{c_{n,T-t}(iz)^n}{n!}} e^{-iz(x_T - x_t)} dz \tag{6.89}$$

From Equation (6.89), we see that differentiating the PDF n times with respect to x_t will simply pull down successive powers of (iz), just as we require for Equation (6.88). Thus we have:

$$(x_T - x_t)\, p[x_T \mid x_t] = \sum_{n=2}^{\infty} \frac{c_{n,T-t}}{(n-1)!} \frac{\partial^{n-1}}{\partial x_t^{\,n-1}} p[x_T \mid x_t] \tag{6.90}$$

Combining Equations (6.85) and (6.90) gives the cumulant expansion of the optimal hedging strategy:

$$\begin{aligned}
\phi_t^*[x_t] &= \frac{1}{\sigma^2 (T-t)} \int_X^{\infty} (x_T - X) \sum_{n=2}^{\infty} \frac{c_{n,T-t}}{(n-1)!} \frac{\partial^{n-1}}{\partial x_t^{\,n-1}} p[x_T \mid x_t]\, dx_T \\
&= \frac{1}{\sigma^2 \tau} \sum_{n=2}^{\infty} \frac{c_{n,\tau}}{(n-1)!} \frac{\partial^{n-1}}{\partial x_t^{\,n-1}} \int_X^{\infty} (x_T - X)\, p[x_T \mid x_t]\, dx_T \\
&= \frac{1}{\sigma^2 \tau} \sum_{n=2}^{\infty} \frac{c_{n,\tau}}{(n-1)!} \frac{\partial^{n-1} V_t}{\partial x_t^{\,n-1}}
\end{aligned} \tag{6.91}$$

In the second line, we used the assumption that the movements of the underlying asset's price are i.i.d., since in this case the cumulants are additive, i.e. $c_{n,T-t} = \frac{(T-t)}{\tau} c_{n,\tau}$. The last line of Equation (6.91) includes V_t, the option price at time t, via Equation (6.21). We now examine our cumulant expansion of the risk-minimizing hedging strategy for the Gaussian distribution, in which case all the cumulants $c_{n,\tau}$ for $n \geq 3$ are identically equal to zero. In this special (Black Scholes) scenario, Equation (6.91) then gives back the Black-Scholes delta hedging strategy:

$$\phi_{t,G}^{\;*}\left[x_t\right] = \frac{\partial V_t}{\partial x_t} \qquad (6.92)$$

However in general this will *not* be true: the presence of kurtosis and skewness etc. in the real distribution of underlying price movements, will lead to the necessity for higher-order corrections to the hedging strategy in order to minimize the risk of writing the option.

This brings us to the end of our technical tour around a generalized treatment of derivatives. This tour took us back to re-examine the foundations of risk, hedging and pricing, and through a step-wise process whereby we could examine the consequences of the various Black-Scholes' approximations. In contrast to the 'beautiful but delicate' Black-Scholes theory, the present formulation could be classified as 'uglier but far more robust' since it does not depend on the real market honoring the underlying approximations of Black-Scholes. Given that it is the accuracy of the final answer that is important in financial practice, and given that the Black-Scholes approximations cannot be guaranteed to hold in any particular market, it is this 'ugly but robust' method which we believe will define the future for portfolio risk management and derivative pricing.

A highly speculative, but potentially very exciting, path for future research would be to use the market models of Chapter 4 to *generate* the PDFs etc. needed to implement the present risk-minimization, hedging and pricing scheme. The methodology would be: (i) build microscopic model, (ii) match model parameters to present state of market, (iii) project market model forward in time by letting it evolve, and (iv) construct PDFs for the future based on the resulting trajectories of the time-evolving market model. Such an approach, if successful, would do away with the idea of having to obtain such measures by analyzing past data. In this case, there would be no reason why the price process would have to be stationary, or even that it have much of a history. For example, the system of interest could be a recent IPO, like our dot-com company *risk-e.com* which featured at the beginning of Chapter 1. In short, one replaces the black-box real market together with its known but limited output from the past, with an approximate output generator for the future.