# 2 Standard finance theory

~~HANDOUT 3 OF 7~~

## 2.1 The problem for standard finance theory

The problem facing *any* theory for managing risk, portfolios, hedging and derivative pricing, is that the theory's applicability will always be limited by the accuracy of the description employed for the underlying market movements. In standard finance theory[1], such market movements are typically described by a stochastic process which then allows one to employ the powerful machinery of stochastic calculus by assuming the continuous-time limit. However, the assumption of a continuous-time limit often yields misleading results as will be discussed in Chapter 6. Furthermore the parametrization and structuring of such stochastic processes is typically very difficult, leading to the creation of ad hoc market models through a process resembling alchemy rather than deductive science. Indeed, it is unclear whether one would ever be able to adapt these standard stochastic prescriptions to include the *full* range of 'stylized facts' exhibited by financial market time-series (see Chapter 3). Having said this, we admit that it is easy to criticize. However these criticisms will help guide our discussion of more generalized approaches later in this book[2]. First though, we need to understand the basics behind standard finance theory.

There is no free lunch in finance. At least, this is what the Efficient Market Hypothesis (EMH) claims (see for example [CLM]). The EMH says that the entire history of information regarding an asset is reflected in its price and that the market responds instantaneously to new information. Thus the EMH implies that if any patterns (such as temporal correlations) do exist, they must be so small that no systematic trading strategy can have a better risk/return profile than the market portfolio. Hence according to the EMH, no profitable information about future movements can be obtained by studying past price-series. Given the no-free-lunch hypothesis, it makes sense that believers in the EMH would choose a model of asset prices that constitutes a 'random walk' in price-space. The theoretical descriptions used in standard finance theory, are typically built around this assumption that asset prices follow some form of random walk. We therefore need to understand the details, and hence limitations, of a random walk.

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[1] As in the rest of this course, we use an all-encompassing term 'standard finance theory'. Although convenient for the present purpose, such labels are always dangerous since they lay one open to criticism about exceptions to the rule. However we would claim that standard finance theory tends to adopt particular classes of assumption concerning, for example, temporal correlations. It is these assumptions, and in particular how one might avoid them, which interest us.

[2] For example, if models which are *differential* with respect to time seem limited, one might think about whether models which involve the *integral* of time will work better. This is indeed the approach adopted in Chapter 6 in order to address the question of risk minimization and hedging in real-world markets.

## 2.2 Taking a random walk

### 2.2.1 Back to basics

We start by revisiting some fairly well-known probability results. Suppose we are tossing a fair coin. Heads (H) and tails (T) have equal probability of occurring, hence $p[\text{heads}] = p[\text{tails}] = 0.5$. We can think of this coin-toss experiment as drawing a random value $\Delta x$ from a probability distribution function (PDF) $p[\Delta x]$. Let's denote $\Delta x = +d$ as outcome H and $\Delta x = -d$ as outcome T. The PDF $p[\Delta x]$ has the following form:

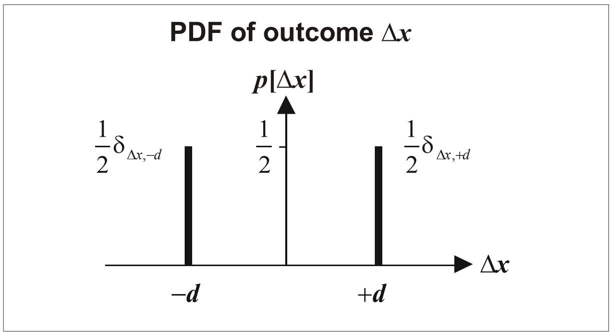


**Figure 2-1 Probability distribution function (PDF) $p[\Delta x]$ for outcomes of a coin-toss experiment or, equivalently, for price-changes in a 'coin-toss' market where up and down price-changes of magnitude $d$ occur with probability 0.5 at each timestep. The Kronecker-delta $\delta_{ij}$ is equal to 1 if $i = j$, otherwise it is zero.**

We can use the outcomes of this coin-toss experiment to generate a price-series for a simple 'coin-toss' market. The price-change at timestep $i$ is given by the coin-toss outcome at that timestep. In accordance with the shorthand notation of Section 1.4, the price-change at timestep $i$ is $\Delta x_{i,i-1} = x_i - x_{i-1}$. We will henceforth use the further abbreviation $\Delta x_{i,i-1} \equiv \Delta x_i$ to denote this price-change at timestep $i$. The price-change over $n$ timesteps is given by $\Delta x_{i,i-n} = x_i - x_{i-n} = \sum_{j=i+1-n}^{i} \Delta x_j$.

Consider, for example, a series of outcomes HHTH which generates a price-series that moves up-up-down-up with step-size *d.* The corresponding price-series is shown below (thick black line) together with the 'tree' formed by all possible such walks:

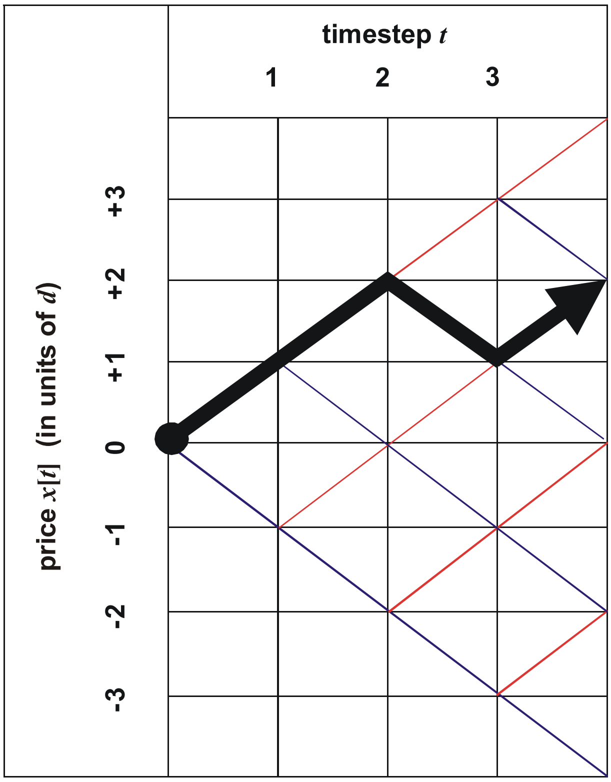


**Figure 2-2 Price-series in a 'coin-toss' market where up and down price-changes of magnitude $d$ occur with probability 0.5 at each timestep. Thick black line corresponds to the sequence of coin-tosses HHTH. All other possible paths form a 'tree' as shown. The price at timestep 0 is defined as the price-origin $x_0 = 0$.**

Broadly speaking, we can think of this simple random-walk model as representing the foundation upon which standard finance theory is built. We say 'broadly' since some finance models are more sophisticated than others. On the other hand, we say 'represents' since the statistical properties related to the correlations between outcomes are indeed quite similar[1]. In particular, this coin-toss price model assumes that the successive outcomes $\{\Delta x_i\}$ are i.i.d. variables, which means *independent and identically distributed.* This makes sense for a coin-toss price mechanism: successive coin-toss outcomes are indeed independent, and since we are using the same coin at each timestep then the PDF $p[\Delta x]$ is identical at every timestep. The coin-toss price model is also consistent with the Efficient Market Hypothesis (EMH) in that it is impossible to forecast the next outcome based on the past outcomes. In other words, if we were betting on future outcomes, we would not be able to gain systematically over time. In short, we have no 'edge' in the coin-toss market because the price-series is random.

To be fair, standard finance theory isn't quite that simplistic. It doesn't necessarily assume that each timestep is a simple coin-toss with two outcomes, up and down of identical step-size. On the other hand, it does assume that the price-process is random -- or so close to random that no systematic profit can be made based on knowledge of past outcomes. Moreover, it does often make the i.i.d.

assumption concerning price-changes. So let's explore this i.i.d. property further for a more general situation. Consider the price-changes[3] in a real market at different timesteps – obviously we should not assume a priori that such price-changes are i.i.d. variables. The PDF for the price-change $\Delta x_i$ at timestep $i$ is $p_i\left[\Delta x_i\right]$. Consider a set of such price-changes[3] measured at different timesteps: $\Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_j, \ldots$ . Standard finance theory then makes one, or invariably both, of the i.i.d. assumptions below:

<u>**Assumption (i)**</u> **The variables $\{\Delta x_i\} \equiv \Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_j, \ldots$ are independent.**

Consider the joint probability distribution $p\left[\Delta x_i, \Delta x_j\right]$ which gives the probability of obtaining the values $\Delta x_i$ *and* $\Delta x_j$ at two particular timesteps $i$ and $j$. These price-changes at timesteps $i$ and $j$ are *independent* if $p\left[\Delta x_i, \Delta x_j\right] = p_i\left[\Delta x_i\right] p_j\left[\Delta x_j\right]$ where $p_i\left[\Delta x_i\right]$ and $p_j\left[\Delta x_j\right]$ are the individual PDFs for timesteps $i$ and $j$. In other words, the joint probability distribution function can be written as a product of the individual PDFs for the price-changes at timesteps $i$ and $j$. The mean value of any product function of $\Delta x_i$ and $\Delta x_j$, for example $f\left[\Delta x_i\right] g\left[\Delta x_j\right]$, then becomes:

$$\begin{aligned} \left\langle f\left[\Delta x_i\right] g\left[\Delta x_j\right]\right\rangle &= \sum_{\Delta x_i, \Delta x_j} f\left[\Delta x_i\right] g\left[\Delta x_j\right] p\left[\Delta x_i, \Delta x_j\right] \\ &= \left\{\sum_{\Delta x_i} f\left[\Delta x_i\right] p_i\left[\Delta x_i\right]\right\}\left\{\sum_{\Delta x_j} g\left[\Delta x_j\right] p_j\left[\Delta x_j\right]\right\} = \left\langle f\left[\Delta x_i\right]\right\rangle\left\langle g\left[\Delta x_j\right]\right\rangle \end{aligned} \qquad (2.1)$$

One important consequence of this independence arises for $f\left[\Delta x_i\right] = \Delta x_i$ and $g\left[\Delta x_j\right] = \Delta x_j$, in which case Equation (2.1) gives

$$\left\langle \Delta x_i \Delta x_j \right\rangle = \left\langle \Delta x_i \right\rangle \left\langle \Delta x_j \right\rangle \qquad (2.2)$$

If Equation (2.2) holds, we say $\Delta x_i$ and $\Delta x_j$ are *uncorrelated.* However this is just one particular statistical property of the variables $\Delta x_i$ and $\Delta x_j$, out of the infinite number of possible choices for

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[3] Apart from the suggested scenario in which the variables $\{\Delta x_i\}$ represent price-changes at timesteps $1, 2 \ldots i$ etc., we could instead take $\{\Delta x_i\}$ to represent any number of other financial data-series, e.g. price-changes for different assets at a given timestep ($i$ becomes an asset label rather than a timestep label); price-changes for different assets at different timesteps; a set of changes in monthly volatilities; daily changes in exchange-rate between different currencies and the US dollar, etc. We just refer to $\Delta x_i$ as a 'price-change' for simplicity. Note that even 'price-change' itself could represent one of many possibilities, e.g. the detrended price-change, the return, the log-return (see Section 1.4). Which of these financial variables is actually closest to being i.i.d. will depend on the financial market itself: there is no way of knowing a priori without performing statistical tests.

$f[\Delta x_i]$ and $g[\Delta x_j]$. In particular, Equation (2.2) can be satisfied for a particular data-set without Equation (2.1) being satisfied. Hence we conclude that *independent variables are always uncorrelated*, while *uncorrelated variables are not necessarily independent.* In a real market it is not clear a priori whether the price-changes are truly independent. Most statistical testing, particularly using standard software packages, will just report on the following correlation measure[4]:

$$c_{ij} = \langle \Delta x_i \Delta x_j \rangle - \langle \Delta x_i \rangle \langle \Delta x_j \rangle \equiv \langle (\Delta x_i - \langle \Delta x_i \rangle)(\Delta x_j - \langle \Delta x_j \rangle) \rangle \qquad (2.3)$$

If $c_{ij} = 0$, then the variables are uncorrelated (see Equation (2.2)). However, this leaves wide open the question of whether *higher-order* correlations[5] exist. Since we have in mind the example where $i$ and $j$ label individual timesteps, these higher-order correlations will represent higher-order *temporal* correlations. If on the other hand $i$ and $j$ labelled different assets at the same timestep, for example, these higher-order correlations would be higher-order *inter-asset* correlations[6]. The correlation measure $c_{ij} = \langle \Delta x_i \Delta x_j \rangle - \langle \Delta x_i \rangle \langle \Delta x_j \rangle$ derived from Equation (2.2), is associated with the $m = 1$ power of $\Delta x_i$ and $\Delta x_j$ (i.e. we have effectively taken $f[\Delta x_i]$ and $g[\Delta x_j]$ proportional to $\Delta x_i$ and $\Delta x_j$ respectively in Equation (2.1)). Higher-order correlations are associated with higher powers of $\Delta x_i$ and $\Delta x_j$ (e.g. $f[\Delta x_i]$ and $g[\Delta x_j]$ proportional to $(\Delta x_i)^m$ and $(\Delta x_j)^m$ respectively with $m > 1$) *or* non-analytic functions[5] of $\Delta x_i$ and $\Delta x_j$ (e.g. $f[\Delta x_i]$ and $g[\Delta x_j]$ proportional to $|\Delta x_i|$ and $|\Delta x_j|$ respectively). In principle, one would have to check *all* such higher-order correlations in order to provide convincing evidence for independence.

<u>**Assumption (ii)**</u> **The variables $\{\Delta x_i\} \equiv \Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_j, \ldots$ are identically distributed**.

If the individual PDFs $p_i$ and $p_j$ are identical functions, then the variables $\Delta x_i$ and $\Delta x_j$ are *identically distributed*. In a real market, this will not generally be true – for example, the PDF of price-changes on Mondays is not exactly the same as that for Fridays.

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[4] $c_{ij}$ is often called the *covariance*. Dividing by the standard deviation of $\Delta x_i$ and $\Delta x_j$ yields the *correlation coefficient*. If $i, j$ refer to timesteps of the same data series, then the correlation coefficient is also known as the *autocorrelation coefficient*.

[5] *Higher-order* correlations can also be referred to as *non-linear* correlations. In this sense, the correlation measure $c_{ij}$ in Equation (2.3) and hence the autocorrelation coefficient, measure *linear* correlations between $\Delta x_i$ and $\Delta x_j$.

[6] Such higher-order inter-asset correlations could have important consequences for assessing the risk in a portfolio. Standard finance theory focuses on assessing the risk associated with the linear correlation $c_{ij}$ between assets.

All market price-processes are therefore classifiable as one of the following four cases. Standard finance theory tends to consider Case I, which is the easiest to deal with mathematically.

**Case I**: assumptions (i) and (ii) hold. Hence the price-process is one with i.i.d. price-changes[7], for example the coin-toss market.

**Case II**: only assumption (i) holds. The price-changes are independent, but are not identically distributed. This would be the case for the coin-toss market if different timesteps had different step-sizes and/or a different number of possible outcomes.

**Case III**: only assumption (ii) holds. The price-changes are not independent, but are identically distributed. This would be the case for the coin-toss market if we were to make the coin-toss outcome at a given timestep conditional on the outcome of previous timesteps. A priori, the unconditional PDFs describing the outcome at each timestep are identical -- however the series of outcomes in a given run will not be independent. An example of such dependence is created by the following rule: *if* we obtain five successive tails TTTTT (i.e. five successive price-changes downwards) *then* we will bias the coin-toss outcome in such a way that there is a very high probability of obtaining T at the next timestep as well (i.e. another price-change downwards). Such a 'memory' effect can, in principle, lead to systematic profit. Believers in the EMH would however claim that such dependence between price-changes must be very small – in fact so small as to be wiped out by transaction costs.

**Case IV**: neither assumption holds. This is the most mathematically complicated case. However it is also the one which seems closest to reality.

### 2.2.2 Price-changes over one timestep

Let's assume for the moment that we do have i.i.d. price-changes, hence we can focus on the properties of the PDF for the single variable $\Delta x_i \equiv \Delta x$ describing the price-change between any two successive time-steps. The *mean, average* or *expectation* value of a function $f[\Delta x]$ is defined as[8]:

$$\begin{aligned}\langle f[\Delta x]\rangle &\equiv \overline{f[\Delta x]} \equiv E[f[\Delta x]] \\ &= \sum_{\Delta x} f[\Delta x]\, p[\Delta x] \equiv \int_{-\infty}^{+\infty} f[\Delta x]\, p[\Delta x]\, d(\Delta x)\end{aligned} \quad (2.4)$$

[7] Recall from Section 1.4 that 'price-change' could mean actual price change $x_i - x_{i-1}$, fractional price-change or return $[x_i - x_{i-1}]/x_{i-1}$, or log-return $\ln x_i/x_{i-1}$.

[8] We will use the words 'mean', 'average', and 'expectation' value interchangeably. We will also tend to use $\langle f[y]\rangle \equiv \overline{f[y]} \equiv E[f[y]]$ interchangeably.

If $\Delta x$ takes discrete values, the mean can be calculated using the summation with $p[\Delta x]$ being a *discrete* PDF; if $\Delta x$ takes continuous values, the mean can be calculated using the integral with $p[\Delta x]$ being a *continuous* PDF (also known as a probability density function). For both discrete and continuous cases, we refer to the associated probability distribution function simply as 'PDF'. Depending on the context, either one may be more useful[9]. To help in discussing the ideas from general probability theory, we will tend to assume discrete $\Delta x$ since the coin-toss example is itself discrete – however the conclusions are the same for both discrete and continuous $\Delta x$.[9]

Choosing $f[\Delta x] = \Delta x$ in Equation (2.4) yields the mean value $\langle \Delta x \rangle$ which represents the $m = 1$ 'moment' of $p[\Delta x]$. The $m$'th 'central moment' is given by $\left\langle \left( \Delta x - \langle \Delta x \rangle \right)^m \right\rangle$. Some of these higher-order moments have specific names. For example the 'skewness' of $p[\Delta x]$ is given by $\left\langle \left( \Delta x - \langle \Delta x \rangle \right)^3 \right\rangle / \sigma^3$ where $\sigma$ is the standard deviation of $\Delta x$ over a single timestep, as defined below. The skewness is equal to zero if $p[\Delta x]$ is symmetric. The 'kurtosis' of $p[\Delta x]$ is given by $\kappa \equiv \left\langle \left( \Delta x - \langle \Delta x \rangle \right)^4 \right\rangle / \sigma^4$ and gives $\kappa = 3$ if $p[\Delta x]$ is Gaussian. We note that the mean $\langle \Delta x \rangle$ may *not* necessarily correspond to a possible experimental outcome. For the coin-toss, $\langle \Delta x \rangle = \frac{1}{2}(-1) + \frac{1}{2}(+1) = 0$ yet '0' is not a possible outcome. So using mean values to represent 'typical' future behaviour may be misleading. In addition, the mean tells us nothing about the *fluctuations* in $\Delta x$. The common measure of such fluctuations in standard finance theory is the *variance*:

$$\sigma_{i,i-1}{}^2 \equiv \sigma^2 = \left\langle \left( \Delta x - \langle \Delta x \rangle \right)^2 \right\rangle \tag{2.5}$$

In the case that the price-process is unbiased, as was the case for the coin-toss market, then $\langle \Delta x \rangle = 0$ and hence $\sigma^2 \equiv \left\langle (\Delta x)^2 \right\rangle$. For the coin-toss market, $\sigma^2 \equiv \left\langle (\Delta x)^2 \right\rangle = \frac{1}{2}(+d)^2 + \frac{1}{2}(-d)^2 = d^2$. Taking the square root gives the *standard deviation* of the price-change $\Delta x$ over a single timestep. This quantity $\sigma$ is arguably the most important quantity in standard finance theory, and is also known as the *volatility*. It is the volatility which is used in standard finance calculations of risk and option pricing, as

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[9] Prices are, strictly speaking, discrete in terms of the way they are quoted on the exchanges. In addition, any numerical modelling and analysis will necessitate discretizing both the price *and* time. For our purposes, it makes no real difference whether we deal with continuous or discrete values of the price-change $\Delta x$. For the time variable, however, it is very convenient for us to consider *discrete* time in order to go beyond the approximations of standard finance theory. In particular, standard finance theory assumes that an investor can trade continuously (see Section 2.4). The microscopic models of Chapter 4 and 5, and the risk analysis in Chapter 6, focus on discrete time: the connection to continuous-time finance can then be examined by setting the actual interval between timesteps to zero.

we will see later in this Chapter. Note that the calculation of the volatility $\sigma$ only involves *two* moments of the PDF $p[\Delta x]$ even though there are an *infinite* number of such moments available. In general, all of these moments will contain new and possibly important information concerning $p[\Delta x]$. Since large price changes generally happen less frequently than smaller price changes, the PDF $p[\Delta x]$ of price-changes will generally decrease monotonically to zero as $|\Delta x| \to \infty$. However there are no universal laws about how rapidly this function will decay to zero as $|\Delta x| \to \infty$ in a real market. Hence $p[\Delta x]$ may have significant 'weight' in the tails. In other words, the value of $p[\Delta x]$ may only decay very slowly to 0 as $|\Delta x|$ becomes large, thereby producing so-called *fat tails* in the distribution. Hence the moments for large *m* may actually be very large. Since information about the market dynamics comes from knowledge of $p[\Delta x]$ for *all* $\Delta x$ (i.e. small and large), we may be missing significant information about the probability of large price-changes if we don't account for this extra weight in the tails somewhere in our calculations. In short, *standard risk calculations based solely on the standard deviation of price-changes*, and hence on the first two moments $m = 1$ and 2, *may give misleading results*. Why is this a fundamental worry as opposed to a minor detail? Well, in contrast to most scientific theory where the behaviour around the mean is usually good enough to describe the system, finance theory is supposed to accurately answer questions about risk. And almost by definition, 'risk' has to mean the risk of large losses and hence *large deviations* in price-change away from the mean. As an example, consider the following two model PDFs for the price-change $\Delta x$, a Lorentzian and a Gaussian:

$$p[\Delta x] = \frac{C}{(\Delta x)^2 + C^2\pi^2} \quad \text{(Lorentzian)} \qquad\qquad p[\Delta x] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\Delta x)^2}{2\sigma^2}} \quad \text{(Gaussian)} \qquad\qquad (2.6)$$

where $C$ is a constant. To the eye, they look fairly similar, as can be seen below:

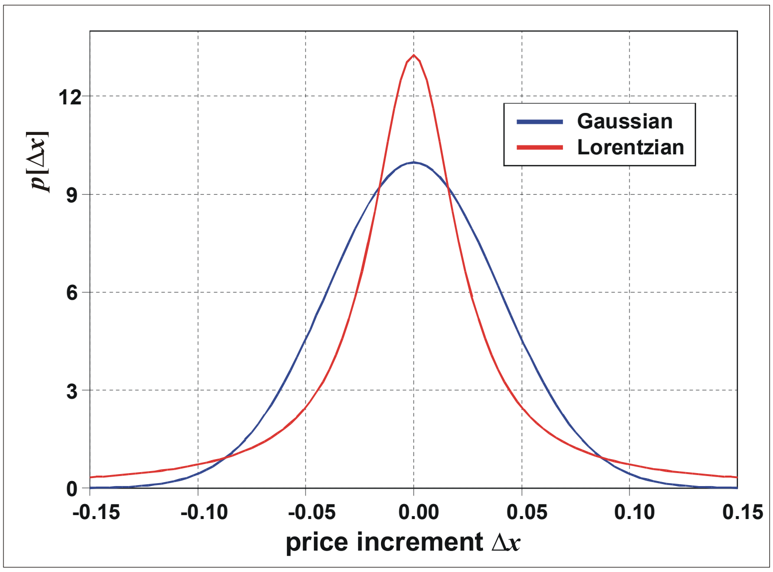


**Figure 2-3 Comparison between a Gaussian PDF for the price-change $\Delta x$ (blue curve) and a Lorentzian PDF (red curve).**

Both are peaked at $\Delta x = 0$ and die off to zero fairly quickly for large $|\Delta x|$, with the Lorentzian having a narrower but higher peak and fatter tails. If one were to produce the histogram of price-changes $p[\Delta x]$ for many real-world price-series on a reasonably short time-scale (e.g. minutes, hours or days) it would be very hard to tell whether the shape was more Gaussian-like or more Lorentzian-like. But surely it can't make that much difference to financial calculations whether our data is more Gaussian-like or more Lorentzian-like? Well, here is the bombshell: Equation (2.6) shows that those fatter tails for the Lorentzian yield a $p[\Delta x] \sim (\Delta x)^{-2}$ dependence for large $|\Delta x|$. Hence while the variance $\sigma^2$ is finite for a Gaussian, the variance is *infinite* for the Lorentzian. This is worrying, since plugging $\sigma^2 = \infty$ into standard finance formulae (e.g. option pricing formulae) gives nonsensical answers. So as far as standard risk calculations which depend on $\sigma$ are concerned, there is a world of difference between a Gaussian distribution for price-changes and a Lorentzian one, despite the fact that both can be reasonable candidates for fitting real-world price-data. And herein lies a problem: any realistic description of financial risk must account correctly for the tails of the distribution of price-changes $p[\Delta x]$. Hence we need to have maximum possible information about the form of $p[\Delta x]$ which most accurately describes the tails. However, the tails correspond to large deviations $|\Delta x|$ and these deviations become increasingly rare as $|\Delta x|$ increases. Hence the tails are where we have least empirical data points, hence least statistical information. In short, risk deals with atypical price-changes, yet the volatility measures typical fluctuations around the mean. Hence the volatility is not sufficient to classify risk – more information is required about $p[\Delta x]$, either through higher-order moments or detailed knowledge of the functional form of the tails.

### 2.2.3 Price-changes over multiple timesteps

### 2.2.3.1 Implications for risk

As we will now discuss, the problems for standard finance theory do not end with the need for a good characterization of the PDF $p[\Delta x]$ for all $\Delta x$. Section 2.2.2 considered the special case where the price-changes are i.i.d. Continuing with i.i.d. price-changes for the moment, the probability of observing a particular sequence of price-changes $\{\Delta x_i\} \equiv \Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_n$ given by $p\left[\{\Delta x_i\} \equiv \Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_n\right]$, simplifies exactly to:

$$p\left[\{\Delta x_i\} \equiv \Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_n\right] = p[\Delta x_1]\, p[\Delta x_2] \ldots p[\Delta x_i] \ldots p[\Delta x_n] \qquad (2.7)$$

This is why it was worth studying the properties of $p[\Delta x]$ in some detail. However for anything other than i.i.d. variables, Equation (2.7) does not hold. For the case of independent but not identically-distributed price-changes (Case II) we have instead:

$$p\left[\{\Delta x_i\} \equiv \Delta x_1, \Delta x_2, \ldots \Delta x_i, \ldots \Delta x_n\right] = p_1[\Delta x_1]\, p_2[\Delta x_2] \ldots p_i[\Delta x_i] \ldots p_n[\Delta x_n] \qquad (2.8)$$

Since the $p$-functions are not identical at consecutive time-steps, this process is effectively *non-stationary*. For dependent variables (Cases III and IV) we cannot split $p\left[\{\Delta x\}\right]$ up into *any* simple product of single-step PDFs. But why should we be worried about such properties of $p\left[\{\Delta x\}\right]$? When one considers 'risk' in practice, one typically thinks of the risk of losing money. Suppose we are trading in a market, and because of our positions any future drop in price of 20% will make us go bankrupt. What is the probability that we will go bankrupt in this market? Suppose we know the PDF for daily price-changes $p[\Delta x]$ and we assume that the daily price-increments are i.i.d. We could then use our knowledge of $p[\Delta x]$ to calculate the probability of the market dropping 20% or more in a given day. We need to add to this the probabilities for the market to drop 20% or more over a period of two days, three days and so on. Note that this could arise in many ways: for example it could fall 10% each day, or 15% on one day and 5% on the next, etc. We need to add up all these possible combinations in order to calculate the probability of the market dropping 20% or more over any time-scale. This is a straightforward calculation, as we have outlined it. However, it is wrong – at least, it will be wrong if successive price-changes are actually dependent (i.e. not i.i.d.). To illustrate this, let's return to the coin-toss example from Case III in Section 2.2.1, and consider the following scenario. Suppose the dependence of outcomes is such that *if* a sequence of five tails TTTTT arises (i.e. five consecutive price-changes downward) *then* the probability that the next outcome is T becomes unity (i.e. another price-change downward). The consequence would be that the price-series would evolve in a seemingly random way until it hit five consecutive downward price-changes. The price would then *continue* to move down indefinitely, producing an enormous crash. Hence instead of just multiplying the probabilities for independent events which lead to a net loss of 20% or more, we need to consider the *conditional* probabilities along each of the paths which could lead to a loss of 20% or more.

### 2.2.3.2 Statistical properties of the moments

We next consider the statistical properties of the moments of price-changes measured over several timesteps. Our notation for the price-change at timestep $i$ was $\Delta x_i = x_i - x_{i-1}$. Hence the price-change between timestep 0 and $n$ is given by $\Delta x_{n,0} = \sum_{j=1}^{n} \Delta x_j = x_n - x_0$. The *mean* price-change between timestep 0 and $n$ is:

$$\left\langle \Delta x_{n,0} \right\rangle = \sum_{j=1}^{n} \left\langle \Delta x_j \right\rangle \tag{2.9}$$

which is the well-known result that the *average of the sum is equal to the sum of the averages.* Equation (2.9) holds *irrespective* of whether the price-changes $\Delta x_j$ are i.i.d. or not. For the special case in which each mean is the same $\left\langle \Delta x_j \right\rangle \equiv \left\langle \Delta x \right\rangle$ (for example, for i.i.d. variables) then we have:

$$\left\langle \Delta x_{n,0} \right\rangle = \sum_{j=1}^{n} \left\langle \Delta x_j \right\rangle = n \left\langle \Delta x \right\rangle \tag{2.10}$$

For the very special case of our coin-toss market, we have $\left\langle \Delta x \right\rangle = 0$ and hence $\left\langle \Delta x_{n,0} \right\rangle = 0$ for an arbitrary time-increment $n$. The *variance* of the price-change between timestep 0 and $n$ can be calculated as follows:

$$\begin{aligned}
\sigma_{n,0}^{\;2} &\equiv \left\langle \left( \Delta x_{n,0} - \left\langle \Delta x_{n,0} \right\rangle \right)^2 \right\rangle = \left\langle \left( \Delta x_{n,0} \right)^2 \right\rangle - \left\langle \Delta x_{n,0} \right\rangle^2 = \left\langle \left( \sum_{j=1}^{n} \Delta x_j \right)^2 \right\rangle - \left\langle \sum_{j=1}^{n} \Delta x_j \right\rangle^2 \\
&= \underbrace{\left\langle \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta x_i \Delta x_j \right\rangle}_{\Downarrow \; \sum_{i=1}^{n} \left\langle \left( \Delta x_i \right)^2 \right\rangle + \sum_{i \neq j} \left\langle \Delta x_i \Delta x_j \right\rangle} \quad - \quad \underbrace{\left\{ \sum_{j=1}^{n} \left\langle \Delta x_j \right\rangle \right\}^2}_{\Downarrow \; \sum_{i=1}^{n} \left\langle \Delta x_i \right\rangle^2 + \sum_{i \neq j} \left\langle \Delta x_i \right\rangle \left\langle \Delta x_j \right\rangle}
\end{aligned} \tag{2.11}$$

If the price-changes $\Delta x_i$ are *uncorrelated*, then $\left\langle \Delta x_i \Delta x_j \right\rangle = \left\langle \Delta x_i \right\rangle \left\langle \Delta x_j \right\rangle$ for $i \neq j$ and hence Equation (2.11) simplifies exactly to:

$$\sigma_{n,0}^{\;2} = \sum_{i=1}^{n} \left\langle \left( \Delta x_i \right)^2 \right\rangle - \sum_{i=1}^{n} \left\langle \Delta x_i \right\rangle^2 = \sum_{i=1}^{n} \left\{ \left\langle \left( \Delta x_i \right)^2 \right\rangle - \left\langle \Delta x_i \right\rangle^2 \right\} = \sum_{i=1}^{n} \sigma_{i,i-1}^{\;2} \tag{2.12}$$

hence the well-known result for *uncorrelated* variables that the *variance of the sum is equal to the sum of the variances.* For the special case in which each variance is the same for each timestep (for example, for i.i.d. variables) then ${\sigma_{i,i-1}}^2 \equiv \sigma^2$ and we have:

$$ {\sigma_{i,i-n}}^2 \equiv \sum_{i=1}^{n} {\sigma_{i,i-1}}^2 = n\,\sigma^2 \qquad (2.13) $$

where ${\sigma_{i,i-n}}^2 = {\sigma_{n,0}}^2$ since the price changes at each timestep have the same variance. Equation (2.13) is the result used often by standard finance theory that the standard deviation (i.e. the volatility) of the price-change over an interval of $n$ timesteps increases as

$$ \sigma_{i,i-n} = n^{\frac{1}{2}}\,\sigma \qquad (2.14) $$

In other words, *the volatility of price-changes over a time-increment* $n$ *increases as the square-root of the time-increment* $n$. For the very special case of our coin-toss market, we have $\sigma = d$ and hence $\sigma_{i,i-n} = n^{\frac{1}{2}}\,d$. Equation (2.14) will reappear in various guises throughout this book. Note that in the *opposite* limit where all the price-changes $\Delta x_i$ are *so correlated* that they all have the same value and sign, $\Delta x$, it then follows that

$$ \begin{aligned} {\sigma_{i,i-n}}^2 &\equiv \left\langle \left(\Delta x_{n,0} - \langle \Delta x_{n,0} \rangle\right)^2 \right\rangle = \left\langle \left(\Delta x_{n,0}\right)^2 \right\rangle - \langle \Delta x_{n,0} \rangle^2 = \left\langle \left( \sum_{j=1}^{n} \Delta x_j \right)^2 \right\rangle - \left\langle \sum_{j=1}^{n} \Delta x_j \right\rangle^2 \\ &= \left\langle \left(n\,\Delta x\right)^2 \right\rangle - \langle n\,\Delta x \rangle^2 = n^2 \left( \left\langle (\Delta x)^2 \right\rangle - \langle \Delta x \rangle^2 \right) \\ &= n^2\,\sigma^2 \end{aligned} \qquad (2.15) $$

and hence the volatility of the price-change after $n$ timesteps now increases as

$$ \sigma_{i,i-n} = n\,\sigma \qquad (2.16) $$

This makes sense: think of walking purposely in a straight line at constant velocity, as opposed to random-walking. The distance moved, and hence your standard deviation, is now proportional to the number of timesteps $n$ as opposed to $n^{\frac{1}{2}}$. In the more general case of some *limited* but *non-zero* level of positive correlation, the corresponding expression for the standard deviation will therefore lie between the uncorrelated case of $n^{\frac{1}{2}}$, and the correlated case of $n^1$. If the price-changes $\Delta x_i$ are *anti-correlated* (i.e. their correlation is negative), then the dependence will be more like $n^0$. It turns out that real price-series will have the property that the volatility of the price-change after $n$ time-steps increases as $\sigma_{i,i-n} \sim n^{\frac{1}{\mu}}\,\sigma$ where $\mu$ is some value to be determined by empirical analysis of the price-series. A 'persistent walk' implies $1 \le \mu \le 2$ while an 'anti-persistent walk' corresponds to $\mu > 2$.

Notice that for the mean (which concerns the $m = 1$ moment of the PDF) and the variance (which concerns the $m = 2$ moment of the PDF) we have made the following statement: *'the M of the sum is equal to the sum of the M'* where *M* represents the mean or variance. For the mean this statement was always true, while for the variance it required uncorrelated variables. It turns out that this statement will also apply to higher-order moments via the so-called cumulants (which are functions of these moments, as shown in [G] and [BP]). However, the higher the moment, the higher the level of independence required for the statement to hold. This makes sense since, as mentioned earlier, the higher-order moments are picking up higher levels of dependence lying buried in the system.

### 2.2.3.3 Probability distribution function: PDF

In Section 2.2.3.2, we obtained results for the moments of the PDF describing the price-change during the time-increment between 0 and $n$, i.e. $\Delta x_{n,0} = x_n - x_0$. However, our goal is now to say something about the functional form of the PDF itself, i.e. $p\left[\Delta x_{n,0}\right]$. Let's start by considering the first non-trivial case, that of $n=2$. By definition $\Delta x_{n=2,0} = \sum_{j=1}^{2} \Delta x_j = \Delta x_1 + \Delta x_2$, hence there may be various combinations of $\Delta x_1$ and $\Delta x_2$ which add to obtain a given value of $\Delta x_{n=2,0}$. This can be seen easily by referring back to Figure 2-2. Hence we can write:

$$p\left[\Delta x_{n=2,0}\right] = \sum_{\substack{\Delta x_1, \Delta x_2 \\ \text{such that} \\ \Delta x_{2,0} = \Delta x_1 + \Delta x_2}} p\left[\Delta x_1, \Delta x_2\right] = \sum_{\Delta x_1} p\left[\Delta x_1, \Delta x_{n=2,0} - \Delta x_1\right] \quad (2.17)$$

where $p\left[\Delta x_1, \Delta x_2\right]$ is the joint probability distribution of $\Delta x_1$ *and* $\Delta x_2$. *If* the two price-changes $\Delta x_1$ and $\Delta x_2$ are i.i.d., then we can factorize the joint probability distribution to obtain:

$$\begin{aligned} p\left[\Delta x_{n=2,0}\right] &= \sum_{\Delta x_1} p\left[\Delta x_1, \Delta x_{n=2,0} - \Delta x_1\right] \\ &= \sum_{\Delta x_1} p\left[\Delta x_1\right] p\left[\Delta x_{n=2,0} - \Delta x_1\right] \equiv p\left[\Delta x\right] * p\left[\Delta x\right] \end{aligned} \quad (2.18)$$

where $p\left[\Delta x\right] * p\left[\Delta x\right]$ is the (discrete) convolution of the two PDFs. Repeating this for $n=3$, we obtain the PDF $p\left[\Delta x_{n=3,0}\right]$ as $p\left[\Delta x_{n=2,0}\right]$ convoluted with $p\left[\Delta x\right]$. This can be repeated for arbitrary $n$. But what does $p\left[\Delta x_{n,0}\right]$ look like? We can see this by thinking back to our coin-toss market, described by the single-timestep PDF $p\left[\Delta x\right]$ in Figure 2-1. The convolution will have the effect of 'blurring' the double-peaked function $p\left[\Delta x\right]$. For $n=2$ we have:

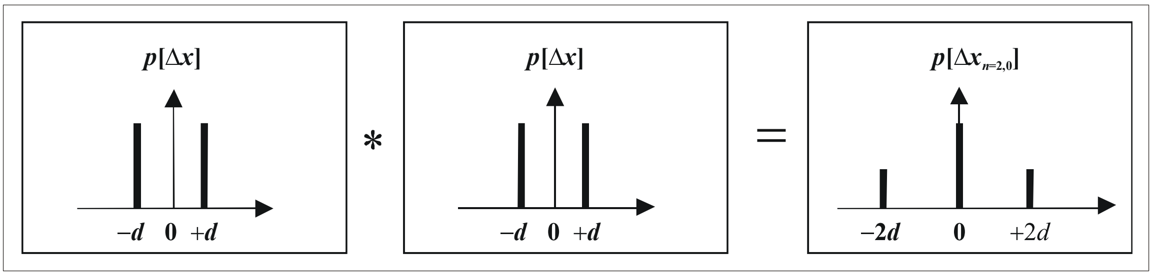


**Figure 2-4 Schematic diagram showing convolution of $p\left[\Delta x\right]$ with itself, to form $p\left[\Delta x_{n=2,0}\right]$.**

Referring back to Figure 2-2, this figure for $p\left[\Delta x_{n=2,0}\right]$ makes sense since there are *two* possible ways of obtaining $\Delta x_{n=2,0} = 0$ and *one* possible way of obtaining $\Delta x_{n=2,0} = +2d$ and $\Delta x_{n=2,0} = -2d$ . As $n$ increases, the repeated convolutions will increase this 'blurring' effect, and $p\left[\Delta x_{n,0}\right]$ will eventually start to look like a bell-shape (i.e. Gaussian). This can even be seen in a crude sense in the above Figure for $n = 2$ . It turns out that we are seeing the *Central Limit Theorem* in action. This theorem plays a crucial role in standard finance theory, so we will discuss it in more detail.

### 2.2.3.4 Central Limit Theorem

The price-change during the time-increment between 0 and $n$ , is given by $\Delta x_{n,0} = \sum_{j=1}^{n} \Delta x_j$ . Let's assume that the price-change variables $\left\{\Delta x_j\right\}$ are i.i.d. . Given the relation demonstrated in Section 2.2.3.3 between probabilities and convolutions, and the connection between convolutions and Fourier transforms, we will exploit the Fourier transform of the probability density function $p\left[y\right]$. We start with the identity

$$\left\langle e^{iqy}\right\rangle = \int_{-\infty}^{+\infty} e^{iqy}\, p\left[y\right] dy \qquad (2.19)$$

Consider the quantity $\left(\Delta x_{n,0} - n\overline{\Delta x}\right)/n \equiv \left(\Delta x_{n,0} - \overline{\Delta x_{n,0}}\right)/n$ , recalling that $\overline{\Delta x_{n,0}}$ is identical to $n$ times the mean price-change per timestep $\overline{\Delta x}$ . We perform the following calculation:

$$\left\langle e^{ik(\Delta x_{n,0}-n\overline{\Delta x})/n}\right\rangle = \left\langle e^{ik(\Delta x_1+\Delta x_2+\Delta x_3\ldots+\Delta x_n-n\overline{\Delta x})/n}\right\rangle$$
$$= \left\langle e^{ik\left(\left[\Delta x_1-\overline{\Delta x}\right]+\left[\Delta x_2-\overline{\Delta x}\right]\ldots+\left[\Delta x_n-\overline{\Delta x}\right]\right)/n}\right\rangle$$
$$= \left[\left\langle e^{ik\left(\Delta x-\overline{\Delta x}\right)/n}\right\rangle\right]^n$$
$$= \left\langle 1+\frac{ik}{n}\left(\Delta x-\overline{\Delta x}\right)-\frac{k^2}{2n^2}\left(\Delta x-\overline{\Delta x}\right)^2+\ldots\right\rangle^n \qquad (2.20)$$
$$= \left[1+0-\frac{k^2\sigma^2}{2n^2}+\ldots\right]^n$$
$$\approx e^{-\frac{k^2\sigma^2}{2n}} \quad \text{as } n\to\infty$$

This calculation requires the assumptions that the price-changes per timestep $\Delta x_j$ are i.i.d., in order to go from the second to the third line, and that we can perform a Taylor expansion around $\overline{\Delta x}$ (fourth line). We have also used the result $e^y = \lim_{n\to\infty}\left[1+y/n\right]^n$. The PDF is obtained by taking the inverse Fourier transform of Equation (2.19), using $q = k/n$:

$$p\left[\Delta x_{n,0}-\overline{\Delta x_{n,0}}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{q^2 n\sigma^2}{2}}\, e^{-iq(\Delta x_{n,0}-\overline{\Delta x_{n,0}})}\, dq$$
$$= \left[\frac{1}{2\pi n\sigma^2}\right]^{\frac{1}{2}} e^{-\frac{(\Delta x_{n,0}-\overline{\Delta x_{n,0}})^2}{2n\sigma^2}} \qquad (2.21)$$

In the limit that $n\to\infty$, we therefore have that the PDF approaches a Gaussian[10] distribution, independent of the specific form of the distribution $p\left[\Delta x\right]$ of the price-changes over one timestep. This is the Central Limit Theorem (CLT). Equation (2.21) can be rewritten as

$$p\left[\Delta x_{n,0}-\overline{\Delta x_{n,0}}\right] = \left[\frac{1}{2\pi\sigma_{n,0}{}^2}\right]^{\frac{1}{2}} e^{-\frac{(\Delta x_{n,0}-\overline{\Delta x_{n,0}})^2}{2\sigma_{n,0}{}^2}} \qquad (2.22)$$

where $\sigma_{n,0} = n^{\frac{1}{2}}\,\sigma$. In other words, the PDF for price-changes $\Delta x_{n,0}$ has the following Gaussian form:

$$p\left[\Delta x_{n,0}\right] = \left[\frac{1}{2\pi\sigma_{n,0}{}^2}\right]^{\frac{1}{2}} e^{-\frac{(\Delta x_{n,0}-\overline{\Delta x_{n,0}})^2}{2\sigma_{n,0}{}^2}} \qquad (2.23)$$

where $\sigma_{n,0}$ is the standard deviation. Therefore for $n\to\infty$, the PDF of price-changes over time-increment *n* becomes Gaussian with a corresponding volatility (i.e. standard deviation) given by $\sigma_{n,0} = n^{\frac{1}{2}}\,\sigma$. In the previous Section, we found that the relation $\sigma_{n,0} = n^{\frac{1}{2}}\,\sigma$ is true if the variables are uncorrelated and identically distributed. In the present case the variables are i.i.d., and hence by

[10] The Gaussian distribution is also known as the normal distribution.

necessity are also uncorrelated: hence we recover the same result. This eventual convergence of the PDF to a Gaussian explains why the Gaussian distribution is used throughout standard finance theory as a model for price-changes.

The implications of the CLT for finance seem remarkable. It seems that the Gaussian distribution, which is arguably the simplest function to manipulate in mathematical terms, is also completely justifiable. However, this whole statement needs to be looked at in more detail. To summarize the CLT story, we have shown that the PDF for price-changes over increments of *n* timesteps (or equivalently, over a real time-increment $\Delta t = n\tau$ where $\tau$ is the time-interval for one timestep) will approach Gaussian if:

*a)* the price-changes $\Delta x_1, \Delta x_2, \ldots \Delta x_n$ over a single timestep (or equivalently, over a real time-interval $\tau$) are i.i.d. variables, *and*

*b)* the PDF $p[\Delta x]$ for price-changes over a single timestep (or equivalently, over a real time-interval $\tau$) has a finite variance[11] $\sigma^2$, *and*

*c)* *n* is so large that the limit $n \to \infty$ has effectively been reached (or equivalently, $\Delta t$ is so large that the limit $\Delta t/\tau \to \infty$ has effectively been reached)

Unfortunately, it turns out that these three conditions are *not* typically met on the timescales of interest in real markets (e.g. hourly, daily, and possibly even weekly). *Hence it cannot be assumed that the PDF for price-changes over the time-increments* $\Delta t$ *of interest, will be Gaussian.* Let's take *a), b)* and *c)* one at a time, starting with *a)*. As will be shown in Chapters 3 and 6, there is strong evidence that the price-changes over a time-interval $\tau$ -- where $\tau$ is of the order of minutes, hours, days, or even possibly weeks -- are not i.i.d. Although the low-order temporal correlations such as $c_{ij}$ defined in Equation (2.3) are usually zero, higher-order temporal correlations typically survive. Hence the price-changes are not i.i.d., and *a)* does not hold. Now we turn to *b)*. We have already seen that there are distributions (e.g. Lorentzian) which might *look* reasonable as models for the PDF of price-changes $p[\Delta x]$, and yet have infinite variance. As will be demonstrated in Chapter 3, the PDFs of price-changes in real markets can look quite similar to a Lorentzian over small time-increments – in particular, they seem to have inverse power-law tails which yield very large higher moments. These large higher moments of $p[\Delta x]$ lead consequently to a very slow convergence to Gaussian. Finally let's consider *c)*. In major markets, the timescale on which trades occur is quite short. However time-increments of at least $\tau = 60$ min are typically needed for real price series in order that conditions *a)* and *b)* hold to a reasonable approximation. The standard financial machinery used in practice tends to assume that the PDF of daily price-changes is Gaussian. However, it is clear that a timescale of

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[11] This is because of the necessity to expand in a Taylor series during the derivation (see Equation (2.20)).

$\Delta t = 1$ day corresponds to less than $10\tau$ and hence does *not* correspond to the $n \to \infty$ limit. There is a related point concerning risk calculations. The CLT was derived by performing a Taylor expansion, and for this reason strictly only applies in the limit $n \to \infty$. What happens in practice is that as *n* increases, the Gaussian form grows in the central portion of the PDF around the mean $\overline{\Delta x_{n,0}}$. However the CLT makes no guarantees about the convergence to a Gaussian in the tails of the PDF $p\left[\Delta x_{n,0}\right]$, i.e. we cannot guarantee that $p\left[\Delta x_{n,0}\right]$ will have Gaussian form for *large* deviations $\Delta x_{n,0}$. Given that the probability of having a large price-change is determined by the functional form of $p\left[\Delta x_{n,0}\right]$ at large $\Delta x_{n,0}$, there is no guarantee that the Gaussian form can be used at all for risk calculations.

Before leaving the discussion of PDFs for price-changes, we note a special case of the above CLT discussion, in which the PDF of price-changes for single timesteps $p[\Delta x]$ is Gaussian, i.e.

$$p[\Delta x] = \left[\frac{1}{2\pi\sigma^2}\right]^{\frac{1}{2}} e^{-\frac{(\Delta x - \overline{\Delta x})^2}{2\sigma^2}} \qquad (2.24)$$

as in Figure 2-3, with a standard deviation $\sigma$. We also assume that the price-changes are i.i.d. which means that we can generate the PDF for $n = 2$ timesteps by just convolving $p[\Delta x]$ with itself (recall Section 2.2.3.3). However, we know from Fourier Transform theory that a Gaussian convolved with itself also yields a Gaussian. Hence the PDF $p\left[\Delta x_{2,0}\right]$ for price-changes over two timesteps is also a Gaussian, with $\sigma_{n=2,0} = 2^{\frac{1}{2}}\,\sigma$. We can repeat this for any *n*, yielding:

$$p\left[\Delta x_{n,0}\right] = \left[\frac{1}{2\pi\sigma_{n,0}{}^2}\right]^{\frac{1}{2}} e^{-\frac{(\Delta x_{n,0} - \overline{\Delta x_{n,0}})^2}{2\sigma_{n,0}{}^2}} \qquad (2.25)$$

as in Equation (2.24) with a standard deviation $\sigma_{n,0} = n^{\frac{1}{2}}\,\sigma$. In this sense, the Gaussian distribution is said to be *stable*. Another way of saying the same thing is that the resulting Gaussian is *self-similar* on all scales: it has the same functional form on all time-scales *n*, i.e. for all values of *n* as can be seen from Equations (2.24) and (2.25). This self-similarity can also be seen by eye if we use Equation (2.24) to generate a price-change at each timestep, as shown below:

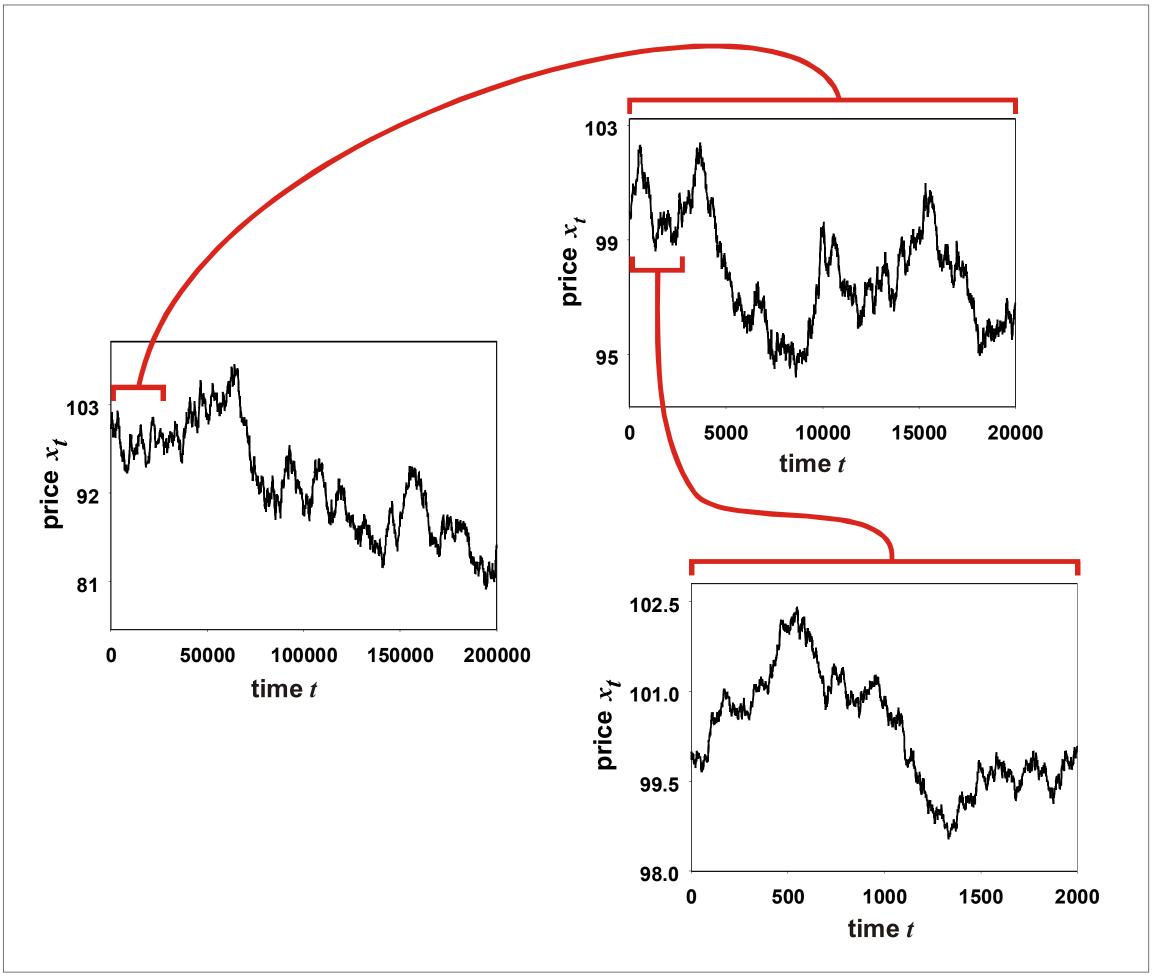



**Figure 2-5 A random walk price-series created by generating a price-change at each timestep using the Gaussian PDF of Equation (2.24). The price-series looks similar over different time windows.**

We can make this self-similarity more obvious by defining a new variable $w_n = n^{-\frac{1}{2}} \Delta x_{n,0}$, and hence $\overline{w_n} = n^{-\frac{1}{2}} \overline{\Delta x_{n,0}}$. Using $\sigma_{n,0} = n^{\frac{1}{2}} \sigma$ and Equation (2.25), we can write the normalization condition for $p\left[\Delta x_{n,0}\right]$ as:

$$\int_{-\infty}^{+\infty} p\left[\Delta x_{n,0}\right] d\left(\Delta x_{n,0}\right) = \left[\frac{1}{2\pi n\sigma^2}\right]^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{(n^{\frac{1}{2}} w_n - n^{\frac{1}{2}} \overline{w_n})^2}{2n\sigma^2}} d\left(n^{\frac{1}{2}} w_n\right) = 1 \qquad (2.26)$$

and hence

$$\left[\frac{1}{2\pi\sigma^2}\right]^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{(w_n - \overline{w_n})^2}{2\sigma^2}} d\, w_n = 1 \qquad (2.27)$$

This means that by *scaling* $\Delta x_{n,0}$, which is the price-change over an increment of $n$ timesteps, we have produced a PDF $p\left[w_n = n^{-\frac{1}{2}} \Delta x_{n,0}\right]$ which is *invariant* under changes in $n$. This PDF is given by:

$$p\left[w_n = n^{-\frac{1}{2}} \Delta x_{n,0}\right] = \left[\frac{1}{2\pi\sigma^2}\right]^{\frac{1}{2}} e^{-\frac{(w_n - \overline{w_n})^2}{2\sigma^2}} \qquad (2.28)$$

We will return to this point in Chapter 3 when we look at self-similarity for real price-change distributions.

### 2.2.4 Continuous-time evolution equation for the PDF of price-changes

We have discussed the PDF of price-changes used in standard finance theory, i.e. the Gaussian distribution. Using this distribution, one can easily generate a price-series by randomly picking values of the price-change variables from the PDF. The resulting price-series constitutes a random walk. This is also commonly referred to as a *Wiener process,* or *Brownian motion*. We had focused on discrete time, in order to properly understand the underlying assumptions behind the Gaussian PDF for price-changes. Now we will bring our discussion closer to standard finance theory, by moving over to *continuous-time* analysis in order to obtain the *same* result of a Gaussian functional form.

Suppose that we are at some particular timestep $n$ during the random walk generated by the coin-toss price model of Figure 2-2. Let's define $p\left[x_t = md\right]$ to be the probability that the price at time $t$ is $x\left[t\right] \equiv x_t = m\,d$. The probability of a positive/negative price change is $p\left[\Delta x_{t,t-1} = \pm d\right] = \frac{1}{2}$. We can thus obtain the probability $p\left[x_t = m\,d\right]$ in the following manner:

$$\begin{aligned} p\left[x_t = m\,d\right] &= p\left[x_{t-1} = (m+1)\,d\right] p\left[\Delta x_{t,t-1} = -d\right] + p\left[x_{t-1} = (m-1)\,d\right] p\left[\Delta x_{t,t-1} = +d\right] \\ &= \frac{1}{2}\left(p\left[x_{t-1} = (m+1)\,d\right] + p\left[x_{t-1} = (m-1)\,d\right]\right) \end{aligned} \tag{2.29}$$

Now we can subtract $p\left[x_{t-1} = md\right]$ from both sides of Equation (2.29) to get:

$$\begin{aligned} p\left[x_t = m\,d\right] - p\left[x_{t-1} = m\,d\right] &= \frac{1}{2}\left(p\left[x_{t-1} = (m+1)\,d\right] + p\left[x_{t-1} = (m-1)\,d\right]\right) - p\left[x_{t-1} = m\,d\right] \\ &= \frac{1}{2}\left(\left(p\left[x_{t-1} = (m+1)\,d\right] - p\left[x_{t-1} = m\,d\right]\right) + \left(p\left[x_{t-1} = (m-1)\,d\right] - p\left[x_{t-1} = m\,d\right]\right)\right) \end{aligned}$$

and hence

$$\begin{aligned} &\frac{\left(p\left[x_t = m\,d\right] - p\left[x_{t-1} = m\,d\right]\right)}{\delta_t} \\ &= \frac{\delta_x^{\,2}}{2\delta_t} \; \frac{\left(p\left[x_{t-1} = (m+1)\,d\right] - p\left[x_{t-1} = m\,d\right]\right) + \left(p\left[x_{t-1} = (m-1)\,d\right] - p\left[x_{t-1} = m\,d\right]\right)}{\delta_x^{\,2}} \end{aligned} \tag{2.30}$$

where $\delta_t, \delta_x$ are the 'mesh-sizes' in time and price. In the coin-toss price model we took these as 1 and $d$ respectively. Equation (2.30) contains the discrete approximation of several partial derivatives. If we take the limit of the 'mesh' of price and time going to zero, i.e. $\delta_t, \delta_x \to 0$ but keep $\frac{\delta_x^{\,2}}{2\delta_t}$ finite, we get

$$\frac{\partial p(x,t)}{\partial t} = D\,\frac{\partial^2 p(x,t)}{\partial x^2} \tag{2.31}$$

where $D \equiv \frac{\delta_x^2}{2\delta_t}$ is a 'diffusion constant'. Note that because we have taken the continuous limit of price as well as time, $p(x,t)$ is a continuous PDF for the price $x$ at time $t$. Equation (2.31) is the *diffusion equation*, and its solution gives the probability $p(x,t)dx$ that the price has a value in the infinitesimal range $x \to x+dx$ at time $t$. It can also be called a *Wiener process*, or a *Fokker-Planck* partial differential equation which governs the evolution of a probability density for an underlying *Markov process*[12]. Physicists also call this *Brownian motion* and use it to describe the random-walk dynamics of particles diffusing in a gas. Implicit in the derivation of Equation (2.31) was the Markov property used in Equation (2.29) whereby the next price-change is independent of all previous price-changes. This Markov property is the *same* as that assumed in Section 2.2.1 for the coin-toss price model in discrete time. Hence it is no surprise that the solution of this diffusion equation is the familiar Gaussian function, i.e.

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-x_0)^2}{4Dt}} \quad (2.32)$$

where the standard deviation is given by $t^{\frac{1}{2}}(2D)^{\frac{1}{2}}$. This represents the PDF for the price $x$ at time $t$ given the price level $x_0$ at time $t=0$. Hence we immediately see the correspondence with the Gaussian form in which the standard deviation was given by $n^{\frac{1}{2}}\sigma$, with $n$ being the number of discrete timesteps as opposed to $t$ which is the continuous time interval. Note that if we had included a bias term in this model, Equation (2.33) would pick up a drift term hence yielding a generalized diffusion, or Fokker-Planck, equation.

### 2.2.5 Stochastic differential equations for the evolution of the price

We have looked at the derivation of standard finance theory's Gaussian PDF for price-changes. In particular we obtained a description of how this PDF evolved as the time interval *t* -- or equivalently the number of timesteps *n* -- increased. One can present an equivalent view of this same process using *stochastic differential equations*, in which a stochastic equation of motion is obtained for the price in continuous time $x(t)$. We will look at this methodology, and then use it later in this Chapter to derive standard option pricing theory.

We start with our coin-toss price model, where the price-change at timestep $i$ is given by $\Delta x_i = x_i - x_{i-1}$. We will denote $\sigma \Delta X_i$ as this price-change, where $\sigma$ controls the step-size and $\Delta X_i$ is

[12] A Markov process is characterized by the property that the outcome at the *next* timestep (i.e. the next price-change) does not depend on past outcomes (i.e. the past price-changes). Hence the value of the price at the next timestep only depends on the present value of the price. Hence our coin-toss price model corresponds to a Markov process.

a stochastic variable describing the coin-toss outcome at timestep $i$. Since we assume i.i.d. price-changes, we can drop the subscripts, hence $\Delta x = \sigma \Delta X$. We then assume that the increment in $x$ is so small that we can replace $\Delta x \to dx$, and similarly $\Delta X \to dX$. Hence we have a stochastic differential equation for the price:

$$dx = \sigma \, dX \tag{2.34}$$

Standard finance theory makes the assumption that $dX$ is a random variable taken from a Gaussian PDF, with zero mean and a standard deviation equal to $(dt)^{\frac{1}{2}}$. In so doing, standard finance theory is basically assuming that $dt$ corresponds to a *small* time-increment which is however sufficiently *large* that the Central Limit Theorem (CLT) is valid; i.e. it is saying that $dt$ is somehow large enough to consist of $n \to \infty$ individual timesteps with associated price-changes per timestep which are i.i.d. and which have finite variance. Hence $dt$ is assumed to be large enough that the PDF for single-step price-changes (e.g. the coin-toss PDF of Figure 2-1) has convolved with itself an 'infinite' number of times. If all this were true, then the CLT would be valid and the PDF for price-changes over the time-increment $dt$ would indeed be Gaussian. Of course if time were truly continuous, then there would indeed exist $n = \infty$ infinitesimal timesteps in any finite time and hence the assumption of Gaussianity would be true. However we know that the actual duration of each timestep cannot realistically be made smaller than the time interval between trades. This time interval is always finite and sometimes quite large, hence explaining why non-Gaussianity can be observed in real financial data in contrast to the predictions of standard finance (see Chapter 3).

We have so far considered price-change processes which comprise a stochastic (i.e. random) term such as a coin-toss, hence our probabilistic description. In standard finance theory, a deterministic *drift* term is often added to account for a possible overall trend in the market. The reasons for a priori inclusion of such a term are debatable – after all by symmetry, both directions (up and down) are equally 'available'. Despite this, such drift terms are included – hence we will also include them in our discussion. Hence Equation (2.34) becomes:

$$dx = \sigma \, dX + \mu dt \tag{2.35}$$

where $\mu$ is a deterministic bias term, i.e. a deterministic rate-of-change of the price. We can integrate Equation (2.35) to give:

$$x(t) = x(0) + \mu t + \phi \sigma \sqrt{t} \tag{2.36}$$

where $\phi$ is a random variable drawn from a Gaussian distribution with mean zero and unit variance: i.e. $\phi \sim \mathrm{N}[0,1]$. Strictly speaking, Equation (2.36) allows $x$ to become negative. Hence it is standard practice to consider the price-change to be the *fractional* change in price, i.e. the return (see Section 1.4):

$$\frac{dx}{x} = \sigma\, dX + \mu dt \tag{2.37}$$

which integrates to give:

$$x(t) = x(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \phi\,\sigma\sqrt{t}\right] \tag{2.38}$$

In order to give more examples of stochastic differential equations, we now briefly mention two further models used in finance, for example for modelling interest rates. The first includes the effect that interest-rates often seem to have of 'returning to the mean'. This process is:

$$dx = \sigma\, dX + (\nu - \mu\, x)\, dt \tag{2.39}$$

which is called a *mean-reverting random walk*. If $x$ is small such that $\nu > \mu x$, the positive coefficient in front of $dt$ means that $x$ will move up on average. If $x$ is large such that $\nu < \mu x$, $x$ will move down on average. Equation (2.39) therefore includes a phenomenological description of a type of temporal correlation which restores $x$ to its mean, and hence lies beyond the basic random walk described in Section 2.2.1. Note however that our earlier claims that such stochastic differential equations can at best only include a limited subset of the actual higher-order temporal correlations exhibited by real market data, still holds. With $r$ instead of $x$, this type of random walk is the so-called Vasicek model for the short-term interest rate. The second model that we will mention has an additional factor in the random part:

$$dx = \sigma\sqrt{x}\, dX + (\nu - \mu\, x)\, dt \tag{2.40}$$

This is the Cox, Ingersoll and Ross model for the short-term interest rate. Obviously an infinite number of such stochastic differential equations can be introduced by making the drift and/or fluctuating parts increasingly complicated. However, in our opinion such an approach lacks a good microscopic understanding, and sometimes even empirical justification in terms of actual data. Nevertheless, the connection between the finance world and the field of stochastic calculus has drawn many applied mathematicians into the study of quantitative finance. This in turn has had a positive feedback effect in terms of generating increasing numbers of modifications to the basic stochastic differential equation for a random walk. However at their heart, these models all make a priori assumptions concerning the type of temporal dynamics in the asset price process. It is precisely *because* of such simplifying assumptions that it is possible to write down stochastic differential equations. Despite this shortcoming, the field of stochastic calculus with application to finance, can and does fill courses and books all by itself, many times over. The standard pricing theory for options, the so-called Black-Scholes theory, arguably provides the showcase for this stochastic calculus (see Section 2.4.3). For more details on the machinery of stochastic calculus, we refer to the excellent discussions in [WDH] and [W].

## 2.3 Risk: tails of the unexpected

There are many possible measures that one might use to quantify the risk in a given market. As mentioned earlier in Chapter 2, the standard measure of risk in the finance industry is the volatility $\sigma$. A numerical value for $\sigma$ can be obtained empirically by calculating the standard deviation of the distribution of empirical price-changes, obtained over a given time-increment $\Delta t$. *If* the PDF of these empirical price-changes was indeed Gaussian, then the use of $\sigma$ as the sole risk measure would make some sense: in an unbiased market where the mean price-change is zero, then $\sigma$ is the only parameter which defines the shape of the Gaussian PDF. However, there are several problems with the methodology of quantifying risk using this one parameter $\sigma$:

- The parameter $\sigma$ is obtained from the empirical price-change distribution for a fixed time-increment $\Delta t$ (e.g. one day). It therefore only measures the risk associated with this fixed, pre-determined time interval $\Delta t$. As mentioned earlier in Section 2.2.3.1, this ignores the risk of losses accumulated over consecutive time-increments $\Delta t$. To an investor, however, such risks are clearly just as important. Whether you lose your money all in one day *or* over a period of three days, you will still be bankrupt. In the case where price-changes have higher-order temporal correlations, the risk due to accumulated losses over several consecutive time-increments can be significantly different from that calculated assuming i.i.d. price-changes with a fixed value of $\sigma$ for time-increment $\Delta t$. Furthermore, drawdowns and crashes cannot be captured by any a priori, fixed time-scale. We return to this discussion in Section 3.6 where we investigate a price-series with higher-order temporal correlations, and in Chapter 7 where we look at crashes using a microscopic market model.
- The parameter $\sigma$ does not take into account the value of the maximal loss *inside* the time-interval $\Delta t$. Consider an example whereby the price-change measured over the whole day would have been just about acceptable to an investor: however in reality he was made bankrupt at 12:15pm when the price plunged by twice that amount. The price then recovered later that afternoon giving a relatively small change on the daily scale – but this was too late for our investor.
- The parameter $\sigma$ treats both upward price movements (i.e. 'market gains') and downward price movements (i.e. 'market losses') equally from the point of view of risk, whereas they should be distinguished. After all, large gains do not represent a risk.
- Empirical price-change PDFs which approximate to a Lorentzian distribution -- or more generally to the Levy distribution to be introduced in Chapter 3 -- will yield a very large numerical value for the standard deviation $\sigma$. While $\sigma$ is finite for a Gaussian PDF, it is strictly infinite for a Lorentzian PDF despite the fact that the two PDFs look similar. Hence it may be difficult to determine $\sigma$ accurately in practice for such a price-series.

- A strict Gaussian functional form is not justified for the tails of the PDF of price-changes regardless of the value chosen for the time-increment $\Delta t$. This is because the Central Limit Theorem (CLT) only applies near the centre of the distribution. Hence the parameter $\sigma$ does not tell us much about the tails of the PDF, where significant risk actually lies.

A more general theoretical approach to risk will be presented in Chapter 6. For now we pursue the standard theory approach, and look briefly at an alternative measure of risk which is sometimes used in the finance industry in an attempt to avoid these problems with $\sigma$. This alternative measure of risk concerns the probability of extreme losses or the so-called Value-at-Risk (VaR). It takes the view that one should choose a measure of risk which focuses on the possibility of large, unexpected downward movements of the market. In particular, the focus is on the probability of large *negative* price-changes. Imagine that we have $n$ samples of a random variable *y*. For example these could correspond to price-changes over successive days, or weeks. We will label these as $\{y_i\} = \{y_1, y_2, \ldots, y_n\}$. We assume that they all correspond to the same PDF $p[y]$ *and* we assume that the drawings are independent (i.i.d.). Of course such assumptions are *not* a priori true, however we will continue. Following the Value-at-Risk philosophy, the risk can be associated with the minimum value contained in $\{y_i\} = \{y_1, y_2, \ldots, y_n\}$, i.e. the largest negative price-change. Making the i.i.d. assumption, the probability that $y_{\min} < -\Omega$ where $\Omega$ is the magnitude of the largest negative price-change that our portfolio can withstand, can be calculated as follows.

$$p[y_{\min} < -\Omega] = 1 - \left[p_>[-\Omega]\right]^n = 1 - \left[1 - p_<[-\Omega]\right]^n \approx 1 - \exp\left[-n\, p_<[-\Omega]\right] \quad \text{for } p_<[-\Omega] << 1 \qquad (2.41)$$

where we have defined the cumulative probability distributions as:

$$p_<[-\Omega] = \int_{-\infty}^{-\Omega} p[y]\, dy \quad \text{and} \quad p_>[-\Omega] = \int_{-\Omega}^{\infty} p[y]\, dy \quad . \qquad (2.42)$$

with $p_<[-\Omega] + p_>[-\Omega] = 1$. Hence $p[y_{\min} < -\Omega] \approx 1 - \exp\left[-n\, p_<[-\Omega]\right]$ gives us the required probability that $y_{\min} < -\Omega$ for a series of *n* trials. For the Gaussian PDF $p[y] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$:

$$p_<[-\Omega] = \frac{1}{2}\mathrm{erfc}\left[\frac{\Omega}{\sqrt{2}\,\sigma}\right] \qquad (2.43)$$

where $\mathrm{erfc}[y]$ is the complementary error function. Therefore $\Omega = \sqrt{2}\,\sigma\, \mathrm{erfc}^{-1}\left[2p_<[-\Omega]\right] \propto \sigma$. This shows us that even in the Value-at-Risk methodology which supposedly focuses on the tails of the distribution, the resulting risk measure may still end up relying indirectly on the volatility $\sigma$.

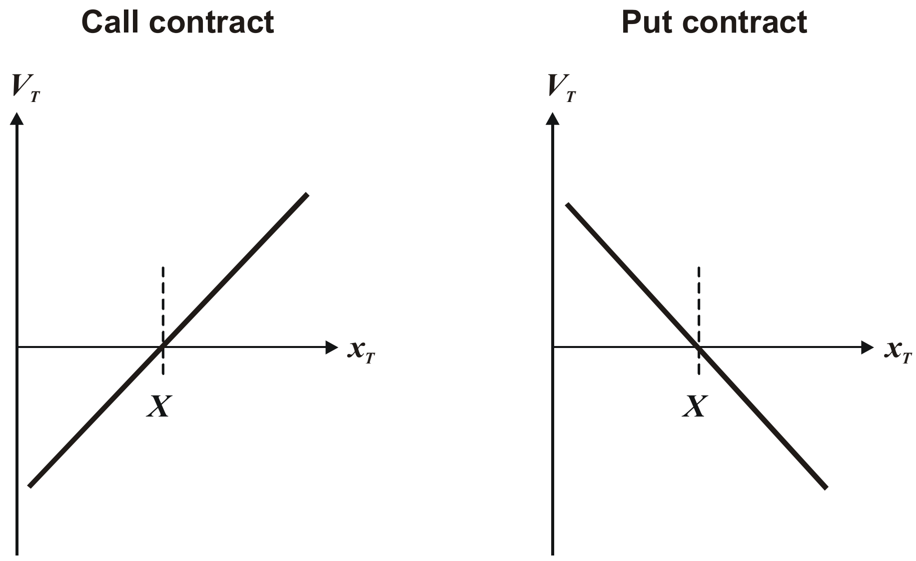
## *2.4 Eliminating risk within the Black-Scholes option pricing theory*

### 2.4.1 Introducing derivatives

Having taken a look at aspects of financial risk, we now turn to look at the *hedging* of risk within standard finance theory using derivatives. This forms the core of much of standard quantitative finance. It is clever stuff – in fact, the theory of option pricing won Merton and Scholes a Nobel prize. However despite its mathematical beauty, there can be some serious pitfalls when trying to implement it in practice. In Chapter 6 we look at how one might go beyond this standard finance theory, or so-called Black-Scholes, treatment. But for now, we content ourselves to sit back, and enjoy the maths. Our treatment borrows heavily from the masters in the field: Wilmott, Dewynne and Howison [WDH]. We start with a brief review of what derivatives actually are.

#### 2.4.1.1 Futures and Forwards

A *forward* contract is a contract entered into by two parties where they *must* fulfil the contract on expiry. This usually means exchanging assets for a pre-determined price. It is particularly popular in the currency markets where the asset is a given amount of currency of a given denomination, and the pre-determined price is the discounted exchange rate for that currency. A *future* contract is essentially the same as a forward contract, though traded in a slightly different way (as discussed in Chapter 1). The forward or future contract must be exercised at expiry. The expiry $T$ (also called the delivery date or maturity date) of the contract is determined at the time of the writing of the contract, as is the exercise price $X$ (also called the forward price or future price). We denote $V_T\left[x_T, X\right]$ as the contract value at expiry $T$ given that the asset price at expiry is $x_T$. Since there is no element of choice, it costs nothing to enter into a forward or futures contract. The payoff diagrams at expiry for the purchaser of a call contract (a contract to buy the asset) and a put contract (a contract to sell the asset) are as follows:




Hence the payoff is given by $(x_T - X)$ for the call contract and $(X - x_T)$ for the put contract. For the other party in this contract, this diagram represents the losses at expiry

### 2.4.1.2 Options

An *option* is like a car insurance. It is a contract that you enter into with another party, e.g. an insuring institution or bank, which writes the contract. In the case of car insurance, this would be the insurance company. You purchase the contract for a relatively small amount of money, typically some fraction of the underlying asset. In the case of car insurance, this is a small fraction of the car's actual worth. The contract is of fixed term (e.g. a few months). The insurance can be against all manner of conditions involving the underlying asset. If the conditions regarding the underlying asset (at the expiry of the contract for a 'European option' or at any time for an 'American option') are unfavourable to you (for example, the car is wrecked; the stock you're holding is worthless), you exercise the option, i.e. you demand that the option writer fulfill his part of the contract by paying out to you. The payoff you would then receive could far exceed the cost of the option, the premium. If the conditions regarding the underlying are favourable to you (for example, the car is fine; the stock you're holding has appreciated in value), you would not exercise the option. You would then lose the initial option cost, corrected by the underlying interest rate to allow for the loss-of-interest during the lifetime of the option contract. The insuring institution, on the other hand, pockets this amount. Of course, they are entering into many of these types of contracts and are hoping that the net profit from a large number of relatively low-cost option contracts will compensate (or maybe even exceed) the possible outlays in the case of options being exercised. Central to the pricing of options, therefore, is an estimate of the underlying risk of the option being exercised. Hence any option pricing model necessarily needs a suitable stochastic/probabilistic model of the behaviour of the underlying asset price during the lifetime of the option. Things are obviously far from simple due to the fact that the underlying asset has a fluctuating value and can vary quite considerably over the lifetime of the option. It is said that a week is a long time in politics – it can be a very long time in the financial markets as well. How can one therefore go about building a model to give a 'price' to such contracts? Clearly one could leave it to 'market forces' to decide the price – the expectation would be that after some trial and error, the market participants would collectively arrive at some agreed prices for these contracts. But how would one then put a price on new forms of contract? And what happens if the market doesn't seem to be behaving 'normally' ?

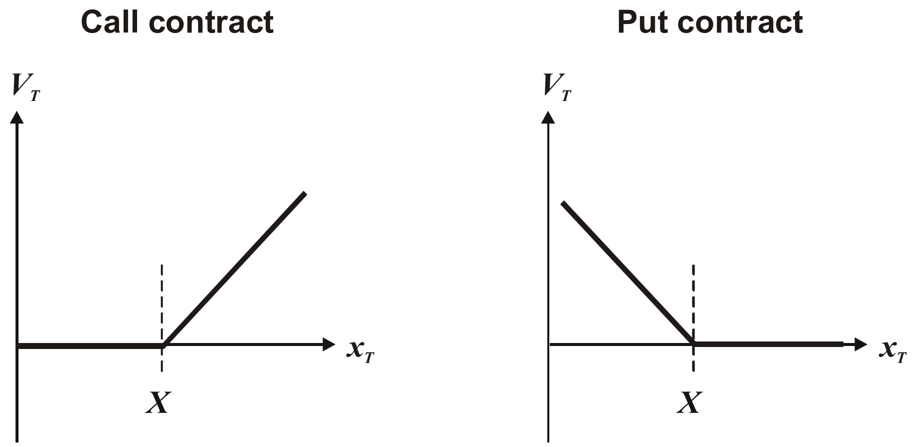
We will look at the standard finance theory approach to pricing such options. This approach is 'standard' in the sense that it rests on the standard models of financial asset price variations that we have been looking at earlier in this Chapter. It is also 'standard' in that it treats time as a continuous variable. A crucial quantity in the pricing of options is a measure of the fluctuations expected during the lifetime of the option. In a Gaussian world, and hence standard finance theory, this is given by the volatility (i.e. standard deviation of price-changes) $\sigma$.

### 2.4.2 Types of options

Options come in many different forms. Loosely speaking, a wide class of these are considered by standard finance theory as being describable by the same formalism. In particular, the prices of these options are obtained by solving the same partial differential equation: the so-called *Black-Scholes equation*. How can so many different option prices be obtained as solutions of the same partial differential equation? The answer lies in the fact that the different characteristics of these options appear as different constraints/boundary conditions on the allowed solutions of this equation. We will take a brief look at some of these options, and hence boundary conditions, and their respective payoffs.

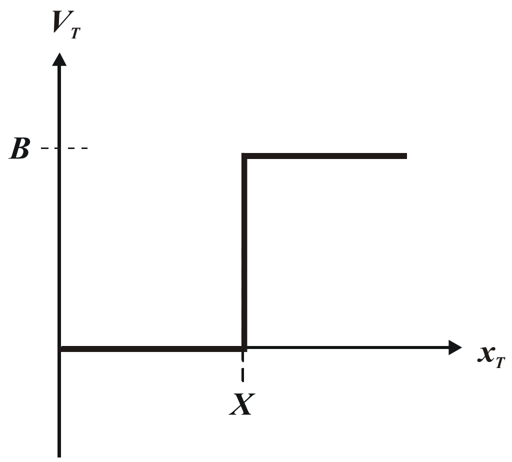
Options are either 'calls' or 'puts'. A *call option* is one where the option-buyer (i.e. holder) is acquiring the option of purchasing a prescribed asset from the option writer, for a prescribed amount (i.e. exercise price or strike price, $X$ ). A *put option* is one where the option-buyer is acquiring the option of selling the asset under the same set of pre-defined conditions. Since the option affords the holder a right, but not an obligation, then clearly an option should cost something.

A 'European option' is one which may *only* be exercised at expiry. The expiry $T$ of the option is determined at the time of writing the contract, and we denote $V_T\left[x_T, X\right]$ as the option payoff at expiry $T$ given that the asset price at expiry is $x_T$. The strike price $X$ is determined at the time of writing of the contract. The simplest type of European option is known as a 'vanilla' option. The payoff diagrams at expiry for the purchaser of a vanilla European call option and vanilla European put option are as follows:

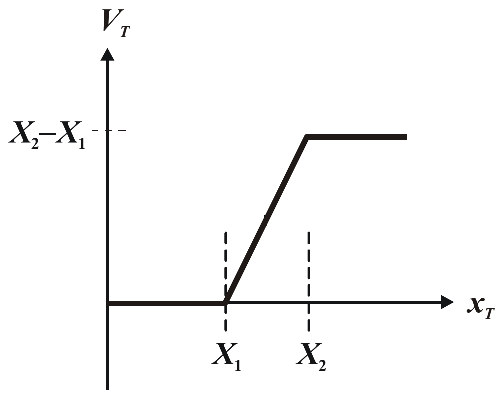



Hence the payoff for a call option is given by $\max\left[x_T - X, 0\right]$ and for a put option by $\max\left[X - x_T, 0\right]$. For the option-writer, these diagrams represent the losses at expiry. Both the above payoff diagrams represent truncated versions of the payoff diagrams for the futures contracts given in Section 2.4.1.1. Obviously, from the point of view of the option-writer these payoff diagrams have a considerable potential loss. So how much should the option-writer charge the option-buyer to compensate for this. In short, how much should an option cost? In the next Section, we answer this within the standard finance framework which relies on random walks. In Chapter 6, we take a more general point of view, relaxing the assumptions of standard finance theory.

Apart from the vanilla option, European options come in other flavours. For example, a cash-or-nothing call is an example of a 'digital' or 'binary' option, and has the following payoff:




Here the payoff is $B\,\mathrm{H}\left[x_T - X\right]$ where $\mathrm{H}\left[\ldots\right]$ is the Heaviside function. We can combine calls and puts with various exercise prices to give portfolios with a variety of payoffs: e.g. a 'bullish vertical spread' as shown below. 'Bullish', because the investor profits from a rise in the asset price: 'vertical' because there are two strike prices involved: 'spread' because it is made up of the same type of option (i.e. calls).




The payoff is given by: $\max\left[x_T - X_1, 0\right] - \max\left[x_T - X_2, 0\right]$. The payoff is achieved by buying one call option, and writing one call option with the same expiry date but larger strike price. This is a way of redirecting risk, and that is what portfolio management is all about. An American option may be exercised at *any* time prior to expiry, in contrast to the European option. Not only must a value be assigned to it, but one would also need to determine *when* it is best to exercise the option. These types of option are some of the hardest to price. Another type of option is the path-dependent (e.g. 'Asian' or lookback) option. For these options the payoff depends on a function of the asset price during the lifetime of the option.

### 2.4.3 Going, going, gone: the magic of zero-risk

We now give the derivation, and solution, of the Black-Scholes equation for option pricing. Our starting point is the stochastic differential equation, Equation (2.37) which describes the asset-price movement comprising a random variable and a drift term:

$$\frac{dx}{x} = \sigma\, dX + \mu dt \qquad (2.44)$$

where the random variable $dX$ is taken from a Gaussian PDF with mean 0 and variance $dt$. As has been the message throughout this Chapter, this equation can only ever be a coarse approximation to the discrete-time, non-Gaussian, temporally correlated movement of asset prices observed in the real financial markets. We accept this for now, but return to the consequences of deviations from these assumptions in Chapter 6.

Assume that at time $t$, the current value of the option is $V$ and the current value of the asset is $x$. We do not need to specify whether this is a call or put option at this stage. The important point is that $V$ is a function of $x$ and $t$, that is, $V(x,t)$. So here we have an example of a function of a stochastic variable $x$. If these were *usual* functions, then we would use our *usual* calculus expansion:

$$dV(x,t) = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial t} dt + \ldots \qquad (2.45)$$

where the dots denote higher order terms which we would *usually* neglect in the limit of very small $dx$ and $dt$. We know that even without the higher order terms, Equation (2.45) works fine for deterministic functions – i.e. we plug in values for the partial derivatives, and out pops a value of $dV$. That's what we all learn in undergraduate calculus, and that's what we *usually* use. However in the case of *stochastic* functions, things are more complicated. Given a value of *t*, we can only make a *probabilistic* statement about the value of $x$ and hence $V$. In this case we'd better hang on to these higher order terms in Equation (2.45) until we can work out what terms are negligible in the limit of small time intervals $dt$. In so doing, we will be implicitly using a type of calculus which is the engine behind all standard, modern financial engineering: *Ito calculus.* In fact we only need one particular result of this Ito calculus, which is the so-called *Ito's lemma*. This is also basically all that is used in the financial world. According to Ito's lemma, the correct form of Equation (2.45) given that $x$ is a stochastic variable, is:

$$dV(x,t) = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}dx^2 \ldots \qquad (2.46)$$

where the dots denote higher order terms. If we plug Equation (2.44) into (2.46), we get

$$dV(x,t) = \frac{\partial V}{\partial x}(x\sigma\, dX + x\mu\, dt) + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x\sigma\, dX + x\mu\, dt)^2 \qquad (2.47)$$

where we have dropped the dots denoting higher-order terms. We now have to expand out the terms in Equation (2.47). The last bracket is where we need to do some work:

$$(x\sigma\, dX + x\mu\, dt)^2 = \underbrace{x^2\sigma^2\,(dX)^2}_{\text{Term 1}} + \underbrace{2x^2\sigma\mu\,(dX)(dt)}_{\text{Term 2}} + \underbrace{x^2\mu^2\,(dt)^2}_{\text{Term 3}} \qquad (2.48)$$

Term 1: As discussed throughout this Chapter, the standard finance theory model is to assume that the stochastic process $dX$ is a random walk with an associated PDF whose standard deviation is equal to $(dt)^{\frac{1}{2}}$. Hence $(dX)^2$ is of order $dt$.

Term 2: We wish to keep terms of order $dt$, hence our specific interest in Term 1. However Term 2 clearly has a higher-order dependence: heuristically we can think of it as something like $(dt)^{\frac{1}{2}}(dt)$ which is hence a higher order term than $dt$. Hence we will neglect it.

Term 3: This clearly has a higher order dependence than $dt$. Hence we will neglect it.

This yields a simplified form of Equation (2.48):

$$(x\sigma\, dX + x\mu\, dt)^2 = x^2\sigma^2\, dt \ldots \qquad (2.49)$$

Putting Equation (2.49) into Equation (2.47) and doing a little bit of tidying up gives:

$$dV(x,t) = \left[\sigma x \frac{\partial V}{\partial x}\right] dX + \left[\mu x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t}\right] dt \qquad (2.50)$$

So far so good, but we still have a stochastic equation for the option price $V(x,t)$, i.e. given the time $t$ and the current asset price we still cannot get a unique value for the option price $V$. Our goal is to obtain a unique price for the option. For this, we need to consider a strategy for holding the option and the asset. This is the process of 'hedging' which will be discussed in detail in Chapter 6. Ideally we would do this in a way that gives minimum, or even zero, risk. The magic of the Black-Scholes approach to option pricing gives us exactly that – and here is how. Suppose we have a portfolio comprising one option and a quantity $-\Delta$ of the underlying asset (note here that we are thus considering the position of the option contract holder). The value of our portfolio at time $t$ is:

$$\Pi(x,t) = V(x,t) - \Delta(x,t)\,x(t) \qquad (2.51)$$

where so far we have shown the implicit dependence on $x$ and $t$ just to keep things formally correct. So as not to make things too messy, we now drop these dependences – let's just remember they are still there. At any given time $t$, the underlying asset price will more than likely change, hence so should the value of the option $V$ and hence the value of the portfolio. Hence the change in the value of the portfolio between time $t$ and time $t+dt$ is given by:

$$d\Pi = dV - \Delta\ dx \qquad (2.52)$$

Notice that the amount of the asset that we hold at time $t$ *does not* change between time $t$ and time $t+dt$ since we do not *a priori* know what will happen to the asset price $x(t)$. So while $\Delta$ stays constant in the time interval $t \to t+dt$, the asset price changes by $dx$, hence the option price changes by $dV$, hence the value of our portfolio changes by $d\Pi$. We can now substitute Equation (2.50) into Equation (2.52) to give:

$$d\Pi = \sigma x\left[\frac{\partial V}{\partial x}\right]dX + \left[\mu x\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t}\right]dt - \Delta\,dx \qquad (2.53)$$

and Equation (2.44) into Equation (2.53) to give

$$d\Pi = \sigma x\left[\frac{\partial V}{\partial x}\right]dX + \left[\mu x\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t}\right]dt - \Delta\,x\left[\sigma\,dX + \mu\,dt\right] \qquad (2.54)$$

Collecting up terms gives:

$$d\Pi = \sigma x\left[\frac{\partial V}{\partial x} - \Delta\right]dX + \left[\mu x\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t} - \mu\Delta x\right]dt \qquad (2.55)$$

The factor $\left[\frac{\partial V}{\partial x} - \Delta\right]$ is a **very important term**! This is because it controls the stochastic element in the portfolio variation $d\Pi$ and hence the portfolio risk. Even though Equation (2.55) appears to still be stochastic, we can *remove* the stochastic term *if* we can engineer the following condition at each time $t$:

$$\Delta = \frac{\partial V}{\partial x} \qquad (2.56)$$

Now, there is a lot that one could say about the practical complications of implementing this seemingly harmless mathematical trick. We will leave this for Chapter 6, and instead just move on to the final answer for the option price in standard finance theory. We assume that the condition in Equation (2.56) holds exactly at every value of $t$ (N.B. $t$ is continuous, hence it needs to hold at an infinite number of points!). Then Equation (2.55) becomes:

$$d\Pi = \left[ \mu x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t} - \mu \Delta x \right] dt \tag{2.57}$$

Substituting in from Equation (2.56), Equation (2.57) becomes:

$$d\Pi = \left[ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t} \right] dt \tag{2.58}$$

which is a completely *deterministic* equation for the change in the value of the portfolio at each time $t$. In other words, there is no longer a random $dX$ term affecting the value of the portfolio. In short, **the risk has been eliminated, yielding a zero-risk portfolio**.

Now, imagine that we hadn't purchased the option or underlying asset. Instead we had chosen the risk-free option of putting our capital in a reputable bank with fixed, guaranteed interest rate of value $r$. In this case our portfolio would instead have consisted entirely of cash and would have increased in value over the same time period $t \rightarrow t + dt$, by an amount $d\Pi = r\Pi\, dt$. Since we presumably have no way of making either a profit or loss systematically in our random-walk market, then these two gains should be equal, i.e. from Equation (2.58) we obtain:

$$r\Pi\, dt = \left[ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t} \right] dt \tag{2.59}$$

Substituting from Equations (2.51) and (2.56) into (2.59) gives:

$$r\left[ V - x\frac{\partial V}{\partial x} \right] dt = \left[ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t} \right] dt \tag{2.60}$$

We want our results to hold for *all* times $t$ during the life of the option, hence the factors multiplying $dt$ on both sides of Equation (2.60) can be equated, i.e. we can effectively cancel out the terms $dt$. This gives the equation

$$r\left[ V - x\frac{\partial V}{\partial x} \right] = \left[ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t} \right] \tag{2.61}$$

or, rearranging slightly, the final equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0 \tag{2.62}$$

which is the famous **'Black-Scholes equation'**. As long as the implicit assumptions made in its derivation are satisfied, any derivative security whose price depends only on the current value of $x$ and $t$, and which is paid for up-front, must satisfy this equation. Notice the amazing property that this equation is *independent* of the drift term $\mu$. Hence two people can disagree on the value of the drift but obtain the same option price. Of course, to solve this equation we need to know the boundary conditions. These boundary conditions define what type of option we are considering, as we will now discuss. For a European call option, the boundary conditions are:

$$V(x,T) = \max(x - X, 0) \qquad V(0,t) = 0 \qquad V(x,t) \xrightarrow{x\to\infty} x$$

If $x = 0$, it remains zero since $dx$ is also now zero. For a European put option, the boundary conditions are:

$$V(x,T) = \max(X - x, 0) \qquad V(0,t) = X\, e^{-r(T-t)} \qquad V(x,t) \xrightarrow{x\to\infty} 0$$

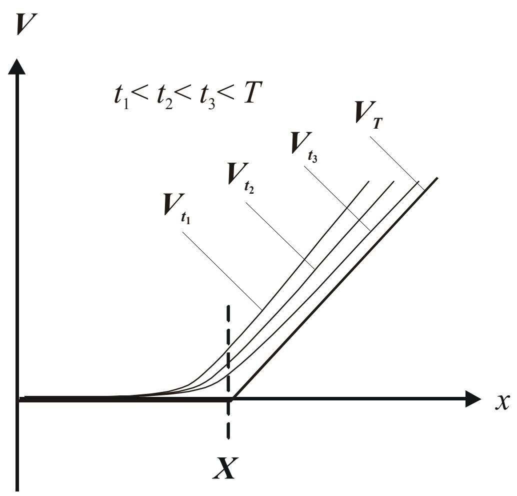
If $x = 0$, it remains zero since $dx$ is also now zero, hence the value of the put becomes the discounted strike price. The solutions of the Black-Scholes equation are as follows:

***European call option***:

$$V(x,t) = x\,\Phi[d_1] - X\, e^{-r(T-t)}\Phi[d_2] \quad \text{where}$$

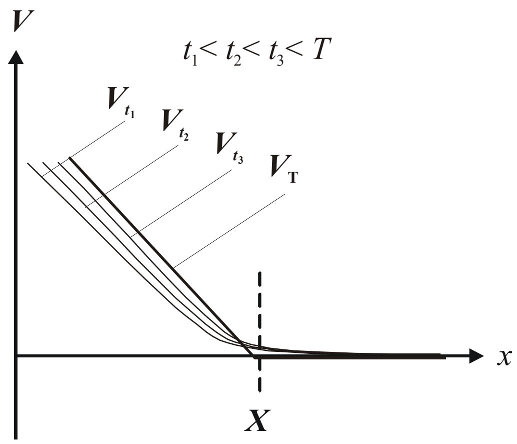
$$\Phi[z] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,dy\ ,\ d_1 = \frac{\ln(x/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\quad d_2 = \frac{\ln(x/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad (2.63)$$

The European call value solution $V(x,t)$ as a function of $x$ has the following form:

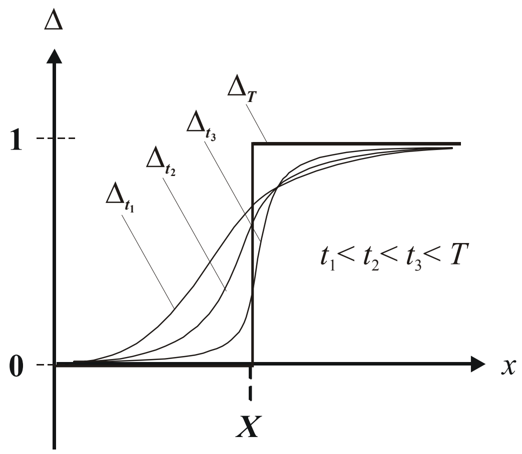



***European put option***:

$$V(x,t) = X\, e^{-r(T-t)}\Phi[-d_2] - x\,\Phi[-d_1] \qquad (2.64)$$




The ***hedging strategy*** for the European call option $\Delta = \frac{\partial V}{\partial x} = \Phi\left[d_1\right]$ has the following form:




For the European put option, the corresponding hedge is given by $\Delta = \frac{\partial V}{\partial x} = \Phi\left[d_1\right] - 1$. We note that these solutions can be checked by direct substitution into the Black-Scholes equation. If the values of the asset just before expiry are close to $X$ then the hedge may change from approximately zero to approximately 1 *many times*. But what about the effect of the associated transaction costs? Yes, that is a problem with this formalism. There is no obvious place to include the effect of transaction costs which, of course, could be enormous if you have to hedge continuously.

The Black-Scholes theory is a mathematical marvel. However even though it is widely used, and indeed has changed the whole face of derivatives trading, we have no guarantee that it really works in practice. Why? Well, it is built upon some strong implicit assumptions concerning random-walk models of asset prices, in addition to the use of continuous-time calculus. It *only* works out so neatly in mathematical terms because these assumptions have been made. As a prelude to Chapter 3, we therefore end with the following summary. If the statistics of the real market and the ideal random-walk market disagree, then standard finance theory may not apply to the market in question. Worse still, standard finance theory may even give misleading answers to important questions of investment, hedging, risk management etc. The extent of this error will depend on the type of financial calculation in question. Even if the actual PDF for price-changes is not too different from a Gaussian, the missing temporal correlations may be strong enough to throw quantitative calculations regarding hedging strategies, risk and portfolio management wildly off-track. Worse still, the ‘missing features’ may not be treatable within a perturbation-type treatment. In short: one cannot quantify the limitations of standard finance theory until it is compared to a more general theory which *doesn’t* make the same approximations.