The binomial tree model: a simple example of pricing financial derivatives

SO9: Financial Physics. Prof. C. J. Foot, Physics Dept, University of Oxford

Assumed background knowledge

This an extension of the 'coin-toss' market (shown in Fig. 2-2 of the book by Johnson *et al.*) in which the two possible outcomes may have unequal probabilities. The Binomial distribution is described in the second-year lectures on probability and the book *Concepts in Thermal Physics* by Profs. Blundell and Blundell. (Sometimes a single step is called a Bernoulli trial.)

This description of the binomial tree model is structured as an answer to the following question (similar to one on the examination paper in 2011).

Question

Consider a binomial tree model for the stock price process $\{x_n : 0 \leq n \leq 3\}$. Let $x_0 = 100$ and let the price rise or fall by 10 % at each time-step. The interest rate is $r = 5\,\%$. The contract we wish to price is a European put option with strike price 110 at time-step 3.

(a) Find the risk neutral probabilities for the tree.

(b) Find the initial value of the option.

(c) After time-step 2, the stock price has three possible values. For each of these three cases, determine what trading strategy the writer of the option should follow to hedge the option.

(d) If the price movements for the asset are up, up and down find the trading strategy required to hedge the option.

Answer

(a) Probability in the binomial model

Denote the risk neutral probability as p for rising, and $1 - p$ for falling. In an arbitrage-free market the increase in share values matches the (riskless) increase from interest. This corresponds to the mathematical expression

$$px_0(1 + 10\%) + (1 - p)x_0(1 - 10\%) = x_0(1 + 5\%).$$

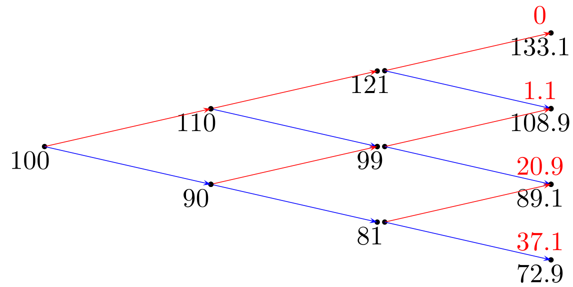
Or more generally, the price goes up by a factor u and down by d, with an interest rate r. Therefore at every time-step $pux_n + (1 - p)dx_n = (1 + r)x_n$ hence

$$\begin{aligned} pu + (1-p)d &= (1+r) \\ p &= \frac{1+r-d}{u-d} = \frac{1+0.05-0.9}{0.2} = 0.75 \end{aligned}$$

[An interest rate r=0 gives $p = 0.5$ which takes us back to the simple 'coin-toss' market.]

(b) Find the price of the option

There are 4 possible states of the market at time $n = 3$. The corresponding stock prices and payoffs of the option are shown in the following figure.



Here the numbers are stock prices (below) and the option payoff (above).

Figure 1: A Binomial tree model with 3 time-steps.

The expectation value of the option payoff in this binomial model is

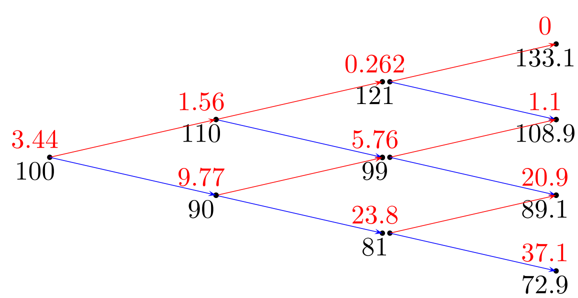
$$\begin{aligned} E(\text{payoff}) &= \left(p^3, 3p^2(1-p), 3p(1-p)^2, (1-p)^3\right) \begin{pmatrix} 0 \\ 1.1 \\ 20.9 \\ 37.1 \end{pmatrix} \\ &= \left(\frac{27}{64}, \frac{27}{64}, \frac{9}{64}, \frac{1}{64}\right) \begin{pmatrix} 0 \\ 1.1 \\ 20.9 \\ 37.1 \end{pmatrix} \\ &= \frac{27 \times 1.1 + 9 \times 20.9 + 37.1}{64} = 3.98 \end{aligned}$$

This is *not* the value of the option V because we have to account for interest. An amount V_0 at time-step 0 becomes worth $V_0(1+r)^3$ at time-step 3, so the value of the option is given by $V_0(1+r)^3 = E(\text{payoff})$, i.e.

$$V_0 = \frac{E(\text{payoff})}{(1+r)^3} = \frac{3.98}{1.158} = 3.44$$

This gives the initial value of the option for this market model. Other ways of writing this expectation value are given below.

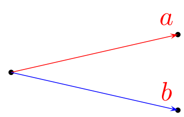
Similarly the value of the option at other nodes can be found by working backwards from the final payoff. (This is not asked for in the exam-style question above but it is useful for the purposes of this lecture, and easily implemented on a spreadsheet).



(c) Determining a suitable hedging strategy for the option writer

The writer of the option will lose money if the stock price goes down so that the holder receives a payoff—the losses or gains of the writer and holder are equal and opposite. Either, or both, of these parties to the contract can hedge to eliminate the risk. In practice it may be only the writer, e.g. a big financial institution such as an investment bank, that has the wherewithal to implement dynamic hedging. Let us consider the option writer's portfolio which consists of ϕ shares of stock and ψ units of risk-free asset. The risk-free asset increases by $(1+r)$ at each step.

At a given step n the market will be at a node in the binomial tree as in the following figure



Here a, b are the two possible values of the derivative after the next step. A portfolio (ϕ, ψ) that replicates the payoff can be found that satisfies the conditions

$$\begin{cases} \phi_n\, ux_n + \psi_n\,(1+r)B_n = a \\ \phi_n\, dx_n + \psi_n\,(1+r)B_n = b \end{cases}$$

where the risk-free asset has a unit price of B_n at step n. Solving this set of two simultaneous equations with two unknowns gives

$$\begin{aligned} \phi_n x_n &= \frac{a-b}{u-d}, \qquad \text{and} \\ \psi_n B_n &= \frac{bu-ad}{u-d} \times \frac{1}{1+r} \end{aligned}$$

Hence for the case where $u = 1 + 10\%$, $d = 1 - 10\%$ and $1 + r = 1 + 5\%$

$$\phi_n x_n = 5(a-b), \qquad \text{and} \tag{1}$$

$$\psi_n B_n = \frac{11b - 9a}{2 \times 1.05} \tag{2}$$

Note that is only the total amount held in riskless assets $\psi_n B_n$ that is important, not the actual number ψ_n, e.g. this could be cash in a bank account with units of either pounds sterling (GBP) or Euro. (We are ignoring any transaction or currency exchange charges.)[1]

We now apply this to the three possible prices of the binomial model after 2 time-steps, *viz.* 81, 99 and 121. From the eqn (1) above, we calculate the number of shares

- For $x_2 = 121$, $\phi_2 = \frac{0-1.1}{121\times0.2} = -0.0455$,
- For $x_2 = 99$, $\phi_2 = \frac{1.1-20.9}{99\times0.2} = -1$,
- For $x_2 = 81$, $\phi_2 = \frac{20.9-37.1}{81\times0.2} = -1$.

The values of $\phi_2 = -1$ for the two lowest prices correspond to the option writer hedging by short selling 1 share (equivalent to owning -1 units of the stock). Short selling is mathematically equivalent to a negative amount of the assets, i.e., the practice of selling assets,

[1] If desired, one could calculate definite values of ψ_n for a particular starting value of B_0, e.g. $(B_0, B_1, B_2, B_3) = (1, 1.05, 1.1025, 1.1576)$.

such as securities, that have been borrowed from a third party (e.g. a broker). The borrowed assets need to be returned to the lender at a later date. The prices $x_2 = 81$ and 99 are so far below the strike price of 110 that there will certainly be a payoff (there is no chance of going above the strike price after the third and final time-step). Thus the holder of the European put option will definitely exercise the option at the expiry date. Correspondingly the writer of the option will certainly have to pay the strike price to the holder for 1 unit of stock. However the $+1$ shares of the stock that the writer receives 'cancels' the -1 units of stock from short selling, in other words the option writer passes on the stock to the lender thus returning what was borrowed. (Note that under conditions for which the option is certain to be exercised we have the same situation as for a forward contract, whose discounted pricing was described in previous lectures.)

Conversely, if the stock price were to be so high that there is no chance that the put option will be exercised then the writer does not need to do any short selling (or anything else). This is not quite true for $x_2 = 121$ but the only possible payoff is small and hence ϕ_2 is close to zero.

A financial institution that sells an option for an initial value V, and then follows a hedging strategy as the stock price varies does not lose or gain (risk-free portfolio). They can actually sell the option for slightly more than V, and make a (riskless) profit on the margin.

Some exercises for the reader

You should use a spread-sheet programme for some of the repetitive calculations, e.g. from iii) onwards.

i) Redo parts b) and c) above for a European put option with strike price 108 at time-step 3. (Answers are not given but obviously there are only small changes cf. a strike price of 110.)

ii) Redo part a) above for an interest rate $r = 4\%$.

iii) Redo b) and c) above for an interest rate $r = 4\%$ and a European put option with strike price 110, or 108, at time-step 3.

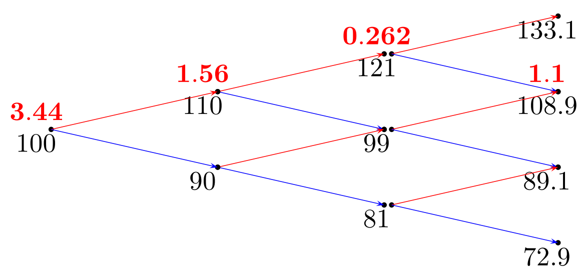
iv) Redo some, or all, of the above for a European call option.

v) Show that there is Put-call parity.

vi) Warning: this part is open-ended and has not been checked—you may need to adjust some of the parameters. Set up a binomial tree model, on a spreadsheet, with about 12, or more, time-steps. [Take the interest rate r as the input and calculate the risk neutral probability p from it.] Compare the value of options that you calculate at various times with the predictions of the Black-Scholes approach. (An Excel version of a B-S calculator is on the course website.) These should agree in the limit of a large number of small time-steps if you match the volatilities.

(d) Determining a suitable hedging strategy for the option writer along a particular path

We now apply eqns (1) and (2) to the path indicated in the following figure.



From the formulae above, we calculate the strategy

- At time 0, $\phi_0 = \frac{1.56-9.77}{100} \times 5 = -0.4107, \quad \psi_0 B_0 = \frac{11\times 9.77 - 9\times 1.56}{2\times 1.05} = 44.51$
- At time 1, $\phi_1 = \frac{0.262-5.76}{110} \times 5 = -0.2500, \quad \psi_1 B_1 = \frac{11\times 5.63 - 9\times 0.262}{2\times 1.05} = 29.06$
- At time 2, $\phi_2 = \frac{-1.1}{121} \times 5 = -0.0455, \quad \psi_2 B_2 = \frac{11\times 1.1}{2.1} = 5.76$

These results can be summarised in a table, where $\Pi_n = \phi_n x_n + \psi_n B_n - V$ is the total value of the portfolio (of the option writer)

step, $n =$	**0**	**1**	**2**	**3**
x_n	100	110	121	108.9
V_n	3.44	1.56	0.262	1.1
$\phi_n x_n$	-41.07	-27.5	-5.5	–
ϕ_n	-0.4107	-0.25	-0.0455	–
$\psi_n B_n$	44.51	29.06	5.76	–
$\phi_n x_n + \psi_n B_n$	3.44	1.56	0.262	–
Π_n	0	0	0	–

Buying and selling shares as dictated by Eqn. (1) and the riskless asset according to Eqn. (2) replicates the value of the option at each time-step, i.e. making $\Pi_n = 0$ at each step means that

$$V = \phi_n x_n + \psi_n B_n$$

The concept of a replicating portfolio is important in financial mathematics.

Comparison with other Financial Mathematics texts

The compounding of interest at each time-step to give a factor of $(1+r)^3$ after 3 steps has been used to keep things simple in the binomial model. Most texts use continuous compounding, i.e., riskless assets increase by e^{3r} after 3 time periods, and the exponential form is mathematically more convenient. The difference is small for $r \ll 1$. For the example in a binomial market with interest rate $r = 5\%$, as used above, a calculation using continuously compounded interest yields a risk neutral probability of $p = 0.7564$ cf. 0.75, and an initial value for the European put option with a strike price of 110 of $V = 3.28$ cf. 3.44.

The expectation value is usually written in a more complicated form; at time t the value for a European put option with a strike price of K which expires at time T is given by

$$V_t(x_t) = e^{-r(T-t)} E\left[\max(K - x_T, 0)|x_t\right]. \tag{3}$$

For a call option, with the same strike price and expiry date,

$$V_t = e^{-r(T-t)} E\left[\max(x_T - K, 0)|x_t\right]. \tag{4}$$

Or for continuous probability distribution functions, eqn 6.20 in the book of Johnson *et al* gives the initial value of an option in terms of the expectation value of the payoff at expiry,

$$V_0[x_0, K, T] = \langle V_T[x_T, K]\rangle_{x_T} = \int_0^\infty V_T[x_T, K] p\left[x_T|x_0\right]\, dx_T. \tag{5}$$

where $p\left[x_T|x_t\right]$ is the conditional probability of a final price x_T for a starting price x_0 at $t = 0$, and K is used for the strike price instead of X; N.B. the interest rate has been set to zero for simplicity (at the beginning of section 6.4.2 of the book). For a European call option this gives

$$V_t = \int_K^\infty (x_T - K) p\left[x_T|x_t\right]\, dx_T. \tag{6}$$

Risk Neutral Pricing
(N.B. the change of notation from x to S for the price of the asset.)

This note aims to clarify the crucially important difference between the risk-neutral probabilities (such as those used in the binomial tree model) and real-world, or physical, probabilities. In a risk-neutral world investors are assumed to require no extra return on average for bearing risks. The valuation of an option, or other derivative, in this risk-neutral world gives the correct price for the derivatives in all worlds (and avoids the need to know real-world, or physical probabilities). This is closely related to the elimination of the real-world drift in the price of the asset in the derivation of the Black-Scholes equation, and its replacement by a risk-neutral drift (which is not the actual expected drift but equals the cost of financing the asset and its yield). To repeat this crucial point: this approach does not try to predict or take expectations of real world events—this was the great insight of Black and Scholes; others had already established similar diffusion equations but thought that drift was determined by risk preferences.

Consider an asset currently priced at S whose price one period later can be Su with probability q, or Sd with probability $1 - q$. How should that asset be priced in a world of risk-averse investors? Any risky asset is priced as the discounted value of its future expectation because of the inherent risk. The expected future price of the asset is $qSu+(1-q)Sd$, and the current price can be written as

$$S = \frac{qSu + (1 - q)Sd}{1 + k}. \tag{7}$$

where k is the risky discount factor consisting of the risk free rate r plus a risk premium. It turns out that there values p and $1 - p$ that can be substituted for q and $1 - q$ which permit us to change k to the risk-free rate r, in the absence of arbitrage. Consider a market with no arbitrage opportunities in which there are only two assets: the stock and a risk-free bond. There would be an arbitrage opportunity if one could borrow an amount equal to the price of the stock at the risk-free rate, purchase the stock and guarantee earning at least the risk-free rate one period later. This would arise if $u > d > 1 + r$ (where the factor u is greater than d by definition). Also $1 + r > u$ cannot arise since then it would be possible to short the stock and use the proceeds to buy the bond to obtain a return of $1 + r - u$. Consequently, no arbitrage requires that

$$u > 1 + r > d. \tag{8}$$

Given these inequalities, it is possible to find weights p and $1 - p$ consistent with $pu + (1 - p)d = 1 + r$. Hence

$$S = \frac{pSu + (1 - p)Sd}{1 + r} \tag{9}$$

Comparison with eqn 7 shows that the price of an asset can be restated by changing the probabilities and discounting at the risk-free rate. The only information required is the volatility (as represented by u and d) and the risk-free rate. This no-arbitrage argument shows that it is possible to state the price of the asset in terms of risk-neutral probabilities with discounting at the risk-free rate. Calling p and $1 - p$ risk neutral probabilities is a source of much confusion.

There is a definite relationship between the probability in the binomial tree model and the actual probability. Define the stocks risk premium as $\phi = (\mathbb{E} - r)/\sigma$ where $\mathbb{E}$ is the expected return on the stock, defined as $\mathbb{E} = qu + (1 - q)d - 1$. The stocks variance is defined as $\sigma^2 = q(1 - q)(u - d)^2$. This is the volatility of a one period binomially distributed variable that can go up to u or down to d (which is straightforward to calculate in the usual way). Substituting these values into the risk premium and noting that $p = (1 + r - d)/(u - d)$, we obtain the result

$$p = q - \phi\sqrt{q(1 - q)} \tag{10}$$

Thus, the binomial probability is the actual probability minus the risk premium times the square root term arising from the volatility of a binomial process. In short, knowing the binomial probability $p = (1+r-d)/(u-d)$ is sufficient without having to know (or estimate) the risk aversion, ϕ, and the actual probability q.

Note that we have not considered options in this treatment. It is easily shown that a call (or put) option can be replicated by positions in the asset and the risk-free bond; therefore options do not change the nature of the market or the treatment given here, provided that they are properly priced allowing no arbitrage. This point has been made using a simple binomial framework but it can also be shown for continuous-time models, albeit with more mathematical complexity, e.g., in the book by S.N. Neftci, *An Introduction to the Mathematics of Financial Derivatives.* San Diego: Academic Press (1996) Chapters 1, 14, and 15. [This way of looking at risk-neutral probabilities borrows extensively from an (online) teaching note by Prof. D.M. Chance, Louisiana University. See also the Hull's book.]

Implied Volatility

The Black-Scholes solution gives the price of an option for a given strike price and volatility. However, if the assumptions of the B-S model are thought not to be valid, or for other reasons, the value of the option can differ from the B-S value. A common way to characterise this is to quote the volatility (assumed constant in time) that would have to be put into the analytic solutions of the B-S equation in order to match the observed value. This is the implied volatility. As a crude example of this the following spreadsheet uses a one-step binomial model to calculate the value of an option for various strike prices, and then evaluates the volatility that would give the same value when input into the solutions of the B-S equation. (A separate spreadsheet that implements the B-S solutions is available on the course website, or at Wolfram alpha). Clearly a model with only two possible final prices is not the same as assuming a Gaussian distribution, hence the implied volatility is not constant. The plot of implied volatility against strike price exhibits a ‘frown’, whereas other cases can lead to a ‘volatility smile’.

Implied volatility = volatility that gives the same price according to the Black-Scholes formula.

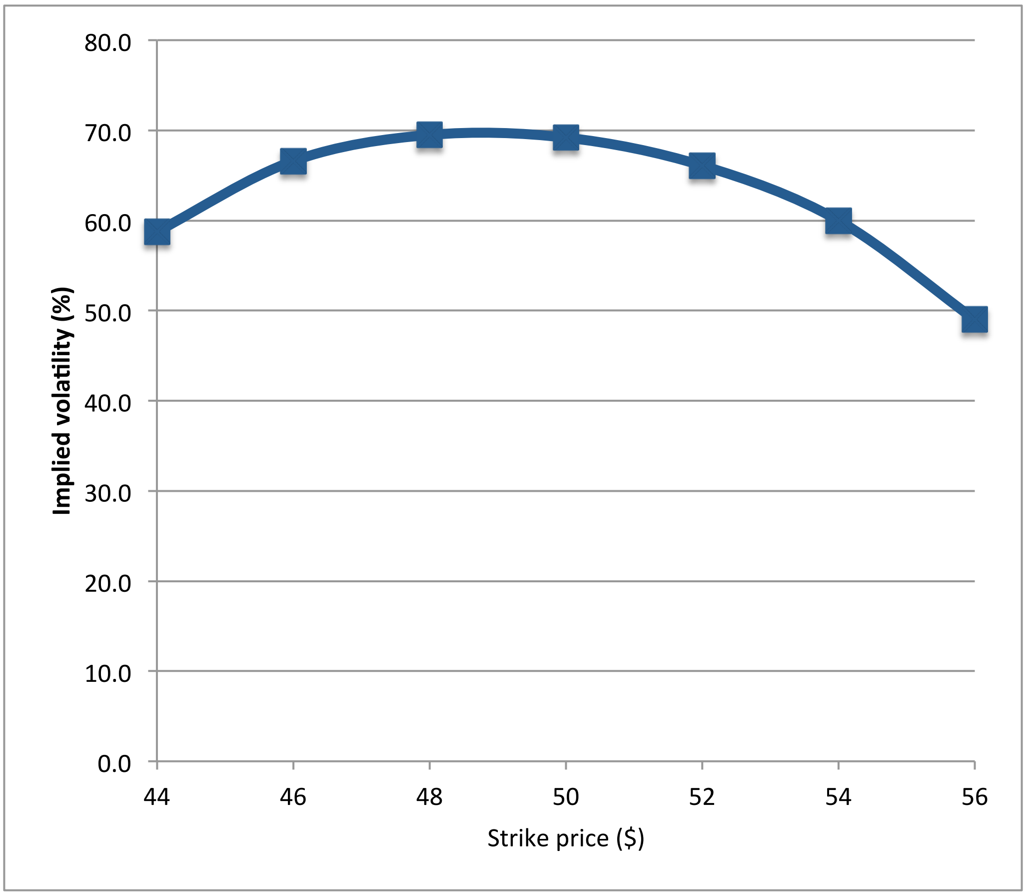
One-step binomial tree model with the following parameters:

u= 1.16
d= 0.84
r= 0.01 for 1 step.
p= 0.5314 p= (exp[rT]-d)/(u-d) = (exp[r]-d)/(u-d)

Call/put prices calculated from the binomial model with starting price =$50.

Strike price ($)	Call price ($)	Put price ($)	Implied Volatility %
42	8.42	0.00	
44	7.37	0.93	58.8
46	6.31	1.86	66.6
48	5.26	2.78	69.5
50	4.21	3.71	69.2
52	3.16	4.64	66.1
54	2.10	5.57	60.0
56	1.05	6.50	49.0
58	0.00	7.42	

From JC Hull, Options, Futures and other Derivatives, Sect. 19.8



CJ Foot, Financial Physics short option, Oxford Physics, 2012.