Brownian Motion

Background material for Financial Physics

Brown’s observation

- While looking through his microscope, the botanist Robert Brown observed small (micron-sized) particles jiggling about in the water.
- These were not pollen grains themselves, which are much bigger, but much smaller ‘bits’ of the plants that come out of the pollen.
- This random motion was not properly understood until Einstein’s paper in 1905.

Einstein-Smoluchowski theory

- Smoluchowski is often forgotten. He published after Einstein.
- Einstein paper was entitled (translated from German):

On the Motion of Small Particles Suspended in a Stationary Liquid, as Required by the Molecular Kinetic Theory of Heat.

- "In this paper it will be shown that, according to the molecular kinetic theory of heat, bodies of a microscopically visible size suspended in liquids must, as a result of thermal molecular motions, perform motions of such magnitudes that they can be easily observed with a microscope. It is possible that the motions to be discussed here are identical with so-called Brownian molecular motion; however, the data available to me on the latter are so imprecise that I could not form a judgment on the question ..."
- This paper was one of the four major contributions to physics that Einstein made in 1905, his Annus Mirabilis.

Random Walk

- Random walks can be modelled using the binomial distribution (see Blundell & Blundell, Ch.3, or Johnson et al. Sect. 2.2.4).
- Well-known result that the standard deviation is proportional to the square root of the number of steps. $\sigma \propto \sqrt{N}$
- For a one-dimensional of fixed step length *L* the r.m.s. displacement is $< x^2 >^{1/2} = \sqrt{N} L$

2D-random walk

A simple way to show the mean square displacement grows linearly in time while the that the mean displacement remains zero: Consider a two-dimensional random walk of N vectors of fixed length l at random angles taken from a uniform probability distribution.

2D-random walk

$$x = \sum_i l \cos\theta_i$$ gives

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = \left\langle \left(\sum_i l \cos\theta_i \right)^2 \right\rangle = l^2 \left\langle \sum_i \cos\theta_i \sum_j \cos\theta_j \right\rangle$$

$$= l^2 \left\langle \sum_i (\cos\theta_i)^2 \right\rangle = l^2 N \frac{1}{2}$$

$$\left\langle \sum_{i,j\neq i} \cos\theta_i \cos\theta_j \right\rangle = 0$$

$$\langle r^2 \rangle = \langle x^2 + y^2 \rangle = N l^2$$

This result is useful for adding complex numbers with random phases.

Diffusion

$$\langle x^2 \rangle = 2Dt$$

Newton's 2nd law.

$$m\ddot{x} = -\alpha\dot{x} + F_0 + F(t)$$

- Forces: damping, steady, random.
- Steady-state solution: $\dot{x} = F_0/\alpha$
- Or, in terms of mobility: $u_{\mathrm{drift}} = \mu \frac{dV}{dx}$
- E.g., for a sphere of radius *a* in a liquid of viscosity *η*

$$\alpha = \mu^{-1} = 6\pi\eta a$$

Einstein relation

Consider a density *n(x)* of independent Brownian particles in a potential *V(x)*. In thermodynamic equilibrium the particle currents from diffusion and mobility balance:

$$-D\frac{dn}{dx} + n\left(-\mu\frac{dV}{dx}\right) = 0$$

At equilibrium $n(x) \propto \exp[-V(x)/k_B T]$

Hence:

$$D = k_B T \mu = \frac{k_B T}{\alpha}$$

A key result derived by Einstein relating diffusion to damping, or equivalently mobility.

Langevin Equation

To understand the Brownian motion more completely, we need to start from the basic physics, i.e. Newton's law of motion. The most direct way of implementing this is to recognize that there is a random, or stochastic, component to the force on the particle, *F(t)*, which we only know through a probabilistic description. N.B. This is not the approach used by Einstein, nor the closely related approach used by the Polish physicist Smoluchowski in 1906. This slightly later method by Langevin is a useful preliminary before discussing stochastic differential equations. The *Langevin equation* is

$$m\ddot{x} = -\alpha\dot{x} + F(t)$$

Forces: damping, random.

(Assume there's no drift for simplicity.)

Langevin Equation

$$m\ddot{x} = -\alpha\dot{x} + F(t)$$

- No random force, *F(t)=0.*

$$\ddot{x} = -\frac{\alpha}{m}\dot{x} \quad \rightarrow \quad \dot{u} = -\frac{1}{\tau}u$$

$$u = u(0)e^{-t/\tau}$$

Langevin Equation

$$\ddot{x} = -\frac{1}{\tau}\dot{x} + \frac{1}{m}F(t) \qquad \text{where } \tau = M/\alpha$$

- Multiply through by *x(t)*

$$x\ddot{x} = -\frac{1}{\tau}x\dot{x} + \frac{1}{m}xF(t)$$

- *x(t) is uncorrelated with F(t).*

$$\langle xF(t)\rangle = 0$$

- Hence taking time-averages

$$\langle x\ddot{x}\rangle + \frac{1}{\tau}\langle x\dot{x}\rangle = 0$$

"Solving" the Langevin Equation for mean square displacement

$$\langle x\ddot{x}\rangle + \frac{1}{\tau}\langle x\dot{x}\rangle = 0$$

$$x\frac{d^2x}{dt^2} = \frac{d}{dt}\left(x\frac{dx}{dt}\right) - \left(\frac{dx}{dt}\right)^2 = \frac{1}{2}\frac{d^2}{dt^2}\left(x^2\right) - \left(\frac{dx}{dt}\right)^2$$

Hence we find (assuming equilibrium value of $\left\langle u^2\right\rangle$

$$\frac{d^2\langle x^2\rangle}{dt^2} + \frac{1}{\tau}\frac{d\langle x^2\rangle}{dt} = 2\langle u^2\rangle = 2\frac{k_BT}{m}$$

- Solution (with initial position and velocity zero).

$$\left\langle x^2(t)\right\rangle = \frac{2k_BT}{\alpha}\tau\left[\frac{t}{\tau} - (1 - e^{-t/\tau})\right]$$

- Ballistic expansion at the thermal speed for short times, and diffusion at longer times.

The **Langevin equation** is nasty to deal with since the forcing term is a random sequence of delta functions—it is not piecewise continuous. However this did not prevent us finding the mean square displacement. It is slightly more involved to find then m.s. velocity starting from:

$$\frac{du}{dt} + \frac{u}{\tau} = \frac{F(t)}{m} = A(t)$$

Integrating from t=0 to t we find

$$u(t) = u(0)e^{-t/\tau} + e^{-t/\tau} \int_0^t e^{-t'/\tau} A(t') dt'$$

Giving the mean velocity found previously $\langle u(t) \rangle = u(0)e^{-t/\tau}$

The mean square velocity is

$$\langle u^2(t) \rangle = u^2(0)e^{-2t/\tau} + e^{-2t/\tau} \int_0^t \int_0^t e^{(t_1+t_2)/\tau} \langle A(t_1)A(t_2) \rangle dt_1 dt_2$$

Finding the m.s. velocity from the **Langevin equation**

$$\langle u^2(t)\rangle = u^2(0)e^{-2t/\tau} + e^{-2t/\tau}\int_0^t\int_0^t e^{(t_1+t_2)/\tau}\langle A(t_1)A(t_2)\rangle dt_1 dt_2$$

$\langle u(0)A(t>0)\rangle = 0$ has been used to eliminate the cross term.

Further details of how to manipulate this double integral can be found in *Statistical Mechanics,* by R.K. Pathria (Pergamon Press, 1972). N.B.

$$\langle A(0)A(\tilde{\tau})\rangle \to 0 \quad \text{as} \quad \tilde{\tau} \to \infty$$