Binomial tree model: alternative derivation based on a no-arbitrage argument
(From the Hull's book.)

Consider a one-step binomial tree with the definitions (as in the Binomial tree notes):

- x_0 is the initial price of underlying asset.
- $x_0 u$ and $x_0 d$ are the prices after time-step 1 on the upper/lower branches.
- V_0 is the initial value of the option.
- V_u and V_d are the values of the option after time-step 1 on the upper/lower branches.
- Π is the value of a portfolio of 1 option and short Δ assets.

Choose Δ such that

$$\begin{aligned} V_u - u x_0 \Delta &= V_d - d x_0 \Delta \\ \Rightarrow \Delta &= \frac{V_u - V_d}{x_0 u - x_0 d} \qquad \text{cf. } \Delta = \frac{\partial V}{\partial x} \quad \text{in Black} - \text{Scholes} \end{aligned}$$

At $t = 0$ the value of the portfolio is $V_0 - \Delta\, x_0 = \Pi(t = 0)$. This must equal the discounted value after 1 time-step (which is the same for the upper/lower branches) otherwise there would be an arbitrage opportunity.
(Continuously compounded interest at rate r is used here giving e^{rT} which is more usual than the $(1 + r)$ used in the lecture notes.)

$$\begin{aligned} V_0 - x_0 \Delta &= e^{-rT}(V_u - u x_0 \Delta) \\ V_0 &= x_0 \Delta - e^{-rT}(u x_0 \Delta - V_u) \\ &= \frac{V_u - V_d}{u - d}\left(1 - u e^{-rT}\right) + e^{-rT} V_u \\ &= e^{-rT}\left\{ V_u \frac{e^{rT} - d}{u - d} + V_d \frac{u - e^{rT}}{u - d} \right\} \\ V_0 &= e^{-rT}\left\{ p V_u + (1 - p) V_d \right\} \end{aligned}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

looks like a probability (the risk-neutral probability), since

$$\frac{e^{rT} - d}{u - d} + \frac{u - e^{rT}}{u - d} = \frac{u - d}{u - d} = 1.$$

CJ. Foot, April 2013